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A NATURAL DEDUCTION SYSTEM OF INDEXICAL LOGIC

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In a previous study,[1] a system **L** of indexical logic sound and complete with respect to a certain semantic theory was developed. However, **L** is an axiomatic logic. As usual, or perhaps even more than usual because of the complexity of indexical reasoning, the logic is unintuitively cumbersome since it is of the axiomatic type. For wrestling with problems of situational dependence in an adequate way, a natural deduction system of indexical logic is therefore needed. So far, no such system seems to exist in the literature. The present study consists of a formulation of the rules of a natural deduction system **N** of indexical logic and of a proof that **N** and **L** are equivalent.

1 *The system* **N** **N** is an improved and extended version of the system **N** of [1]. S is the auxiliary word "Show". It is assumed that no variable or constant occurs in S. A *show line* is a sequence $\mathrm{S}F$ and a *line* is either a show line or a formula. Crossing out the "Show" in front of F gives us F; that is, $\cancel{\mathrm{S}}F = F$. Given a finite sequence p of lines, the *conjunction of* p is the c such that c is $F \rightarrow F$ with F the first sentential constant if no line of p is a formula, F if F is the only line of p which is a formula, and the result of conjoining in order those lines of p which are formulas otherwise. p *consists of a show line* just in case the only line of p is a show line. If q is also a finite sequence of lines, then the following terminology is assumed:

1. q is obtainable from p by adding a show line just when q is p with a show line added at its end.

2. q is obtainable from p by adding an assumption just when there are formulas F_1 and F_2 such that the last line of p is $\mathrm{S}F_1$, q is p with F_2 added at its end, and one of the following holds (each clause is prefixed with its notation, name, and diagram):

a Rule of assumption for the proof of conditionals and disjunctions

$$\frac{\mathrm{S}F \rightarrow G}{F} \quad \frac{\mathrm{S}\,{\sim}F \vee G,\ \mathrm{S}F \vee G.}{{\sim}F}$$

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For some F and G, either $F_1 = F \rightarrow G$ and $F_2 = F$ or $F_1 = F \vee G$ and F_2 and F are contrary ($F_2 = \sim F$ or $F = \sim F_2$).

ca Rule of contrary assumption

$\frac{SF}{G}$ F and G contrary.

F_1 and F_2 are contrary.

3. q is obtainable from p by an inference rule just when there are formulas F_1 through F_5 such that F_1 through F_4 are lines of p, q is p with F_5 added at its end, and one of the following holds:

ei Existential instantiation

$\frac{\bigvee xF}{{}^x_yF}$ y a new variable.

For some x, y, and F such that y does not occur in p, $F_1 = \bigvee xF$ and $F_5 = {}^x_yF$.

si Simplification of implications or *modus ponens*

$$\frac{\begin{matrix} F \rightarrow G \\ F \end{matrix}}{G}$$

$F_1 = F_2 \rightarrow F_5$.

c Conjunction

$$\frac{\begin{matrix} F \\ G \end{matrix}}{F \wedge G}.$$

$F_5 = F_1 \wedge F_2$.

sc Simplification of conjunctions

$$\frac{F \wedge G \quad G \wedge F}{F}.$$

There is a G such that F_1 is one of $F_5 \wedge G$ and $G \wedge F_5$.

sd Simplification of disjunctions

$$\frac{F \vee G \;\; {}_{\sim F} \;\; G \vee F}{G}, \frac{\sim F \vee G \;\; {}_{F} \;\; G \vee \sim F}{G}.$$

There is a G such that F_1 is one of $G \vee F_5$ and $F_5 \vee G$, but F_5 and G are contrary.

se Simplification of equivalences

$$\frac{F \leftrightarrow G \;\; {}_{F} \;\; G \leftrightarrow F}{G}.$$

F_1 is one of $F_2 \leftrightarrow F_5$ and $F_5 \leftrightarrow F_2$.

ex General existence rule[2]

$$\frac{t\mathrm{I}u \;\; u\mathrm{I}t \;\; t\mathrm{A} \;\; t\mathrm{B} \;\; t\mathrm{M} \;\; t \vdash F \;\; t \ulcorner u\mathrm{E}}{t\mathrm{E}}, \; \frac{t \vdash u\mathrm{A} \qquad t \vdash u\mathrm{B}}{t \vdash u\mathrm{E}}.$$

There are t and u such that one of the following holds:
i. F_1 is one of $t\mathrm{I}u$, $u\mathrm{I}t$, $t\mathrm{A}$, $t\mathrm{B}$, $t\mathrm{M}$, $t \vdash F$, and $t \ulcorner u\mathrm{E}$ and $F_5 = t\mathrm{E}$.
ii. F_1 is one of $t \vdash u\mathrm{A}$ and $t \vdash u\mathrm{B}$ and $F_5 = t \vdash u\mathrm{E}$.

exv Existence rule for variables

$$\frac{\sim\!\!\wedge x F \;\; \vee x F \;\; t\mathrm{E}}{y\mathrm{E}}.$$

There are x, F, t, and y such that F_1 is one of $\sim\!\!\wedge xF$, $\vee xF$, and $t\mathrm{E}$ and F_5 is $y\mathrm{E}$.

id Rule of the identity with something of existents

$$\frac{t\mathrm{E}}{\vee x \;\, t\mathrm{I}x}\, x \text{ not free in } t.$$

There are t and x such that x is not free in t, $F_1 = t\mathrm{E}$, and $F_5 = \vee x \;\, t\mathrm{I}x$.

ui Universal instantiation

$$\frac{\begin{array}{l}\wedge y\langle y\mathrm{M} \rightarrow y \ulcorner t \;\mathrm{I}\; t\rangle \\ \quad t\mathrm{E} \\ \quad \wedge xF\end{array}}{{}^{x}_{t}F}\; y \text{ not free in } t, \; \frac{\begin{array}{c}z\mathrm{E} \\ \wedge xF\end{array}}{{}^{x}_{z}F}.$$

For some x, t, and F, either t is a variable or there is a y not free in t such that $F_1 = \wedge y\langle y\mathrm{M} \rightarrow y \ulcorner t \;\mathrm{I}\; t\rangle$, $F_2 = t\mathrm{E}$, $F_3 = \wedge xF$, and $F_5 = {}^{x}_{t}F$.

eg Existential generalization

$$\frac{\begin{array}{l}\wedge y\langle y\mathrm{M} \rightarrow y \ulcorner t \;\mathrm{I}\; t\rangle \\ \quad t\mathrm{E} \\ \quad {}^{x}_{t}F\end{array}}{\vee xF}\; y \text{ not free in } t, \; \frac{\begin{array}{c}z\mathrm{E} \\ {}^{x}_{z}F\end{array}}{\vee xF}.$$

For some x, t, and F, either t is a variable or there is a y not free in t such that $F_1 = \wedge y\langle y\mathrm{M} \rightarrow y \ulcorner t \;\mathrm{I}\; t\rangle$, $F_2 = t\mathrm{E}$, $F_3 = {}^{x}_{t}F$, and $F_5 = \vee xF$.

pd Properness of existentialized descriptions

$$\frac{\iota xF\mathrm{E}}{\vee y \wedge x\langle F \leftrightarrow x\mathrm{I}y\rangle}\; y \neq x \text{ and not free in } F.$$

There are x, F and y such that $x \neq y$, y is not free in F, $F_1 = \iota xF\mathrm{E}$, and $F_5 = \vee y \wedge x\langle F \leftrightarrow x\mathrm{I}y\rangle$.

int Interchangeability of coextensional terms and formulas

$$\frac{\begin{array}{l} \wedge\!\!\!\vee t\mathrm{E} \\ \wedge\!\!\!\vee u\mathrm{E} \qquad\qquad\qquad\qquad t\mathrm{I}u \\ \qquad \wedge y\langle y\mathrm{M} \to y \vdash \langle t\mathrm{E} \vee u\mathrm{E} \to t\mathrm{I}u\rangle\rangle \\ \qquad\qquad\qquad {}^{u}_{t}F \end{array}}{F} \quad y \text{ not free in } t \text{ or } u,$$

$$\frac{\begin{array}{l} \wedge\!\!\!\vee x\mathrm{E} \\ \wedge\!\!\!\vee y\mathrm{E} \quad x\mathrm{I}y \\ \quad {}^{y}_{x}F \end{array}}{F}, \quad \frac{\begin{array}{c} G \leftrightarrow H \\ \wedge y\langle y\mathrm{M} \to y \vdash \langle G \leftrightarrow H\rangle\rangle \\ {}^{H}_{G}F \end{array}}{F} \quad y \text{ not free in } G \text{ or } H,$$

$$\frac{t\mathrm{I}u \quad u\mathrm{I}t \quad t\mathrm{A}}{u\mathrm{A}}, \quad \frac{t\mathrm{I}u \quad u\mathrm{I}t \quad t\mathrm{B}}{u\mathrm{B}}, \quad \frac{t\mathrm{I}u \quad u\mathrm{I}t \quad t\mathrm{M}}{u\mathrm{M}}, \quad \frac{t\mathrm{I}u \quad u\mathrm{I}t \quad t\mathrm{I}v}{u\mathrm{I}v \quad v\mathrm{I}t} \quad \frac{v\mathrm{I}t}{t\mathrm{I}v \quad v\mathrm{I}u}, \quad \frac{t\mathrm{I}u \quad u\mathrm{I}t \quad t \vdash F}{u \vdash F},$$

$$\frac{t\mathrm{I}u \quad u\mathrm{I}t \quad t \ulcorner v\mathrm{E}}{t \ulcorner v \;\mathrm{I}\; u \ulcorner v}, \quad \frac{t\mathrm{I}u \quad u\mathrm{I}t \quad u \ulcorner v\mathrm{E}}{t \ulcorner v \;\mathrm{I}\; u \ulcorner v}, \quad \frac{v \vdash t\mathrm{I}u \quad v \vdash u\mathrm{I}t \quad v \vdash t\mathrm{A}}{v \vdash u\mathrm{A}}, \quad \frac{v \vdash t\mathrm{I}u \quad v \vdash u\mathrm{I}t \quad v \vdash t\mathrm{B}}{v \vdash u\mathrm{B}}.$$

There are t, u, v, y, F, G, and H such that one of the following holds:

i. y is not free in t or u, either t and u are variables or $F_1 = \wedge y\langle y\mathrm{M} \to y \vdash \langle t\mathrm{E} \vee u\mathrm{E} \to t\mathrm{I}u\rangle\rangle$, either F_2 and F_3 are $\wedge\!\!\!\vee t\mathrm{E}$ and $\wedge\!\!\!\vee u\mathrm{E}$ respectively or $F_2 = t\mathrm{I}u$, and $F_4 = {}^{u}_{t}F_5$.

ii. y is not free in G or H, $F_1 = \wedge y\langle y\mathrm{M} \to y \vdash \langle G \leftrightarrow H\rangle\rangle$, $F_2 = G \leftrightarrow H$, and $F_4 = {}^{H}_{G}F_5$.

iii. F_1 is one of $t\mathrm{I}u$ and $u\mathrm{I}t$, F_2 is one of $t\mathrm{A}$, $t\mathrm{B}$, and $t\mathrm{M}$, and F_5 is one of $u\mathrm{A}$, $u\mathrm{B}$, and $u\mathrm{M}$ respectively.

iv. F_1 is one of $t\mathrm{I}u$ and $u\mathrm{I}t$, F_2 is one of $t\mathrm{I}v$ and $v\mathrm{I}t$, and F_5 is one of $u\mathrm{I}v$, $v\mathrm{I}t$, $t\mathrm{I}v$, and $v\mathrm{I}u$.

v. F_1 is one of $t\mathrm{I}u$ and $u\mathrm{I}t$, $F_2 = t \vdash F$, and $F_5 = u \vdash F$.

vi. F_1 is one of $t\mathrm{I}u$ and $u\mathrm{I}t$, F_2 is one of $t \ulcorner v\mathrm{E}$ and $u \ulcorner v\mathrm{E}$, and $F_5 = t \ulcorner v \;\mathrm{I}\; u \ulcorner v$.

vii. F_1 is one of $v \vdash t\mathrm{I}u$ and $v \vdash u\mathrm{I}t$, F_2 is one of $v \vdash t\mathrm{A}$ and $v \vdash t\mathrm{B}$, and F_5 is one of $v \vdash u\mathrm{A}$ and $v \vdash u\mathrm{B}$ respectively.

intb Interchangeability of coextensional terms and formulas in variable binder expressions

$$\frac{\begin{array}{c} \wedge x_1 \ldots \wedge x_k \langle \wedge y\langle y\mathrm{M} \to y \vdash \langle u\mathrm{E} \vee t_i\mathrm{E} \to u\mathrm{I}t_i\rangle\rangle \wedge \langle u\mathrm{E} \vee t_i\mathrm{E} \to u\mathrm{I}t_i\rangle\rangle \\ b(xtF) \end{array}}{b(xt(^{i}_{u})F)},$$

$$\frac{\begin{array}{c} \wedge x_1 \ldots \wedge x_k \langle \wedge y\langle y\mathrm{M} \to y \vdash \langle u\mathrm{E} \vee t_i\mathrm{E} \to u\mathrm{I}t_i\rangle\rangle \wedge \langle u\mathrm{E} \vee t_i\mathrm{E} \to u\mathrm{I}t_i\rangle\rangle \\ b(xt(^{i}_{u})F) \end{array}}{b(xtF)},$$

$$\frac{\begin{array}{c} \wedge x_1 \ldots \wedge x_k \langle \wedge y\langle y\mathrm{M} \to y \vdash \langle u\mathrm{E} \vee t_i\mathrm{E} \to u\mathrm{I}t_i\rangle\rangle \wedge \langle u\mathrm{E} \vee t_i\mathrm{E} \to u\mathrm{I}t_i\rangle\rangle \\ v \vdash b(xtF) \end{array}}{v \vdash b(xt(^{i}_{u})F)},$$

$$\frac{\begin{array}{c}\wedge x_1 \ldots \wedge x_k \langle \wedge y \langle y\mathrm{M} \rightarrow y \vdash \langle u\mathrm{E} \vee t_i\mathrm{E} \rightarrow u\mathrm{I}t_i \rangle\rangle \wedge \langle u\mathrm{E} \vee t_i\mathrm{E} \rightarrow u\mathrm{I}t_i \rangle\rangle \\ v \vdash b(xt(\tbinom{i}{u})F)\end{array}}{v \vdash b(xtF)}$$

$$\frac{\begin{array}{c}\wedge x_1 \ldots \wedge x_k \langle \wedge y \langle y\mathrm{M} \rightarrow y \vdash \langle G \leftrightarrow F_j \rangle\rangle \wedge \langle G \leftrightarrow F_j \rangle\rangle \\ b(xtF)\end{array}}{b(xtF(\tbinom{j}{G}))},$$

$$\frac{\begin{array}{c}\wedge x_1 \ldots \wedge x_k \langle \wedge y \langle y\mathrm{M} \rightarrow y \vdash \langle G \leftrightarrow F_j \rangle\rangle \wedge \langle G \leftrightarrow F_j \rangle\rangle \\ b(xtF(\tbinom{j}{G}))\end{array}}{b(xtF)},$$

$$\frac{\begin{array}{c}\wedge x_1 \ldots \wedge x_k \langle \wedge y \langle y\mathrm{M} \rightarrow y \vdash \langle G \leftrightarrow F_j \rangle\rangle \wedge \langle G \leftrightarrow F_j \rangle\rangle \\ v \vdash b(xtF)\end{array}}{v \vdash b(xtF(\tbinom{j}{G}))},$$

$$\frac{\begin{array}{c}\wedge x_1 \ldots \wedge x_k \langle \wedge y \langle y\mathrm{M} \rightarrow y \vdash \langle G \leftrightarrow F_j \rangle\rangle \wedge \langle G \leftrightarrow F_j \rangle\rangle \\ v \vdash b(xtF(\tbinom{j}{G}))\end{array}}{v \vdash b(xtF)},$$

$$\frac{\begin{array}{c}b(xtF)\mathrm{E} \qquad\qquad\qquad\qquad\qquad\qquad b(xt(\tbinom{i}{u})F)\mathrm{E} \\ \underline{\wedge x_1 \ldots \wedge x_k \langle \wedge y \langle y\mathrm{M} \rightarrow y \vdash \langle u\mathrm{E} \vee t_i\mathrm{E} \rightarrow u\mathrm{I}t_i \rangle\rangle \wedge \langle u\mathrm{E} \vee t_i\mathrm{E} \rightarrow u\mathrm{I}t_i \rangle\rangle}\end{array}}{b(xt(\tbinom{i}{u})F)\ \mathrm{I}\ b(xtF)},$$

$$\frac{\begin{array}{c}v \vdash b(xtF)\mathrm{E} \qquad\qquad\qquad\qquad\qquad v \vdash b(xt(\tbinom{i}{u})F)\mathrm{E} \\ \underline{\wedge x_1 \ldots \wedge x_k \langle \wedge y \langle y\mathrm{M} \rightarrow y \vdash \langle u\mathrm{E} \vee t_i\mathrm{E} \rightarrow u\mathrm{I}t_i \rangle\rangle \wedge \langle u\mathrm{E} \vee t_i\mathrm{E} \rightarrow u\mathrm{I}t_i \rangle\rangle}\end{array}}{v \vdash b(xt(\tbinom{i}{u})F)\ \mathrm{I}\ b(xtF)},$$

$$\frac{\begin{array}{c}b(xtF)\mathrm{E} \qquad\qquad\qquad\qquad b(xtF(\tbinom{j}{G}))\mathrm{E} \\ \underline{\wedge x_1 \ldots \wedge x_k \langle \wedge y \langle y\mathrm{M} \rightarrow y \vdash \langle G \leftrightarrow F_j \rangle\rangle \wedge \langle G \leftrightarrow F_j \rangle\rangle}\end{array}}{b(xtF(\tbinom{j}{G}))\ \mathrm{I}\ b(xtF)},$$

$$\frac{\begin{array}{c}v \vdash b(xtF)\mathrm{E} \qquad\qquad\qquad v \vdash b(xtF(\tbinom{j}{G}))\mathrm{E} \\ \underline{\wedge x_1 \ldots \wedge x_k \langle \wedge y \langle y\mathrm{M} \rightarrow y \vdash \langle G \leftrightarrow F_j \rangle\rangle \wedge \langle G \leftrightarrow F_j \rangle\rangle}\end{array}}{v \vdash b(xtF(\tbinom{j}{G}))\ \mathrm{I}\ b(xtF)}.$$

In all these schemas, y is not free in a value of x, t, F, or $\langle uG \rangle$.

There are k, l, m, b, x, t, F, i, j, u, G, y, and v such that $\mathsf{C}\mathrm{N}klmbxtF$, $1 \leqslant k$, there is no value of x or t or F or $\langle uG \rangle$ in which y is free, and one of the following holds:

i. b is formula-making, $1 \leqslant i \leqslant l$, $F_1 = \mathsf{C}(\wedge x \wedge y \langle y\mathrm{M} \rightarrow y \vdash \langle u\mathrm{E} \vee t_i\mathrm{E} \rightarrow u\mathrm{I}t_i \rangle\rangle \wedge \langle u\mathrm{E} \vee t_i\mathrm{E} \rightarrow u\mathrm{I}t_i \rangle)$, and one of the following holds:

   a. $F_2 = b(xtF)$ and $F_5 = b(xt(\tbinom{i}{u})F)$ or vice versa
   b. $F_2 = v \vdash b(xtF)$ and $F_5 = v \vdash b(xt(\tbinom{i}{u})F)$ or vice versa.

ii. b is formula-making, $1 \leqslant j \leqslant m$, $F_1 = \mathsf{C}(\wedge x \wedge y \langle y\mathrm{M} \rightarrow y \vdash \langle G \leftrightarrow F_j \rangle\rangle \wedge \langle G \leftrightarrow F_j \rangle)$, and one of the following holds:

   a. $F_2 = b(xtF)$ and $F_5 = b(xtF(\tbinom{j}{G}))$ or vice versa
   b. $F_2 = v \vdash b(xtF)$ and $F_5 = v \vdash b(xtF(\tbinom{j}{G}))$ or vice versa.

iii. b is term-making, $1 \leqslant i \leqslant l$, $F_1 = \mathrm{C}(\bigwedge x \bigwedge y \langle y\mathrm{M} \to y \vdash \langle u\mathrm{E} \vee t_i\mathrm{E} \to u\mathrm{I}t_i\rangle\rangle \wedge \langle u\mathrm{E} \vee t_i\mathrm{E} \to u\mathrm{I}t_i\rangle)$, and one of the following holds:

a. F_2 is one of $b(xtF)\mathrm{E}$ and $b(xt(^{i}_{u})F)\mathrm{E}$ and $F_5 = b(xt(^{i}_{u})F)\ \mathrm{I}\ b(xtF)$

b. F_2 is one of $v \vdash b(xtF)\mathrm{E}$ and $v \vdash b(xt(^{i}_{u})F)\mathrm{E}$ and $F_5 = v \vdash b(xt(^{i}_{u})F)\ \mathrm{I}\ b(xtF)$.

iv. b is term-making $1 \leqslant j \leqslant m$, $F_1 = \mathrm{C}(\bigwedge x \bigwedge y \langle y\mathrm{M} \to y \vdash \langle G \leftrightarrow F_j\rangle\rangle \wedge \langle G \leftrightarrow F_j\rangle)$, and one of the following holds:

a. F_2 is one of $b(xtF)\mathrm{E}$ and $b(xtF(^{j}_{G}))\mathrm{E}$ and $F_5 = b(xtF(^{j}_{G}))\ \mathrm{I}\ b(xtF)$

b. F_2 is one of $v \vdash b(xtF)\mathrm{E}$ and $v \vdash b(xtF(^{j}_{G}))\mathrm{E}$ and $F_5 = v \vdash b(xtF(^{j}_{G}))\ \mathrm{I}\ b(xtF)$.

rwr Rewriting of bound variables

$$\frac{b(xtF)}{b(x(^{i}_{y})\ {}^{x_i}_{y}t\ {}^{x_i}_{y}F)},\quad \frac{b(x(^{i}_{y})\ {}^{x_i}_{y}t\ {}^{x_i}_{y}F)}{b(xtF)},\quad \frac{v \vdash b(xtF)}{v \vdash b(x(^{i}_{y})\ {}^{x_i}_{y}t\ {}^{x_i}_{y}F)},$$

$$\frac{v \vdash b(x(^{i}_{y})\ {}^{x_i}_{y}t\ {}^{x_i}_{y}F)}{v \vdash b(xtF)},\quad \frac{b(xtF)\mathrm{E}\quad b(x(^{i}_{y})\ {}^{x_i}_{y}t\ {}^{x_i}_{y}F)\mathrm{E}}{b(x(^{i}_{y})\ {}^{x_i}_{y}t\ {}^{x_i}_{y}F)\ \mathrm{I}\ b(xtF)},$$

$$\frac{v \vdash b(xtF)\mathrm{E}\quad v \vdash b(x(^{i}_{y})\ {}^{x_i}_{y}t\ {}^{x_i}_{y}F)\mathrm{E}}{v \vdash b(x(^{i}_{y})\ {}^{x_i}_{y}t\ {}^{x_i}_{y}F)\ \mathrm{I}\ b(xtF)}.$$

In all of these schemas, y is not free in a value of x, t, or F.

There are k, l, m, b, x, t, F, i, y, and v such that $\mathrm{CN}klmbxtF$, $1 \leqslant i \leqslant k$, there is no value of x or t or F in which y is free, and one of the following holds:

i. b is formula making and one of the following holds:

a. $F_2 = b(xtF)$ and $F_5 = b(x(^{i}_{y})\ {}^{x_i}_{y}t\ {}^{x_i}_{y}F)$ or vice versa

b. $F_2 = v \vdash b(xtF)$ and $F_5 = v \vdash b(x(^{i}_{y})\ {}^{x_i}_{y}t\ {}^{x_i}_{y}F)$ or vice versa.

ii. b is term making and one of the following holds:

a. F_2 is one of $b(xtF)\mathrm{E}$ and $b(x(^{i}_{y})\ {}^{x_i}_{y}t\ {}^{x_i}_{y}F)\mathrm{E}$ and $F_5 = b(x(^{i}_{y})\ {}^{x_i}_{y}t\ {}^{x_i}_{y}F)\ \mathrm{I}\ b(xtF)$

b. F_2 is one of $v \vdash b(xtF)\mathrm{E}$ and $v \vdash b(x(^{i}_{y})\ {}^{x_i}_{y}t\ {}^{x_i}_{y}F)\mathrm{E}$ and $F_5 = v \vdash b(x(^{i}_{y})\ {}^{x_i}_{y}t\ {}^{x_i}_{y}F)\ \mathrm{I}\ b(xtF)$.

ipd Identification of proper descriptions[3]

$$\frac{\bigvee y \bigwedge x \langle F \leftrightarrow x\mathrm{I}y\rangle}{\bigvee x \langle F \wedge \iota xF\ \mathrm{I}\ x\rangle}\quad y \neq x \text{ and not free in } F.$$

There are x, F, and y such that $x \neq y$, y is not free in F, $F_1 = \bigvee y \bigwedge x \langle F \leftrightarrow x\mathrm{I}y\rangle$, and $F_5 = \bigvee x \langle F \wedge \iota xF\ \mathrm{I}\ x\rangle$.

ac Actuality

$$\frac{\begin{array}{c} t\mathrm{E} \\ \mathcal{N}t\mathrm{B} \end{array}}{t\mathrm{A}}, \quad \frac{\begin{array}{c} u \vdash t\mathrm{E} \\ \mathcal{N}u \vdash t\mathrm{B} \end{array} \qquad u \ulcorner t\,\mathrm{I}\,u}{u \vdash t\mathrm{A}}.$$

There are t and u such that one of the following holds:

i. $F_1 = t\mathrm{E}$, $F_2 = \mathcal{N}t\mathrm{B}$, and $F_5 = t\mathrm{A}$.
ii. Either $F_1 = u \vdash t\mathrm{E}$ and $F_2 = \mathcal{N}u \vdash t\mathrm{B}$ or $F_2 = u \ulcorner t\mathrm{I}u$. Also, $F_5 = u \vdash t\mathrm{A}$.

sac Simplification of actuality

$$\frac{t\mathrm{A}}{\mathcal{N}t\mathrm{M} \quad \mathcal{N}t\mathrm{B}}, \quad \frac{u \vdash t\mathrm{A}}{\mathcal{N}u \vdash t\mathrm{B}}, \quad \frac{\begin{array}{c} u \vdash t\mathrm{A} \\ u \vdash t\mathrm{M} \end{array}}{u \ulcorner t\,\mathrm{I}\,u}.$$

There are t and u such that one of the following holds:

i. $F_1 = t\mathrm{A}$ and F_5 is one of $\mathcal{N}t\mathrm{M}$ and $\mathcal{N}t\mathrm{B}$.
ii. $F_1 = u \vdash t\mathrm{A}$ and $F_5 = \mathcal{N}u \vdash t\mathrm{B}$.
iii. $F_1 = u \vdash t\mathrm{A}$, $F_2 = u \vdash t\mathrm{M}$, and $F_5 = u \ulcorner t\,\mathrm{I}\,u$.

mom Momenthood

$$\frac{t \vdash F \quad t \ulcorner u\mathrm{E}}{t\mathrm{M}}.$$

There are t, F, and u such that F_1 is one of $t \vdash F$ and $t \ulcorner u\,\mathrm{E}$ and $F_5 = t\mathrm{M}$.

inv Invariance of variables

$$\frac{t\mathrm{M}}{t \ulcorner x\,\mathrm{I}\,x}.$$

There are t and x such that $F_1 = t\mathrm{M}$ and $F_5 = t \ulcorner x\,\mathrm{I}\,x$.

idx Indexing

$$\frac{\begin{array}{c} t\mathrm{M} \\ \mathcal{N}t \vdash F \end{array}}{t \vdash \mathcal{N}F}, \quad \frac{\begin{array}{c} t\mathrm{M} \\ \mathcal{N}t \vdash F \end{array} \quad t \vdash G}{t \vdash \langle F \to G \rangle}, \quad \frac{\begin{array}{c} t \vdash F \\ t \vdash G \end{array}}{t \vdash \langle F \wedge G \rangle}, \quad \frac{t \vdash F \quad t \vdash G}{t \vdash \langle F \vee G \rangle},$$

$$\frac{\begin{array}{c} t\mathrm{M} \\ t \vdash F \leftrightarrow t \vdash G \end{array}}{t \vdash \langle F \leftrightarrow G \rangle}, \quad \frac{\begin{array}{c} x\mathrm{E} \\ \Lambda x \quad t \vdash F \end{array}}{t \vdash \Lambda x F} \begin{array}{l} x \text{ not} \\ \text{free in } t \end{array}, \quad \frac{\bigvee x \quad t \vdash F}{t \vdash \bigvee x F} \begin{array}{l} x \text{ not} \\ \text{free in } t \end{array},$$

$$\frac{t \ulcorner u\,\mathrm{E}}{t \vdash u\,\mathrm{E}}, \quad \frac{t \ulcorner u\,\mathrm{M}}{t \vdash u\,\mathrm{M}}, \quad \frac{t \ulcorner u\,\mathrm{I}\, t \ulcorner v}{t \vdash u\mathrm{I}v}, \quad \frac{\iota x\; t \vdash F\;\mathrm{E} \quad t \ulcorner \iota x F\;\mathrm{E}}{t \ulcorner \iota x F\;\mathrm{I}\,\iota x\; t \vdash F} \begin{array}{l} x \text{ not} \\ \text{free in } t \end{array}.$$

There are t through v, F and G, and x such that x is not free in t and one of the following holds:

i. $F_1 = t\mathrm{M}$, $F_2 = \mathcal{N}t \vdash F$, and $F_5 = t \vdash \mathcal{N}F$.
ii. Either $F_1 = t\mathrm{M}$ and $F_2 = \mathcal{N}t \vdash F$ or $F_1 = t \vdash G$, and $F_5 = t \vdash \langle F \to G \rangle$.

iii. $F_1 = t \vdash F$, $F_2 = t \vdash G$, and $F_5 = t \vdash \langle F \wedge G\rangle$.
iv. F_1 is one of $t \vdash F$ and $t \vdash G$ and $F_5 = t \vdash \langle F \vee G\rangle$.
v. $F_1 = t\mathrm{M}$, $F_2 = t \vdash F \leftrightarrow t \vdash G$, and $F_5 = t \vdash \langle F \leftrightarrow G\rangle$.
vi. $F_1 = x\mathrm{E}$, $F_2 = \bigwedge x\ t \vdash F$, and $F_5 = t \vdash \bigwedge xF$.
vii. $F_1 = \bigvee x\ t \vdash F$ and $F_5 = t \vdash \bigvee xF$.
viii. $F_1 = t \ulcorner u$ E and $F_5 = t \vdash u\mathrm{E}$.
ix. $F_1 = t \ulcorner u$ M and $F_5 = t \vdash u\mathrm{M}$.
x. $F_1 = t \ulcorner u$ I $t \ulcorner v$ and $F_5 = t \vdash u\mathrm{I}v$.
xi. F_1 is one of $\iota x\ t \vdash F$ E and $t \ulcorner \iota xF$ E and $F_5 = t \ulcorner \iota xF$ I $\iota x\ t \vdash F$.

sidx Simplification of indices

$$\frac{t \vdash \sim F}{\sim t \vdash F},\quad \frac{t \vdash \langle F \rightarrow G\rangle \quad t \vdash F}{t \vdash G},\quad \frac{t \vdash \langle F \wedge G\rangle \quad t \vdash \langle G \wedge F\rangle}{t \vdash F},$$

$$\frac{t \vdash \langle F \vee G\rangle \quad t \vdash \langle G \vee F\rangle \quad \sim t \vdash F}{t \vdash G},\quad \frac{t \vdash \langle \sim F \vee G\rangle \quad t \vdash \langle G \vee \sim F\rangle \quad t \vdash F}{t \vdash G},$$

$$\frac{t \vdash \langle F \leftrightarrow G\rangle \quad t \vdash \langle G \leftrightarrow F\rangle \quad t \vdash F}{t \vdash G},\quad \frac{t \vdash \bigwedge xF}{\bigwedge x\ t \vdash F}\ \begin{matrix} x \text{ not} \\ \text{free in } t\end{matrix},$$

$$\frac{t \vdash \bigvee xF}{\bigvee x\ t \vdash F}\ \begin{matrix} x \text{ not} \\ \text{free in } t\end{matrix},\quad \frac{t \vdash u\mathrm{E}}{t \ulcorner u\ \mathrm{E}},\quad \frac{t \vdash u\mathrm{M}}{t \ulcorner u\ \mathrm{M}},\quad \frac{t \vdash u\mathrm{I}v}{t \ulcorner u\ \mathrm{I}\ t \ulcorner v}.$$

There are t through v, F and G, and x such that x is not free in t and one of the following holds:

i. $F_1 = t \vdash \sim F$ and $F_5 = \sim t \vdash F$.
ii. $F_1 = t \vdash \langle F \rightarrow G\rangle$, $F_2 = t \vdash F$, and $F_5 = t \vdash G$.
iii. F_1 is one of $t \vdash \langle F \wedge G\rangle$ and $t \vdash \langle G \wedge F\rangle$ and $F_5 = t \vdash F$.
iv. F_1 is one of $t \vdash \langle F \vee G\rangle$ and $t \vdash \langle G \vee F\rangle$, $F_2 = \sim t \vdash F$, and $F_5 = t \vdash G$.
v. F_1 is one of $t \vdash \langle \sim F \vee G\rangle$ and $t \vdash \langle G \vee \sim F\rangle$, $F_2 = t \vdash F$, and $F_5 = t \vdash G$.
vi. F_1 is one of $t \vdash \langle F \leftrightarrow G\rangle$ and $t \vdash \langle G \leftrightarrow F\rangle$, $F_2 = t \vdash F$, and $F_5 = t \vdash G$.
vii. $F_1 = t \vdash \bigwedge xF$ and $F_5 = \bigwedge x\ t \vdash F$.
viii. $F_1 = t \vdash \bigvee xF$ and $F_5 = \bigvee x\ t \vdash F$.
ix. $F_1 = t \vdash u\mathrm{E}$ and $F_5 = t \ulcorner u$ E.
x. $F_1 = t \vdash u\mathrm{M}$ and $F_5 = t \ulcorner u$ M.
xi. $F_1 = t \vdash u\mathrm{I}v$ and $F_5 = t \ulcorner u$ I $t \ulcorner v$.

aidx Association of indices

$$\frac{v \vdash t \vdash F}{v \ulcorner t \vdash F},\quad \frac{v \ulcorner t \vdash F}{v \vdash t \vdash F},\quad \frac{v \ulcorner \langle t \ulcorner u\rangle\ \mathrm{E}\ \langle v \ulcorner t\rangle \ulcorner u\ \mathrm{E}}{v \ulcorner \langle t \ulcorner u\rangle\ \mathrm{I}\ \langle v \ulcorner t\rangle \ulcorner u}.$$

There are t through v and F such that one of the following holds:

i. $F_1 = v \vdash t \vdash F$ and $F_5 = v \ulcorner t \vdash F$ or vice versa.
ii. F_1 is one of $v \ulcorner \langle t \ulcorner u\rangle$ E and $\langle v \ulcorner t\rangle \ulcorner u$ E and $F_5 = v \ulcorner \langle t \ulcorner u\rangle$ I $\langle v \ulcorner t\rangle \ulcorner u$.

4. q is obtainable from p by a proof method just when there are an index m of p[4] and formulas F_1 through F_3 such that there is no index n of p greater than m such that p_m is a show line, F_1 and F_2 are lines of p, $p_m = \not{S}\ F_3$, $q = p_m$ cut off from the m^{th} line with $F_3 = q_m$ added at its end, and one of the following holds.

p Direct proof[5]

$$\begin{array}{l} F \\ \hline \not{S}\ F \\ |\ \vdots \end{array} .$$

$F_3 = F_1$.

ip Indirect proof

$$\begin{array}{l} G \\ \sim G \\ \hline \not{S}\ F \\ |\ \vdots \end{array} .$$

F_1 and F_2 are contrary.

cp Conditional proof

$$\begin{array}{l} \sim F \quad G \\ \hline \not{S}\ F \rightarrow G \\ | \quad \vdots \end{array} .$$

There are F and G such that F_1 is one of $\sim F$ and G and $F_3 = F \rightarrow G$.

dp Disjunction proof

$$\begin{array}{l} F \quad G \\ \hline \not{S}\ F \vee G \\ | \quad \vdots \end{array} .$$

There are F and G such that F_1 is one of F and G and $F_3 = F \vee G$.

ep Equivalence proof

$$\begin{array}{l} F \rightarrow G \\ G \rightarrow F \\ \hline \not{S}\ F \leftrightarrow G \\ | \quad \vdots \end{array} .$$

There are F and G such that $F_1 = F \rightarrow G$, $F_2 = G \rightarrow F$, and $F_3 = F \leftrightarrow G$.

up Universal proof

$$\frac{F}{\not{S}\wedge x_1 \ldots \wedge x_i F} \quad x_1 \text{ through } x_i \text{ not free above.}$$

$$\begin{array}{l} | \quad \vdots \end{array}$$

For some nonempty sequence x of variables, there is no index k of p smaller than m such that a value of x is free in the largest formula which occurs in p_k and $F_3 = \mathsf{C}(\wedge x F_1)$.

The above clauses list the ways in which q *is obtainable from* p. A *proof sequence* is a nonempty and countable sequence s of finite sequences of lines such that s_1 consists of a show line and s_i is obtainable from s_{i-1} if i is an index of s. F is **N**-*provable* just when there exists a finite proof sequence whose last sequence of lines has F as its only line. That is, $\mathsf{S}F$ can be transformed into F by means of the rules and proof methods of **N**.

To make **N** applicable to an axiom set, the definition of obtainability can be extended by adding an additional axiom rule. Given finite sequences of lines p and q and a set A of formulas, q is obtainable from p by A just when either q is obtainable from p or the following holds:

ax Axiom rule

$$\frac{\vdots}{G} \quad G \text{ an axiom.}$$

q is p with a member of A added at its end.

The proof sequences constructed by this more general notion of obtainability are the *proof sequences in A*. F is **N**-*provable in A* just when there exists a finite proof sequence in A whose last line sequence has F as its only line. Thus, F is **N**-provable just when F is **N**-provable in every set of formulas and so just when F is **N**-provable in the empty set.

It can be shown that no new formulas are **N**-provable in A even if lines sequences are allowed to be extended by adding formulas which follow from previous lines by means of provable formulas. That is, if p and q are finite sequences of lines and q is p with G added at its end, then the following inference rule is derivable in **N** applied to A.

t Theorems rule

$$\frac{\begin{array}{l} F_1 \\ \vdots \\ F_m \end{array}}{G} \quad \begin{array}{l} \text{where } F_1 \wedge \ldots \wedge F_m \wedge G_1 \wedge \ldots \wedge G_n \rightarrow G \\ \text{and } G_1 \text{ through } G_n \text{ are provable.} \end{array}$$

There are positive integers m and n and formulas $F_1 \ldots F_m$, $G_1 \ldots G_n$ such that F_1 through F_m are lines of p, G_1 through G_n are **N**-provable in A, and $F_1 \wedge \ldots \wedge F_m \wedge G_1 \wedge \ldots \wedge G_n \rightarrow G$ is **N**-provable in A.

It is very useful to have the theorems rule among the inference rules even though it is redundant. In fact, the rules of **N** often overlap each other. This lack of economy is for the sake of ease and naturalness of application.

2 *The equivalence of* **N** *with* **L**

Lemma 1 *Every axiom of* **L** *is* **N**-*provable.*

Proof: The proof is through the construction of a proof sequence which has F as the only line of its last line sequence for any instance F of each of the schemas of **L**. All of the constructions are relatively simple. The following is an annotated line sequence designation which indicates a proof sequence for the main description principle of **L**. It is assumed that y is not free in F, t, or x and that the distinct y' and y'' do not occur in F through y.

1	$\not{S}\; t \text{ I } \text{℩} xF \leftrightarrow \vee y\langle \wedge x\langle F \leftrightarrow x\text{I}y\rangle \land t\text{I}y\rangle$	2, 19 **ep**
2	$\not{S}\; t \text{ I } \text{℩} xF \rightarrow \vee y\langle \wedge x\langle F \leftrightarrow x\text{I}y\rangle \land t\text{I}y\rangle$	18 **cp**
3	$t \text{ I } \text{℩} xF$	2 **a**
4	$\text{℩} xF \text{ E}$	3 **ex**
5	$\vee y \wedge x\langle F \leftrightarrow x\text{I}y\rangle$	4 **pd**
6	$\wedge x\langle F \leftrightarrow x\text{I}y'\rangle$	5 **ei**
7	$\vee x\langle F \land \text{℩} xF \text{ I } x\rangle$	5 **ipd**
8	${}^{x}_{y''}F \land \text{℩} xF \text{ I } y''$	7 **ei**
9	$\text{℩} xF \text{ I } y''$	8 **sc**
10	$y''\text{E}$	9 **ex**
11	${}^{x}_{y''}F \leftrightarrow y'' \text{ I } y'$	6, 10 **ui**
12	${}^{x}_{y''}F$	8 **sc**
13	$y'' \text{ I } y'$	11, 12 **se**
14	$\text{℩} xF \text{ I } y'$	9, 13 **int**
15	$t\text{I}y'$	3, 14 **int**
16	$\wedge x\langle F \leftrightarrow x\text{I}y'\rangle \land t\text{I}y'$	6, 15 **c**
17	$y'\text{E}$	15 **ex**
18	$\vee y\langle \wedge x\langle F \leftrightarrow x\text{I}y\rangle \land t\text{I}y\rangle$	16, 17 **eg**
19	$\not{S} \vee y\langle \wedge x\langle F \leftrightarrow x\text{I}y\rangle \land t\text{I}y\rangle \rightarrow t \text{ I } \text{℩} xF$	34 **cp**
20	$\vee y\langle \wedge x\langle F \leftrightarrow x\text{I}y\rangle \land t\text{I}y\rangle$	19 **a**
21	$\wedge x\langle F \leftrightarrow x\text{I}y'\rangle \land t\text{I}y'$	20 **ei**
22	$\wedge x\langle F \leftrightarrow x\text{I}y'\rangle$	21 **sc**
23	$y'\text{E}$	20 **exv**
24	$\vee y \wedge x\langle F \leftrightarrow x\text{I}y\rangle$	22, 23 **eg**
25	$\vee x\langle F \land \text{℩} xF \text{ I } x\rangle$	24 **ipd**
26	${}^{x}_{y''}F \land \text{℩} xF \text{ I } y''$	25 **ei**
27	$\text{℩} xF \text{ I } y''$	26 **sc**
28	$y''\text{E}$	27 **ex**
29	${}^{x}_{y''}F \leftrightarrow y'' \text{ I } y'$	22, 28 **ui**
30	${}^{x}_{y''}F$	26 **sc**
31	$y'' \text{ I } y'$	29, 30 **se**
32	$\text{℩} xF \text{ I } y'$	27, 31 **int**
33	$t\text{I}y'$	21 **sc**
34	$t \text{ I } \text{℩} xF$	32, 33 **int** .

Lemma 2 *If F and $F \to G$ are* **N**-*provable, then G is* **N**-*provable.*

Proof: Assume that there are proof sequences whose last line sequences are designated by $\begin{array}{l}\not S\ F\\ |\ \vdots\end{array}$ and $\begin{array}{l}\not S\ F \to G\\ |\quad\ \vdots\end{array}$. By means of **si** and **p**, these proof sequences can be combined into a proof sequence whose structure is indicated by the following designation and shows that G is **N**-provable:

$$\begin{array}{l}\not S\ G\\ \left|\begin{array}{l}\not S\ F\\ |\ \vdots\\ \not S\ F \to G\\ |\quad\ \vdots\\ G\end{array}\right.\end{array}\quad .$$

Lemma 3 *If F is* **N**-*provable, then $\land xF$ is* **N**-*provable.*

Proof: If there is a proof sequence whose last line sequence is designated by $\begin{array}{l}\not S\ F\\ |\ \vdots\end{array}$, it follows by **up** that there is a proof sequence whose structure is indicated by $\begin{array}{l}\not S\ \land xF\\ \left|\begin{array}{l}\not S\ F\\ |\ \vdots\end{array}\right.\end{array}$ and shows that $\land xF$ is **N**-provable.

Lemma 4 *If s is a proof sequence, m is an index of s, $1 < l \leq m$, and s_l is obtainable from s_{l-1} by a proof method, then the conjunction of (s_l without its last line) $\to$ the last line of s_l is* **L**-*provable.*

Proof: Assume the antecedent. For any l and m, let Alm hold just when m is an index of s, $1 < l \leq m$, and s_l is obtainable from s_{l-1} by a proof method. Also, for any index l of s, let Cl hold just when the conjunction of (s_l without its last line) $\to$ the last line of s_l is **L**-provable. Let P be the set of all m such that, if m is an index of s, then, for any l, Alm only if Cl. 1 is in P since not $Al1$. So assume that m is in P and $Alm + 1$. Hence, m is an index of s and so Cl if Alm. If s_{m+1} is not obtainable from s_m by a proof method, then s_l is not longer than s_{l-1} while s_{m+1} is longer than s_m and so $l \neq m + 1$ and $l \leq m$. That is, Alm and so Cl. Hence, assume that s_{m+1} is obtainable from s_m by a proof method. In other words, there is an index k of s_m and an F such that $(s_m)_k = \mathrm{S}F$, there is no index n of s_m greater than k such that $(s_m)_n$ is a show line, and s_{m+1} is s_m cut off at the k^{th} line with $(s_m)_k$ changed to F. If $l \neq m + 1$, $l \leq m$ and Cl since Alm. Assume then that $l = m + 1$. Also, if $s_{m+1} = s_l$ is obtainable from $s_m = s_{l-1}$ by **up** and $F = \land x_1 \ldots \land x_i G$ with i a positive integer, let $H = G$. Otherwise, $H = F$. Clearly,

a. the conjunction of $s_m \to H$ is **L**-provable,

for the formulas needed to obtain s_l from s_{l-1} by a proof method are present in the conjunction of s_m and tautologically imply H.

Let n = the largest index of s_m. By induction, it can be shown that

b. For any natural number z, if $z \leqslant n - k$, then the conjunction of (s_m cut off from the $(n + 1) - z^{\text{th}}$ line) $\rightarrow H$ is **L**-provable.

By a, b holds for $z = 0$. For any natural number $z \leqslant n - k$, let Kz = the conjunction of (s_m cut off from the $(n + 1) - z^{\text{th}}$ line). Assume both that z is a natural number such that K$z \rightarrow H$ is **L**-provable if $z \leqslant n - k$ and that $z + 1 \leqslant n - k$. Since $z \leqslant n - k$, K$z \rightarrow H$ is **L**-provable. Let $b = s_m$ cut off from the $(n + 1) - (z + 1) = n - z^{\text{th}}$ line and let $c = s_m$ cut off from the $(n + 1) - z = (n - z) + 1^{\text{th}}$ line. We must show that K$z + 1 \rightarrow H$ is **L**-provable.

It is not possible that c is obtainable from b by adding a show line since then there is an index n of s_m greater than k such that $(s_m)_n$ is a show line.

Assume that c is obtainable from b by adding an assumption. If there are J and K such that $(s_m)_{n-z} = \mathsf{S}\ J \rightarrow K$, then, since $(s_m)_k$ is the last show line of c, $n - z = k$. Hence, $H = J \rightarrow K$ and $(s_m)_{(n-z)+1} = J$. Since Kz = K$z + 1 \wedge J$ and K$z \rightarrow \langle J \rightarrow K\rangle$ is **L**-provable by assumption, it follows by tautological implication that K$z + 1 \rightarrow H$ is **L**-provable. Similarly, if $(s_m)_{n-z} = \mathsf{S}\ J \vee K$, $H = J \vee K$ and there is a G contrary to J such that $(s_m)_{(n-z)+1} = G$. Since Kz = K$z + 1 \wedge G$ and K$z \rightarrow J \vee K$ is **L**-provable by assumption, it follows again by tautological implication that K$z + 1 \rightarrow H$ is **L**-provable. Finally, if there are contrary J and K such that $(s_m)_{n-z} = \mathsf{S}\ J$ and $(s_m)_{(n-z)+1} = K$, $H = J$ and K$z + 1 \rightarrow H$ is **L**-provable by tautological implication since Kz = K$z + 1 \wedge K$ and K$z \rightarrow J$ is **L**-provable by assumption.

If c is obtainable from b by an inference rule other than **ei**, K$z + 1 \rightarrow$ Kz is clearly **L**-provable via the structure of **L** and Theorem 25 of [2]. Hence, K$z \rightarrow H$ is as well by tautological implication. Assume then that c is obtainable from b by **ei**. Hence, for some G, x, and y, $\vee x G$ is a line of b, y does not occur in b, and $(s_m)_{(n-z)+1} = {}^{x}_{y}G$. But y occurs in neither K$z + 1$ nor G nor H while Kz = K$z + 1 \wedge {}^{x}_{y}G$ and K$z \rightarrow H$ is **L**-provable by assumption. Hence, both K$z + 1 \wedge {}^{x}_{y}G \rightarrow H$ and K$z + 1 \rightarrow \vee x G$ are **L**-provable and so K$z + 1 \rightarrow H$ is **L**-provable via Corollary 12 and Theorem 25 of [2].

Assume finally that there is an index $j < m$ of s such that $c = s_{j+1}$ and c is obtainable from s_j by a proof method. Since $1 < j + 1 \leqslant m$, A$j + 1\ m$ and so C$j + 1$. This means that the conjunction of (c without its last line) $\rightarrow$ the last line of c is **L**-provable. But $c = s_m$ cut off from the $(n - z) + 1^{\text{th}}$ line and the last line of $c = (s_m)_{n-z} = J$ for some J. Hence, K$z + 1 \rightarrow J$ is **L**-provable while Kz = K$z + 1 \wedge J$ and so K$z + 1 \rightarrow H$ is again **L**-provable since K$z \rightarrow H$ is.

This exhausts the ways in which c is obtainable from its predecessors and so b holds. Putting $z = n - k$ in b, it follows that the conjunction of (s_m cut off from the $k + 1^{\text{th}}$ line) $\rightarrow H$ is **L**-provable. Since $(s_m)_k$ is a show line, the conjunction of (s_m cut off from the $k + 1^{\text{th}}$ line) = the conjunction of (s_m cut off from the k^{th} line). Also, s_m cut off from the k^{th} line = s_{m+1} without its last line and H = the last line of s_{m+1}. If J = the conjunction of (s_{m+1} without its last line), it follows that $J \rightarrow H$ is **L**-provable. If s_{m+1} is obtainable from s_m by **up**, there are a positive integer i and x through x_i such that $F = \wedge x_1 \ldots \wedge x_i\ H$ and none of x_1 through x_i is free in J. Since

$J \rightarrow H$ is **L**-provable, it follows that $J \rightarrow F$ is by repeated applications of the axioms of **L** corresponding to Theorem 13 of [2]. On the other hand, if s_{m+1} is obtainable from s_m by a proof method other than **up**, $F = H$ and $J \rightarrow F$ is again **L**-provable. Since $l = m + 1$, Cl. But then $m + 1$ is in P and the lemma holds.

Theorem 1 *F is* **N**-*provable just when F is* **L**-*provable.*

By Lemmas 1-3, every **L**-provable formula is **N**-provable. Assume then that there is a finite nonempty proof sequence s whose last line sequence has F as its only line. Let m be the greatest index of s. By analyzing cases, it is clear that s_m is only obtainable from s_{m-1} by a proof method. But then the conjunction of (s_m without its last line) $\rightarrow$ the last line of s_m is **L**-provable by Lemma 4. Since s_m has F as its only line, it follows that $(G \rightarrow G) \rightarrow F$ is **L**-provable where G is the first sentential constant and so F is **L**-provable by tautology and *modus ponens*.

Corollary 1 *F is* **N**-*provable just when F is valid.*

This follows from Theorem 1 together with Theorems 24 and 27 of [2].

NOTES

1. See [2]. That study was summarized at the Royal Institute of Technology in Stockholm in May of 1973 and presented in full at the Salzburg Colloquium on Logic and Ontology in September of 1973. The terminology of [2] is here presupposed. The system **N** and Theorem 1 were also referred to in an abstract in The *Bulletin of the Section of Logic* 5,(1976),pp. 16-19.

2. Observe that the rule only implies that logical predicates and operation symbols are existence implying. This is because intensional predicates and operation symbols are allowed for in **N** and **L**. Observe also that $t \ulcorner u$ E need not imply uE (for, like $\vdash$, $\ulcorner$ is intensional).

3. If the Hilbert selection variable binder ϵ were included among the logical constants of **N** and **L**, it could be dealt with by rules like **pd** and **ipd** in **N** and by the corresponding axioms in **L**. For example, ϵxFE $\rightarrow \mathsf{V}xF$ and $\mathsf{V}xF \rightarrow \mathsf{V}x\langle F \wedge \epsilon xF \mathrm{I}\, x\rangle$ could be added to the schemas of **L** together with the absoluteness principle $t \ulcorner \epsilon xF$E $\vee$ $\epsilon x\ t \vdash F$ E $\rightarrow t \ulcorner \epsilon xF \mathrm{I}\, \epsilon x\ t \vdash F$ (x not free in t). Notice that it is not one of the nonstandard existence rules for descriptions, but rather all of the usual substitution rules for descriptions which break down in indexical logics.

4. An index of a sequence is one of the objects in the domain of the sequence.

5. The line to the left of the schema indicates omission of the sequence below $\mathbb{S}F$.

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[2] Schock, R., "A complete system of indexical logic," *Notre Dame Journal of Formal Logic*, vol. XXI (1980), pp. 293-315.

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