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NORMAL DERIVABILITY AND FIRST-ORDER ARITHMETIC

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Jervell [3] proved a normal form for derivations in a first-order formal system of arithmetic (say **HA**, Heyting's arithmetic) in the following way: from every formal derivation in **HA** a possibly infinite structure is generated (included in the relations that defines the structure is some form of the ω-rule) and shown to be well founded. From the properties of such infinite structures, one goes back to **HA** and proves a normal form. From such a normal form many proof theoretical applications relative to **HA** can be given, among them consistency.

Two remarks can be made. First, the normal form for **HA** is not provable in a complete sense; hence the significance given to the normalization theorem by Prawitz ([8], III,2), (the operational interpretation of the logical constants), is weakened in the case of **HA**. Second, the proof theoretical applications relative to **HA** do not need normal form for **HA**, in the sense that they are already possible from the normal form for the induced infinite derivations. Moreover, that something is lost in going back to the normal form of **HA** can be deduced from the proof of the uniform reflection principle for **HA**, which is possible by the normal form of the infinite derivations, but not possible by Jervell's normal form.

In the present work we will follow a different way. We will start from the same **HA** (Section **1**), and generate infinite derivations (also by some form of the ω-rule); but, at this point, we do not aim to establish a normal form for **HA**. Instead, we will study the infinite derivations as a *sui generis* infinitary system, which we call ω-**HA** (Section **2**). We will establish the normal form for ω-**HA** (Section **5**) and, after that, go back to **HA** for applications (Sections **6** and **7**). The separate treatment of ω-**HA** is simply a matter of convenience, to make clear the object to be studied: it is not, properly speaking, a system of independent interest. Section **3** is the step required for extending to **HA** the proof theoretical properties of ω-**HA**. Section **4** is given because, in proving theorems, we find it preferable to make use of an assignment of ordinals to derivations instead of simply using bar induction. In fact, in this way we have a sharper measure for the

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principles applied in the proofs, and for the systems in which proofs are given (see, for example Section **5.26**).

The second remark above—that the applications do not need normal form for **HA**—was inspired to us by considering Lopez-Escobar [6], where the "extremely restricted ω-rules" are given for a sequent system of first-order arithmetic. Lopez-Escobar's ω-rules have the following form: derive $\forall xAx$ from the sequence of derivations $\{(A\overline{n})_n\}$ (all n), provided that $\forall xAx$ is derivable in **HA** from the same set of open assumptions. This saves us from defining the form of the ω-derivations, and avoids the complication of Section **2.2** below. However, we find it worthwhile to characterize more sharply the ω-derivations, since better results can be obtained—in particular the results mentioned in Kreisel-Mints-Simpson [5], p. 100, lines 1-3. Such results are contained in the proofs of (ii) and (iii) of Section **7.1**.

The direct object of the present work will be **HA**, because it allows the full generality of the proof theoretical results for first-order arithmetic. In fact, all results valid for **HA** extend to **PA** (Peano arithmetic, classical) by imbedding **PA** into the negative fragment of **HA**; while the converse is not true. If we would consider only **PA**, all the proofs of the present work would be extremely simplified; in fact, taking $\exists$ and $\vee$ as defined logical constants, in proving normalization only a very simple subset of proper reduction figures and no permutative reduction figure at all should be considered (leaving out all troublesome cases).

1 *The formal system* HA

1.1 **HA** is a formalization of arithmetic based on first-order intuitionistic predicate calculus in a natural deduction framework (see Prawitz [8], II,1). We have symbols for individual parameters and (bound) individual variables, *logical inference rules* for introducing and eliminating logical constants ($\vee$, $\wedge$, $\rightarrow$, $\forall$, $\exists$), and the intuitionistic Λ-rule. In addition we have the Post system with the symbols and the *basic rules* for 0 (zero), $'$ (successor), = (equality), + (addition), and $\cdot$ (multiplication). We also have the induction rule

$$\textbf{IND} \qquad \frac{A0 \qquad \begin{array}{c}[Aa]\\ Aa'\end{array}}{\forall xAx}$$

a is the proper parameter of the rule, subject to the usual restrictions; $\overline{n}$ is the symbol for the numeral $0' \cdots '$ where there are n occurrences of $'$.

1.2 A derivation is in normal form if no introduction rule or **IND** is followed by the inverse elimination rule. The normal form is obtained by applying reduction figures. As it is shown by Prawitz ([8], p. 263) all possible reduction figures are not enough for a complete normalization of **HA**. This is due to the possible presence of a parameter in the term t, when At is obtained by $\forall$**E** from $\forall xAx$, which is a consequence of **IND**.

2 *The infinitary system* ω-**HA**

2.1 ω-*rules* A natural way out for the situation described in Section **1.2** is given by the following considerations. In arithmetic we are dealing with a fixed denumerable domain ω (the natural numbers) and we have in the language of **HA** a closed term as the name for each of the individuals in ω (the numerals). It is then possible to replace all open rules of **HA** (i.e., **∀I**, **IND**, and **∃E**, containing proper parameters) by closed ones (without parameters). Since ω is denumerable, the resulting rules have necessarily denumerable premisses (ω-rules). With such ω-rules new reduction figures are now needed (see Section **2.4**) and they are enough for a complete normalization; i.e., Prawitz's inversion principle hold.

2.2 *Constructive and equivalent infinite derivations* It should not be difficult to recognize the plausibility of the ω-rules (once the pure formalist position is overcome): if the natural numbers provide the intended model for the numerals, the ω-rules should be considered as sound rules. Objections may arise if they are not constructively given (i.e., if there is no method for constructing each of the premisses) and, even if they are, if they are stronger than the corresponding open rules of **HA**, as in the case of the recursively restricted ω-rules (if so, one cannot extend to derivability in **HA** all the results stated about derivability in the system with ω-rules).

2.2.1 A natural way out for both the objections can be found by restructuring the ω-rules so that their premisses are constructed from derivations yet given in **HA**: i.e., by giving instructions on how to transform every open rule of **HA** into an ω-rule. This is done in Section **2.3**. We call the infinite derivations so obtained ω^*-derivations; the set of all ω^*-derivations is ω^*-**HA**.

2.2.2 Once we have the ω^*-derivations, we are able to normalize; i.e., any given Π of ω^*-**HA** can be transformed by applications of reduction figures into a Π' which is normal and equivalent to Π (i.e., it has the same conclusion and no more open assumptions). But at this point a difficulty arises in the characterization of our infinite derivations: the restricted ω-rules of ω^*-**HA** are not stable under normalization. In fact the reduction figures preserve the derivations of formulas, but not the form of such derivations. Since we have defined the ω^*-derivations only with respect to their form, we are led to widen (starting from ω^*-**HA**) the admissible infinite derivations. (This is done in Section **2.5**; and the steps toward such enlarged characterization are given in **2.4**.) We call the infinite derivations so obtained ω-derivations; the set of all ω-derivations is ω-**HA**.

2.2.3 The unusual characterization of ω-**HA** as a system follows by opposition to the usual one. In the latter a system is first defined using standard (finite or infinite) rules; then a set of reduction figures is given; and, third, the system is normalized: the reduction figures yield derivations belonging to the system itself. In the former the reduction figures enter into the definition of the system itself.

Both the characterizations above are given by defining a structure; but, while in the usual definition the relations defining the structure have the shape of the standard rules, in defining ω-**HA** they have a different shape (reduction figures vs rules). Reminding that a system is a structure with *specific* relations, the purists may then call ω-**HA** a structure (its universe being ω^*-**HA**). We prefer to call it a system since we are allowed by the analogy noted above (with adaptations left implicit) to extend to it the usual terminology such as *derivations, derivability*, etc., and the usual notations: $\Gamma \vdash A\,[\Gamma \vdash_{\omega^*} A,\ \Gamma \vdash_{\omega} A]$, $\vdash A\,[\vdash_{\omega^*} A,\ \vdash_{\omega} A]$ denote derivations in **HA** [ω^*-**HA**, ω - **HA**] of A from Γ and from null assumptions.

2.3 *The ω^*-derivations*

2.3.1 We give here some notions and notations to be used in the sequel.

1. $\begin{matrix}\Sigma_1\\ [A]\\ \Sigma_2\end{matrix}$ is the *composition* of $\begin{matrix}\Sigma_1\\ A\end{matrix}$ (with conclusion A) and $\begin{matrix}[A]\\ \Sigma_2\end{matrix}$ (with open assumption(s) A in Γ).

2. $\Sigma(a/\bar{n})$ is the *$\bar{n}$-substitution* of any numeral $\bar{n}$ for a in $\Sigma(a)$, Σ containing the parameter a.

3. Remark. If a is *the* proper parameter of an application α of **∃E**, **∀I**, **IND**, then the subderivation $\Sigma(a/\bar{n})$ *above* α is still a derivation in **HA**. The same applies when an improper parameter is replaced by some numeral.

4. **2.3.3** *Inductive clause in the definition of the ω^*-derivations.* In **2.3.2** and **2.3.3** below, in the derivation

$$\Sigma(a/\bar{n})$$

(eventually with open assumptions and/or conclusion in the pictorial notation) a is *the* proper parameter of an application α of **∀I**, **IND**, **∃E**, and $\Sigma(a/\bar{n})$ is obtained from the subderivation of one of the premisses of α, by substituting all open rules above α by ω-rules and all improper parameters by some numeral (note also that the proper parameters of some application below α are improper parameters of the subderivation above). In this way, $\Sigma(a/\bar{n})$ is an ω^*-derivation.

2.3.2 *The ω-rule for the universal quantifier* Suppose the following denumerable sequence of derivations is given in ω^*-**HA**:

$$\left\{\begin{pmatrix}\Sigma_n\\ A\bar{n}\end{pmatrix}_n\right\} \qquad (\text{all } n),$$

then we can conclude $\forall xAx$ only if one of the two following cases holds:

Case 1. For all n, $\begin{pmatrix}\Sigma_n\\ A\bar{n}\end{pmatrix}_n = \dfrac{\Sigma(a/\bar{n})}{A(a/\bar{n})}$, where Aa is the premiss of an application of **∀I** in **HA**.

If it is $\Gamma \vdash_{\omega^*} A(a/\bar{n})$, then it is $\Gamma \vdash_{\omega^*} \forall xAx$ (note that Γ is the same for all n, since a do not occur in Γ).

Case 2a. For $n = 0$, $\begin{pmatrix}\Sigma_0\\ A0\end{pmatrix}_0 = \begin{matrix}\Sigma\\ A0\end{matrix}$ where $A0$ is the zero-premiss of **IND**;

Case 2b. For $n > 0$, $\begin{pmatrix}\Sigma_n\\ A\bar{n}\end{pmatrix}_n$ is obtained by n compositions in the following way

$$\begin{pmatrix}\Sigma_1\\ A\bar{1}\end{pmatrix}_1 = \begin{matrix}\Sigma_0\\ [A(a/0)]\\ \Sigma(a/0)\\ A(a'/\bar{1})\end{matrix}, \quad \begin{pmatrix}\Sigma_2\\ A\bar{2}\end{pmatrix}_2 = \begin{matrix}\Sigma_1\\ [A(a/\bar{1})]\\ \Sigma(a/1)\\ A(a'/\bar{2})\end{matrix}, \ldots, \begin{pmatrix}\Sigma_n\\ A\bar{n}\end{pmatrix}_n = \begin{matrix}\Sigma_{n-1}\\ [A(a/\overline{n-1})]\\ \Sigma(a/\overline{n-1})\\ A(a'/\bar{n})\end{matrix}$$

where Aa' is the induction-premiss of **IND**.

If it is $\Gamma_0 \vdash_{\omega^*} A0$ and $\Gamma_1 \cup \{A\bar{n}\} \vdash_{\omega^*} A\overline{n+1}$, then $\Gamma_0 \cup \Gamma_1 \vdash_{\omega^*} \forall x Ax$ (corresponding to the discharge of Aa in **IND**).

For both cases we represent the rule compactly as:

$$\omega\forall\mathbf{I} \qquad \frac{\{(A\bar{n})_n\}}{\forall x Ax} \text{ with proviso.}$$

2.3.3 *The ω-rule for the existential quantifier* In ω^*-**HA**, if $\begin{matrix}\Sigma\\ \exists x A_x\end{matrix}$ and $\left\{\begin{pmatrix}[A\bar{n}]\\ \Sigma_n\\ B\end{pmatrix}_n\right\}$, then we can conclude B not depending on $A\bar{n}$ (all n), if

$$\begin{pmatrix}[A\bar{n}]\\ \Sigma_n\\ B\end{pmatrix}_n = \begin{matrix}[A\ (a/\bar{n})]\\ \Sigma(a/\bar{n})\\ B\end{matrix}$$ where B is minor premiss of $\exists\mathbf{E}$ in **HA**.

If $\Gamma_0 \vdash_{\omega^*} \exists x Ax$ and $\Gamma_1 \cup \{A\bar{n}\} \vdash_{\omega^*} B$, then $\Gamma_0 \cup \Gamma_1 \vdash_{\omega^*} B$. We represent the rule compactly as

$$\omega\exists\mathbf{E} \qquad \frac{\exists x Ax \quad \left\{\begin{pmatrix}[A\bar{n}]\\ B\end{pmatrix}_n\right\}}{B} \text{ with proviso.}$$

2.4 *Reduction figures*

2.4.1 We assume from Prawitz the notions of *maximal formula, maximal segment,* and *reduction figure* (including immediate simplifications). Reduction figures for infinite derivations are given as in Prawitz [8], pp. 251-254; except that now (instead of $\forall$-reduction, $\exists$-reduction, $\exists\mathbf{E}$-reduction and the immediate simplification for $\exists$) the following ω-reduction figures are given, corresponding to the ω-rules ($\Pi_1 \succ_1 \Pi_2$: Π_2 is the result of the application of a reduction figure to Π_1)

$$\omega\forall\,\mathbf{red}\cdot \qquad \cfrac{\cfrac{\left\{\begin{pmatrix}\Sigma_n\\ \overline{A\bar{n}}\end{pmatrix}_n\right\}}{\forall x Ax}}{A\bar{k}} \succ_1 \begin{pmatrix}\Sigma_k\\ \overline{A\bar{k}}\end{pmatrix}_k, \text{ for a fixed } k.$$

$$\omega\exists\text{red}\qquad \frac{\dfrac{\dfrac{\Sigma'_1}{A\bar{k}}}{\exists xAx}\qquad \left\{\left(\dfrac{\begin{matrix}[A\bar{n}]\\ \Sigma_n(\bar{n})\end{matrix}}{B}\right)_n\right\}}{B} \succ_1 \frac{\begin{matrix}\dfrac{\Sigma'_1}{[A\bar{k}]}\\ \Sigma_k(\bar{k})\end{matrix}}{B}\text{, for fixed } k.$$

$$\omega\exists\text{Ered}\qquad \frac{\dfrac{\dfrac{\Sigma'_1}{\exists xAx}\qquad \left\{\left(\dfrac{\begin{matrix}[A\bar{i}]\\ \Sigma_i\end{matrix}}{B}\right)_i\right\}}{B}\qquad \Sigma'_2}{C} \succ_1 \frac{\dfrac{\Sigma'_1}{\exists xAx}\qquad \left\{\left(\dfrac{\dfrac{\begin{matrix}[A\bar{i}]\\ \Sigma_i\end{matrix}}{B}\quad \Sigma'_2}{C}\right)_i\right\}}{C}\ .$$

$$\omega\exists\text{simpl}\qquad \frac{\dfrac{\Sigma'_1}{\exists xAx}\qquad \left\{\left(\dfrac{\Sigma_n}{B}\right)_n\right\}}{B} \succ_1 \left(\frac{\Sigma_i}{B}\right)_i\quad (\text{any } i)\ .$$

2.4.2 *Definitions*

Definition $\succ$ is the transitive relation generated by all $\succ_1$.

Definition Π is in $\succ$-relation with Π' if there is a sequence of derivations $\Pi_1, \Pi_2, \ldots, \Pi_n$ such that $\Pi_i \succ_1 \Pi_{i+1}$ and $\Pi = \Pi_n$ and $\Pi' = \Pi_1$.

2.5 *The infinitary system* ω-**HA**

2.5.1 *The infinitary system* ω^*-**HA** ω^*-**HA** is the system whose *language* is as in **HA**, except that individual parameters do not occur. Individual terms are all closed. The *logical inference rules* are as in **HA**, except that $\forall$**I** is replaced by $\omega\forall$**I**, and $\exists$**E** by $\omega\exists$**E**. The *basic inference rules* are as in **HA** (but all terms are closed). The *induction rule* does not appear; it is included in $\omega\forall$**I** (see Section **3.2.3**).

2.5.2 *The infinitary system* ω-**HA** The infinitary system ω-**HA** consists of all derivations in ω^*-**HA**, plus all derivations in $\succ$-relation with $\Pi \in \omega^*$-**HA**.

3 *Equivalence between* **HA** *and* ω-**HA** In this section we will see together what could be seen as immediate consequences of **2.3**, **2.4**, and **2.5**.

ω^*-**HA** is generated in such a way that an ω^*-derivation is given if and only if a finite equivalent derivation of a sentence is given in **HA**. This is what is shown in **3.1**-**3.3**.

ω-**HA** is generated in such a way that an ω-derivation is given if and only if an equivalent ω^*-derivation is given. This fact follows by investigation of the reduction figures.

3.1 Theorem *If* $\Gamma \vdash_{\overline{\omega^*}} A$, *then* $\Gamma \vdash A$.

Proof: By induction on the length of $\Gamma \vdash_{\overline{\omega^*}} A$. The basis and the steps for the rules which are in both systems are immediate. To show the theorem for conclusions of $\omega\forall$**I** and $\omega\exists$**E**, in these cases the proof is made possible by their restrictions. Let us consider $\omega\forall$**I**: let $\Gamma \vdash_{\overline{\omega^*}} \forall xAx$; to show $\Gamma \vdash \forall xAx$.

(i) Case 1 of Section **2.3.2** applies: $\Gamma \vdash_{\overline{\omega^*}} A\bar{n}$ is uniformly given for all

$\bar{n}$. By the restriction, $\Gamma \vdash Aa$ (both the Γ's are the same by the way $\Gamma \vdash_{\omega*} A\bar{n}$ is given, using the induction hypothesis on the closed formulas not containing a). Apply, in **HA**, **∀I**.

(ii) Case 2 of Section **2.3.2** applies: there are in **HA** both $\Gamma_0 \vdash A0$ and $\Gamma_1 \cup \{Aa\} \vdash Aa'$. Apply **IND**: $\Gamma_0 \cup \Gamma_1 \vdash \forall x Ax$.

3.2 Theorem *Let A be a closed formula. If $\Gamma \vdash A$, then $\Gamma \vdash_{\omega*} A$.*

Proof: By induction on the length of $\Gamma \vdash A$. The not immediate steps are for conclusions of **∃E**, **∀I**, and **IND**:

3.2.1 ∃E Suppose $\Gamma_0 \vdash \exists x Ax$ and $\Gamma_1 \cup \{Aa\} \vdash B$.
By Remark 3 of **2.3.1** $\Gamma_0 \vdash \exists x Ax$ and $\Gamma_1 \cup \{A(a/\bar{n})\} \vdash B$ (all n).
By induction hyp. $\Gamma_0 \vdash_{\omega*} \exists x Ax$ and $\Gamma_1 \cup \{A(a/\bar{n})\} \vdash_{\omega*} B$ (all n).
Apply ω**∃E** (the restrictions are clearly satisfied): $\Gamma_0 \cup \Gamma_1 \vdash_{\omega*} B$.

3.2.2 ∀I Suppose $\Gamma \vdash Aa$ (a proper parameter).
By Remark 3 of **2.3.1** $\Gamma \vdash A(a/\bar{n})$ (all n)
By induction hyp. $\Gamma \vdash_{\omega*} A(a/\bar{n})$ (all n)
By ω**∀I** $\Gamma \vdash_{\omega*} \forall x Ax$.

3.2.3 IND Suppose $\Gamma_0 \vdash A0$ and $\Gamma_1 \cup \{Aa\} \vdash Aa'$.
By Remark 3 of **2.3.1** $\Gamma_0 \vdash A0$ and $\Gamma_1 \cup \{A(a/\bar{n})\} \vdash A\overline{n+1}$ (all n).
By induction hyp. $\Gamma_0 \vdash_{\omega*} A0$ and $\Gamma_1 \cup \{A(a/\bar{n})\} \vdash_{\omega*} A\overline{n+1}$ (all n).
Build up, following the instruction of Case 2, Section **2.3.2**, the denumerable sequence of derivations $\{\Sigma_n\}$. Apply ω**∀I**.

3.3 Theorem $\Gamma \vdash A$ *if and only if* $\Gamma \vdash_{\omega*} A$ *(A closed).*

3.4 *Constructivity of* ω-**HA** "It must be remembered that an infinite derivation . . . is in certain respects an incomplete representation of an argument. In order to be conclusive, each application of **∀I** in such a derivation should be supplemented by an argument showing that for each n, Π_n is a derivation of $A\bar{n}$. It is by leaving out this supplementary argument in the representation of the proofs that the derivations get such a simple structure" (Prawitz [8], p. 267).

As to the "supplementary argument" in **2.3**, for Case 1 and **2.3.3** no problem arise, since all the premisses are uniformly derived from a constructively given derivation; for Case 2, each premiss is derived from two constructively given derivations. For the Case **2.4**, reduction figures preserve constructibility: in the not trivial cases, composition of given derivations is applied, just as in Case 2.

4 *Assignment of ordinals to ω-derivations* In this section we first present a system of ordinal numbers which can be assigned to every ω-derivation to measure its length and to make possible proofs by induction on the length of the well-founded derivations. After this, the normalization theorem for ω-**HA** is proved.

4.1 *Ordinals* For our purposes it is not necessary, but the full power of set theoretical ordinals, but simply a constructive segment of them and very few simple properties and functions. We use Gentzen's ordinals and the function # (natural sum) (see Gentzen [1], p. 277), and show that they are enough for our pruposes.

4.1.1 *Recursive definition of ordinals, equality and order relation between them* The system $\mathbf{S}_0$ consist of the number 0. We define $0 = 0$ and not $0 < 0$. Suppose that the ordinals of the system $\mathbf{S}_m$ $(0 \leqslant m < \omega)$ are already defined, as well as = and the <-relation. The system $\mathbf{S}_{m+1}$ consists of 0 and all the ordinals represented as

$$\omega^{\alpha_1} + \omega^{\alpha_2} + \ldots + \omega^{\alpha_n}$$

where $1 \leqslant n < \omega$ and the α_i's are ordinals of the system $\mathbf{S}_m$ and $\alpha_i \geqslant \alpha_{i+1}$. (Each system includes, then, all preceding ones.) Let β and γ belong to $\mathbf{S}_{m+1}$: $\beta = \gamma$ if their representations coincide. $\beta < \gamma$ $[\beta > \gamma]$ if the first noncoinciding exponent α_i in the representation of β is smaller [larger] than the corresponding exponent in the representation of γ.

With the usual set theoretical ordinals and functions on them we have the following correspondence (we shall make use notationally of this correspondence in the following):

$\mathbf{S}_1$ consist of $0, \omega^0, \omega^0 + \omega^0, \ldots$

that is

$$0, 1, 2, \ldots \text{ (all the natural numbers)}$$

The limit number of the system is ω.

$\mathbf{S}_2$ consist of

$$0, \omega^0, \omega^0 + \omega^0, \ldots, \omega^{\omega^0}, \omega^{\omega^0} + \omega_0, \ldots, \omega^{\omega^0} + \omega^{\omega^0}, \ldots, \omega^{\omega^0 + \omega^0}, \ldots$$

that is

$$0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega \cdot 2, \ldots, \omega^2, \ldots$$

The limit number is ω^ω.

$\mathbf{S}_3$ contains all numbers below ω^{ω^ω}(i.e., $\omega^{(\omega^\omega)}$); etc.

The limit number of all $\mathbf{S}_i$ is the number ε_0, the first ε-number.

Immediate properties of our ordinals are

(i) *if* $\beta = \gamma + \ldots$, *then* $\beta > \gamma$
(ii) *if* $\alpha < \beta$ *then* $\omega^\alpha < \omega^\beta$
(iii) *if* α *belongs to a* $\mathbf{S}_i$, *then* ω^α *belongs to* $\mathbf{S}_{i+1}$, *hence* $\omega^\alpha < \varepsilon_0$.

4.1.2 *Natural sum* (#) *of two nonzero ordinals* Let $\alpha = \omega^{\gamma_1} + \ldots + \omega^{\gamma_n}$ and $\beta = \omega^{\delta_1} + \ldots + \omega^{\delta_m}$ and $\zeta_1 \geqslant \zeta_2 \geqslant \ldots \geqslant \zeta_{m+n}$ be all the γ_i and δ_i ordered by $\geqslant$. Then

$$\alpha \# \beta = \omega^{\zeta_1} + \ldots + \omega^{\zeta_{m+n}}.$$

The following properties of # are easily proved:

(i) $\alpha \# \beta = \beta \# \alpha$
(ii) $\alpha \# \beta > \alpha$
(iii) *if* $\alpha_1, \alpha_2 < \alpha$, *then* $\omega^{\alpha_1} \# \omega^{\alpha_2} < \omega^{\alpha}$
(iv) *if* $\alpha_1 < \alpha$ *then* $\alpha_1 \# \beta < \alpha \# \beta$
(v) $\#$ *is strictly increasing in both arguments.*

4.1.3 *Assignment of ordinals to* ω*-derivations* ω-derivations are, in the presence of ω-rules, represented in the form of infinitely generated trees; i.e., some nodes of the trees have denumerably many nodes immediately above. As we shall show, trees are well founded; i.e., each thread is finite. At each node there corresponds a formula in the derivation.

4.1.3.1 We assign ordinals to the nodes of the tree in the following way:

(i) to a top formula A we assign an ordinal $\alpha > 0$.
Notation: $\mathrm{ord}(A) = \alpha$.

(ii) Let A_i be the premisses of A. If $\mathrm{ord}(A_i) = \alpha_i$, then $\mathrm{ord}(A) \geqslant \sup(\alpha_i) = \alpha$, where α is strictly greater than every α_i.
Notation: $\mathrm{ord}\begin{pmatrix}\Sigma \\ A\end{pmatrix} = \mathrm{ord}(A) = \alpha$.

4.1.3.2 Remark. Let $\alpha = \mathrm{ord}\begin{pmatrix}\Sigma_1 \\ A\end{pmatrix}$, $\beta = \mathrm{ord}\begin{pmatrix}[A] \\ \Sigma_2\end{pmatrix}$. Then $\mathrm{ord}\begin{pmatrix}\Sigma_1 \\ [A] \\ \Sigma_2\end{pmatrix} \leqslant \alpha \# \beta$.

This is an immediate consequence of **4.1.3.1** and the properties of $\#$: in fact, if β_i was an ordinal of $\begin{matrix}[A] \\ \Sigma_2\end{matrix}$, it becomes $\beta_i \# \alpha$ in $\begin{matrix}\Sigma_1 \\ [A]. \\ \Sigma_2\end{matrix}$

4.1.4 *Gentzen's ordinals are enough* In order to show that to every (not normal) ω-derivation a Gentzen's ordinal can be assigned, we first consider ω^*-**HA**. Here we argue again by induction on the length of derivations. Top formulas have ordinals $\alpha < \varepsilon_0$. Suppose that the premisses A_i (i depending on the number of the premisses) of some rule have ordinals $\alpha_i < \varepsilon_0$. To show that the conclusion A has ordinal $\alpha < \varepsilon_0$.

As to the finite rules, ω**∃E** and ω**∀I** corresponding to **∀I** in **HA**, there is no problem, since for every α_i there is an α such that $\alpha_i < \alpha < \varepsilon_0$ (take, for example, $\max(\alpha_i) + 1 = \alpha$, where $\max(\alpha_i)$ is the larger ordinal in the set $\{\alpha_i\}$; in the case of the infinite rules above, $\max(\alpha_i)$ exist: all α_i are in fact the same. As to ω**∀I** corresponding to **IND** in **HA**, there is no such $\max(\alpha_i)$. Considering this last case we show in **4.1.4.3** that to every given derivation in ω^*-**HA** an ordinal $\alpha < \varepsilon_0$ can be assigned. If we then consider ω-**HA**, we examine each reduction figure. The result follows either trivially or, when $\Pi >_1 \Pi'$ by composition, in the following way: $\mathrm{ord}(\Pi') = \alpha = \alpha_1 \# \alpha_2$ (see **4.1.3.2**), where α_1 and α_2 are ordinals of subderivations of Π. If $\alpha_1, \alpha_2 < \varepsilon_0$, then $\alpha < \varepsilon_0$. Gentzen's ordinals are also enough for further increases of ordinals in Section **5.2** by **4.1.1** (iii).

4.1.4.1 *Depth of a derivation in* **HA**

Definition $\mathrm{depth}(\Pi) = n(0 \leq n)$ if n is the highest number of occurrences of **IND** in every thread of Π.

4.1.4.2 Lemma (corollary of **3.2.3**) *Let* Π *be a derivation in* **HA** *such that* $\mathrm{depth}(\Pi) = k$. Then *there is an equivalent* ω^**-derivation* Π' *such that* $\mathrm{ord}(\Pi') < \omega^k + \omega$.

Proof: By induction on k. Basis: $k = 0$. Then $\mathrm{ord}(\Pi') = \mathrm{ord}(\Pi) < \omega$.

Step: $k = m + 1$. Let $\dfrac{\Sigma_0}{A0}$ and $\dfrac{\begin{matrix}[Aa]\\ \Sigma_1(a)\end{matrix}}{Aa'}$ be the derivations of the two premisses of an occurrence of **IND** in Π such that at least one of them is of $\mathrm{depth} = m$. Apply **2.3.1.3**; by induction hypothesis there are in ω^*-**HA**

$$\frac{\Sigma'_0}{A0} \text{ and, for each } n,\ \frac{\begin{matrix}[A\bar{n}]\\ \Sigma'_1(\bar{n})\end{matrix}}{A\overline{n+1}}$$

such that their ordinals are, respectively, $\alpha_0 < \omega^m + \omega$ and $\alpha_1 < \omega^m + \omega$. Hence the sequence $\{\Sigma_n\}$ of **3.2.3** has the following ordinals:

$$\alpha'_0 < \omega^m + \omega,\ \alpha'_1 < (\omega^m + \omega)\cdot 2, \ldots, \alpha'_{i-1} < (\omega^m + \omega)\cdot i, \ldots .$$

Then $\sup((\omega^m + \omega)\cdot n) = \omega^{m+1}$ (all n) is the ordinal of the conclusion of $\{\Sigma_n\}$. From this point below, till the premisses of the next **IND** below (if any, otherwise till the conclusion of the derivation) the ordinals are finitely increased; hence their ordinals are $<\omega^{m+1} + \omega$. This concludes the proof.

4.1.4.3 Theorem *For every* Π *of* **HA** *there is an equivalent* Π' *of* ω^*-**HA** *such that* $\mathrm{ord}(\Pi') < \omega^\omega$.

Proof: Immediate from **4.1.4.2**, being $\mathrm{depth}(\Pi) < \omega$ in every Π of **HA**.

5 *Normalization theorem for* ω-**HA**

5.1 A *normalization theorem* for ω-**HA** is the constructive proof that, by repeated applications of the reduction figures for ω-**HA**, for any ω-derivation Π one can find an equivalent Π' to which no reduction figure can be applied.

5.2 *Normalization theorem*

5.2.1 Let $\Pi \succ_1 \Pi'$ be a *proper* reduction figure; Π_0 the derivation of the maximal formula M; Π_i the derivations of the co-premisses F_i of M (if any); $\alpha_0 = \mathrm{ord}(\Pi_0)$, $\alpha_i = \mathrm{ord}(\Pi_i)$. The (sub)derivation Π_n of F_n which occurs in Π' is said to be the *active subderivation of* $\succ_1$.

Lemma *Under the conventions above, if* $\mathrm{ord}(\Pi) = \alpha$, *then* $\mathrm{ord}(\Pi') \leq \alpha_0 \# \alpha_n$, *where* α_n *is the ordinal of the active subderivation* Π_n ($\alpha_n = 0$ *if there is no such* Π_n).

Proof: Immediate by the inspection of the various reduction figures, applying **4.1.3.2**.

5.2.2 *Definitions and notations*

5.2.2.1 *Definition* The *degree* d of a derivation Π in ω-**HA** is the highest degree of a maximal segment in Π.

Notation: $\mathrm{deg}(\Pi) = d$. If it is $\begin{smallmatrix}\Pi\\ A\end{smallmatrix}$, then $\mathrm{deg}(A) = \mathrm{deg}(\Pi)$. (Note that $d(A)$ is the logical degree of a formula, while $\mathrm{deg}(A)$ is the degree of a derivation with conclusion A).

5.2.2.2 *Remark on the existence of* $\mathrm{deg}(\Pi)$ From the way ω-**HA** is generated one can see that the degrees of the formulas in **HA** are left unchanged in the equivalent ω-derivations: hence, as in **HA**, also in ω-**HA** there is a maximum degree for the formulas (and, a fortiori, for all maximal systems) in every given derivation. This is not guaranteed with unrestricted ω-rules in fact it may be possible, for example, that in an ω-rule the derivation of $A\bar{n}$ contains a maximal formula of degree n (all n). In this case there would be no *maximum* for $\mathrm{deg}(\Pi)$, but only the *supremum* (which is ω).

5.2.2.3 *Definition* The *maximal length* m *of a derivation* Π in ω-**HA** is the highest length of a maximal segment in Π.

Notation: $\mathrm{mlth}(\Pi) = m$.

5.2.2.4 *Remark on the existence of* $\mathrm{mlth}(\Pi)$. From the way ω-**HA** is generated one can see that the length of the segments in **HA** is left unchanged in the equivalent ω-derivation, except, eventually, for **2.3.2**, Case 2b and in applications of $\succ_1$: when composition of derivation is applied new segments may arise; but in the first case the length of the segments is only finitely increased, since $A\bar{n}$ is different from $A\overline{n+1}$ and, then, they cannot belong to the same segment. In the second case we may also assume without loss of generality, that assumptions and conclusion of the active subderivation are different

Notation: $\Gamma \vdash_{\omega} A[d, \alpha]$: there is a derivation Π of A from Γ such that $\mathrm{deg}(\Pi) = d$ and $\mathrm{ord}(\Pi) = \alpha$.

$\Gamma \vdash_{\omega} A[0, \alpha]$: the derivation is in normal form.

5.2.3 *Reduction lemma* *Let* Π *be* $\Gamma \vdash_{\omega} A$; $\mathrm{deg}(\Pi) \geqslant 1$; $\mathrm{mlth}(\Pi) = m$. *Then there is a* $d' < d$ *such that*

$$\text{if } \Gamma \vdash_{\omega} A[d, \alpha], \text{ then } \Gamma \vdash_{\omega} A[d', \omega^{\alpha}].$$

Proof: By induction on α. Basis: A is a top formula: $d = 0$ and the theorem is vacuously proved. Step: suppose the theorem for all $\bar{\alpha} < \alpha$. To show for α: Starting from A we go up through the branches of Π. Every subderivation Π_1 of F we meet, fits to the following cases:

Case 1. F is not the consequence of a maximal segment.

Case 2. F is consequence of a maximal segment σ such that $\mathrm{lth}(\sigma) = 1$ (i.e.: σ is a maximal formula).

Case 3. F is consequence of a maximal segment σ such that $1 < \mathrm{lth}(\sigma)$.

For each of these three cases we apply induction hypothesis and show that there is a derivation Π_1' of F (where σ does not appear), which is equivalent to Π_1 and satisfies the lemma. Then we substitute Π_1' for Π_1; the final result is Π'; if all subderivations of Π' satisfy the lemma, then so does Π': i.e., $\Gamma \vdash_{\overline{\omega}} A[d', \omega^{\alpha}]$.

In Cases 1 and 2 we do not consider subcases where immediate simplifications can be applied: it is intended that such simplifications are applied always when possible, and it is immediate to show the lemma.

For Case 1 the lemma is proved immediately: $\Pi_1 = \Pi_1'$.

For Case 2 the lemma is proved by applying a suitable proper reduction figure, according to the form of σ.

For Case 3 the strategy is the following. Let B be the formula in σ: all the segments with length >1 branch upward, hence the downmost occurrence of B in σ is the root of a tree (of segments) τ. Our aim is to eliminate τ. We cannot do this directly; we then: (i) apply a permutative reduction figure to eliminate the root of τ and get other trees of segments, such that all their segments have length less than 1 with respect to the length of all the segments in τ; (ii) repeat such operation a sufficient number of times for each of the roots of the new reduced trees of segments. We get finally a situation in which all the segments have length 1 and we do as in Case 2. From these points below, till a derivation of F which is equivalent to Π_1, we show how ordinals can be assigned. As to the proof of Case 3 we give step (i) of the above strategy; i.e., suppose $m' \leq m$ to be the length of the longest segment in τ: we take m' as induction value (we suppose the lemma to hold for $m' - 1$) and show how to lower m'.

Let then F_i be the premisses of F. We use the following ordinals: $\mathrm{ord}(F_i) = \alpha_i$, $\mathrm{ord}(F) = \alpha$.

Case 1 F is not the consequence of a maximal segment. We have

$$\frac{\begin{matrix}\Sigma_i \\ F_i\end{matrix}}{F} \quad (i \text{ depending on the number of the premisses})$$

Degree: by induction hypothesis being $\mathrm{deg}\begin{pmatrix}\Sigma_i \\ F_i\end{pmatrix} < d$, it is $\mathrm{deg}(F) < d$.
Ordinal: by induction hypothesis being $\mathrm{ord}(F_i) = \omega^{\alpha_i}$, it is $\mathrm{ord}(F) = \omega^{\alpha}$, which is greater than ω^{α_i} (cf. **4.1.1** (ii)).

Case 2 F is consequence of a maximal formula M.

Subcase 2.1. $M = F_1 \wedge F_2$. Apply $\wedge$-**red**.

$$\frac{\dfrac{\begin{matrix}\Sigma_1 \\ F_1\end{matrix} \quad \begin{matrix}\Sigma_2 \\ F_2\end{matrix}}{F_1 \wedge F_2}}{F_i} \succ_1 \begin{matrix}\Sigma_i \\ F_i\end{matrix} \qquad (i = 1, 2)$$

Degree: immediate, since, by induction hypothesis, $\deg\begin{pmatrix}\Sigma_i \\ F_i\end{pmatrix} < d$.
Ordinal: by induction hypothesis it is $\mathrm{ord}\begin{pmatrix}\Sigma_i \\ F_i\end{pmatrix} < \omega^{\alpha_i} < \omega^{\alpha}$.

Subcase 2.2. $M = \forall x Ax$. Apply $\omega\forall$-**red**. The proof is completely analogous to Subcase 2.1.

Subcase 2.3. $M = A_1 \vee A_2$. Apply $\vee$-**red**.

$$\frac{\dfrac{\begin{matrix}\Sigma_i' \\ A_i\end{matrix}}{A_1 \vee A_2} \quad \begin{matrix}[A_1] \\ \Sigma_1 \\ F\end{matrix} \quad \begin{matrix}[A_2] \\ \Sigma_2 \\ F\end{matrix}}{F} \succ_1 \begin{matrix}\Sigma_i' \\ [A_i] \\ \Sigma_i \\ F\end{matrix} \qquad (i = 1, 2)$$

Degree: by induction hypothesis $\deg\begin{pmatrix}\Sigma_i' \\ A_i\end{pmatrix} < d$ and $\deg\begin{pmatrix}A_i \\ \Sigma_i \\ F\end{pmatrix} < d$. In Π_1', A_i can arise as a new maximal-formula (or part of a new maximal segment), but $\mathrm{d}(A_i) < \mathrm{d}(A_1 \vee A_2) \leqslant d$. Hence $\deg(\Pi_1') < d$.

Ordinal: set $\alpha_0 = \mathrm{ord}\begin{pmatrix}\Sigma_i' \\ A_i\end{pmatrix}$, $\alpha_i = \mathrm{ord}\begin{pmatrix}[A_i] \\ \Sigma_i \\ F\end{pmatrix}$. By induction hypothesis and **5.2.1**, $\mathrm{ord}(\Pi_1') \leqslant \omega^{\alpha_0} \# \omega^{\alpha_i} < \omega^{\alpha}$.

Subcase 2.4. $M = A_1 \to A_2$ and Subcase 2.5: $M = \exists x Ax$, are completely analogous to Subcase 2.3, applying the suitable proper reduction figure.

Case 3 F is consequence in Π_1 of a maximal segment σ such that $1 < \mathrm{lth}(\sigma) = m' \leqslant m$. We suppose the theorem proved for $m' - i (1 \leqslant i \leqslant m')$ an show how to get a Π_1'' equivalent to Π_1, in which $\mathrm{lth}(\sigma) = m' - 1$.

Subcase 3.1 The last occurrence of B in σ is a consequence of $\vee$**E**. Apply $\vee$**E-red**.

$$\frac{\dfrac{\dfrac{\Sigma_0}{A_1 \vee A_2} \quad \begin{matrix}[A_1] \\ \Sigma_1 \\ B\end{matrix} \quad \begin{matrix}[A_2] \\ \Sigma_2 \\ B\end{matrix}}{B} \quad \{\Pi_n\}}{F} \succ_1 \frac{\dfrac{\Sigma_0}{A_1 \vee A_2} \quad \dfrac{\begin{matrix}[A_1] \\ \Sigma_1 \\ B\end{matrix} \quad \{\Pi_n\}}{F} \quad \dfrac{\begin{matrix}[A_2] \\ \Sigma_2 \\ B\end{matrix} \quad \{\Pi_n\}}{F}}{F}$$

(n depending on the number of the minor premisses of F)

Degree: By induction hypothesis $\deg(A_1 \vee A_2) < d$. In the derivation to the right $\mathrm{lth}(\sigma) = m' - 1$; by the supposition on $m' - i$ there are derivations of the uppermost occurrences of F where B is eliminated by a proper reduction. By this facts and induction hypothesis there is a Π_1' such that $\deg(\Pi_1') < d$.

Ordinal: let the ordinals in Π_1 be the following

$$\frac{\dfrac{\beta_0 \quad \beta_1 \quad \beta_2}{\alpha_0} \quad \{\alpha_n'\}}{\alpha}.$$

Combining induction hypothesis and **5.2.1**, we assign to Π_1' the following ordinals

$$
\dfrac{\omega^{\beta_0} \qquad \begin{array}{c}\omega^{\beta} \# \, \omega^{\gamma} \\ \vdots \\ \omega^{\beta} \# \, \omega^{\gamma} \# \, (m' - 1)\end{array} \qquad \begin{array}{c}\omega^{\beta} \# \, \omega^{\gamma} \\ \vdots \\ \omega^{\beta} \# \, \omega^{\gamma} \# \, (m' - 1)\end{array}}{\omega^{\alpha}}
$$

where (i) $\omega^\beta \# \omega^\gamma$ are the ordinals of the uppermost occurrences of F after the elimination of B by proper reduction figures, where
(ii) $\beta = \max(\beta_i')$, and β_i' are the ordinals (in Π_1) of the uppermost occurrences of B in τ (such max exists being $\{\beta_i\}$ finite)
(iii) $\gamma = \max(\gamma_i)$, where γ_i are the ordinals (in Π_1) of the active subderivations of the proper reductions which eliminate B;

Being $\beta \leqslant \beta_i < \alpha$ $(i = 1, 2)$; $\gamma < \alpha$ and $m' = \omega^0 \# \ldots \# \omega^0$ and $0 < \alpha$, it is $\omega^\beta \# \omega^\gamma \# (m' - 1) < \omega^\alpha$ by **4.1.2** (iii).

Subcase 3.2 The last occurrence of B in σ is consequence of ω**∃E**. The proof is analogous to Subcase 3.1, applying ω**∃E-red**.

This case concludes the proof.

5.2.4 *Definition* We define the following sequence of ordinals $\{\omega_n^\alpha\}$

$$\omega_0^\alpha = \alpha,\ \omega_1^\alpha = \omega^{\omega_0^\alpha},\ \ldots,\ \omega_{n+1}^\alpha = \omega_n^{\omega^\alpha},\ \ldots$$

If $\alpha < \varepsilon_0$, then it is immediate, applying **4.1.1** (iii), that the limit number of $\{\omega_n^\alpha\}$ is ε_0.

5.2.5 *Normalization Theorem* *If* $\Gamma \vdash_{\overline{\omega}} A[d, \alpha]$, *then* $\Gamma \vdash_{\overline{\omega}} A[0, \omega_d^\alpha]$.

Proof: Apply at most d times **5.2.3**. In particular, given an ω^*-derivation Π, then there is an equivalent $\Pi' \in \omega$-**HA** which is normal.

5.2.6 *Transfinite Induction* By Theorem **4.1.4.3**, for every ω-derivation Π, $\alpha = \mathrm{ord}(\Pi) < \omega^\omega < \varepsilon_0$. Consider now Definition **5.2.4**: transfinite induction up to ε_0 is *sufficient* as a principle of proof for **5.2.5**. It is *necessary* since there is not a finite upper bound for d in ω-**HA**; i.e., any given Π has degree $d < \omega$, but there is no d such that every Π has degree $\leqslant d$. However, for any given derivation Π of A, the existence of the corresponding normal Π' can be proved without full ε_0-induction, but using induction up to a definite $\alpha < \varepsilon_0$; hence the proof is possible in **HA** itself by Gentzen [2]. By this remark, the applications in **6.1** and **6.2** below do not need ε_0-induction, and are given in **HA**.

ε_0-induction is needed for **6.13** and for establishing that something is *not* provable in a normal derivation (as required for proving the consistency of **HA**).

6 *Applications of the normalization and equivalence theorems*

6.1 We give some definitions and theorems to be used in the sequel. Proofs are not given because they can be taken from the corresponding statements for first-order intuitionistic predicate calculus (see Prawitz [7], [8]), with the only modifications due to the presence of the basic rules. For a clear presentation see also Torelstra [9], IV,2 (where a path has a different definition, but the same properties).

6.1.1 *Definition* A *path* in a normal ω-derivation Π is a sequence $A_1, \ldots, A_n$ of formulas such that:

(i) A_1 is a top formula not discharged by $\omega\exists$**E**, $\vee$**E**

(ii) if A_i is not major premiss of $\omega\exists$**E**, $\vee$**E**, then A_{i+1} is a formula immediately below A_i; A_i is not minor premiss of $\rightarrow$**I** if $i < n$

(iii) if A_i is major premiss of $\omega\exists$**E**, $\vee$**E**, then A_{i+1} is one of the assumptions discharged by this rule

(iv) A_n is the conclusion of Π or the minor premiss of $\rightarrow$**I**.

Remark: In a normal Π every formula of Π belongs to a path.

6.1.2 Lemma (on the form of a path in a normal Π) *In a normal derivation, a path can be divided into segments* $\sigma_1, \ldots, \sigma_n$; *the segments can be divided into the following three parts* ((1) *and* (3) *possibly empty*):

(1) *an elimination part* (**E-part**) $\sigma_1, \ldots, \sigma_{m-1}$, *where each segment* σ_i $(1 \leq i < m - 1)$ *is major premiss of an* **E-rule**, *and* σ_{i+1} *is subformula of* σ_i;

(2) *a minimum part* $\sigma_m, \ldots, \sigma_{m+k-1}$, *in which each segment except the last one is premiss of* $\wedge$ *or a basic rule;*

(3) *an introduction part* (**I-part**) $\sigma_{m+k}, \ldots, \sigma_n$, *where each segment is the conclusion of an* **I-rule**, *and* σ_i *is a subformula of* σ_{i+1}.

6.1.3 Theorem (subformula property for ω-**HA**) *Let* Π *be a normal derivation in* **HA** *of* A *from* Γ. *Then every formula in* Π *is either a subformula of* A *or a subformula of some* F *in* Γ *or an atomic formula belonging to the minimal part of a path of* Π.

6.2 *Properties of* ω-**HA** As a consequence of the form of normal ω-derivations, we can derive *for* ω-**HA** the same properties which follow for the first-order predicate calculus from its normal form, and the properties which follow for **HA** from Jervell's normal form (see Troelstra [9], IV,2): in particular the consistency, the disjunction property, the explicit definability property (for derivations from null assumptions and for Harrop sentences), and other proof theoretical closure conditions.

6.3 *Properties of* **HA** By making use of (3) we can now extend to **HA** the same properties stated in **6.2** for ω-**HA**. As an example we show the following theorem.

6.3.1 Theorem (explicit definability property for **HA**) *If* Π *is a proof* (i.e., *a derivation from null assumptions*) *of* $\exists xAx$ *in* **HA**, *then there is a proof* Π' *of* $A\bar{n}$ *for a suitable* n.

Proof: Let Π be a derivation in **HA**. By the equivalence theorem, an equivalent ω-proof Π_1 can be constructed, and for Π_1 the property holds by **6.2**. Apply again the equivalence theorem and get the desired Π'.

7 *The uniform reflection principle for* **HA** Throughout this section we suppose the syntax of **HA** and ω-**HA** arithmetized in **HA** by a standard Gödel numbering; i.e., all syntactical properties and object are identified with arithmetical predicates and natural numbers. In particular, in the case of ω-**HA**, ordinals are represented by a "natural" well ordering of order type ε_0 and the provability predicate includes information about derivability in **HA**, so that the restrictions to the ω-rules can be arithmetically expressed.

$\mathsf{Proof}(x, y)$ is the provability predicate for **HA**, whose intuitive meaning is: x is the (Gödelnumber) of a proof of the formula (with Gödelnumber) y.

$\mathsf{Pr}(y)$ is the proof predicate for **HA**. $\mathsf{Pr}(y) := \exists x \mathsf{Proof}(x, y)$.

Analogously for ω-**HA**:

$\omega\mathsf{Proof}(x, y, z)$: x is a proof of y and its ordinal is z.
$\omega\,\mathsf{Pr}(y) := \exists x \exists z\, \omega\,\mathsf{Proof}(x, y, z)$.
$\mathsf{Norm}(y) := \omega\,\mathsf{Pr}(y)$ and the proof is normal.

The uniform reflection principle for **HA** is:

RFN $$\forall y(\mathsf{Pr}(\ulcorner A\bar{y}\urcorner) \to Ay),$$

where y is the only free variable in A and $\ulcorner A\bar{y}\urcorner$ is the Gödelnumber of the sentence obtained from Ay by substituting $\bar{n}$ for y.

Kreisel and Levy [4] have shown the significance of the reflection principles and in Section 10 they asked for a proof of the following (which is only possible using a cut-free or a normal semiformal system):

$$\mathbf{HA} + \mathbf{TI}_{\varepsilon_0} \vdash \mathbf{RFN}.$$

The proof was first given in Lopez-Escobar [6], using a cut-free sequent system. We will sketch rather informally the proof of the same result using normalization. The correspondence between cut elimination and normalization is well known; however cut-free and normal derivations have a rather different structure: while in cut-free systems the subformula property (which is used in the proof of **RFN**) is immediate, in normal systems it has to be derived by some intermediate steps. The main lines in the proof of **7.1** are taken from Lopez-Escobar's work, which, in general, makes much more careful statements about the formalization of the semiformal system and the system in which proofs are given.

7.1 Theorem $\mathbf{HA} + \mathbf{TI}_{\varepsilon_0} \vdash \forall y(\mathsf{Pr}(\ulcorner A\bar{y}\urcorner) \to Ay)$.

Proof: The proof is given in $\mathbf{HA} + \mathbf{TI}_{\varepsilon_0}$ by the following steps:

i. $\forall y(\mathsf{Pr}(\ulcorner A\bar{y}\urcorner) \to \omega\,\mathsf{Pr}(\ulcorner A\bar{y}\urcorner))$
ii. $\forall y(\omega\,\mathsf{Pr}(\ulcorner A\bar{y}\urcorner) \to \mathsf{Norm}(\ulcorner A\bar{y}\urcorner))$
iii. $\forall y(\mathsf{Norm}(\ulcorner A\bar{y}\urcorner) \to \mathbf{T}_n(\ulcorner A\bar{y}\urcorner))$ (n depending on the logical degree of A)
iv. $\forall y(\mathbf{T}_n(\ulcorner A\bar{y}\urcorner) \to Ay)$.

Step i is achieved by formalizing the proofs of Theorem 3.2 and 4.1.3. Step ii by formalizing the proof of **5.2**. Step iii will be given in **7.3** and Step iv in **7.2**.

7.2 *Truth definition*

7.2.1 We give recursively the following truth definition $\mathbf{T}_n$ for closed formulas. The obvious valuation function Val for *closed* terms is supposed to have been given.

Definition (by induction $\mathrm{d}(A)$; i.e., the logical degree of A):

(i) for atomic $A \equiv t_1 = t_2$

$$\mathbf{T}_0(\ulcorner t_1 = t_2 \urcorner) \leftrightarrow \mathrm{Val}(t_1) = \mathrm{Val}(t_2)$$

(ii) $\mathbf{T}_{n+1}(\ulcorner A \circ B \urcorner) \leftrightarrow \mathbf{T}_n(\ulcorner A \circ B \urcorner) \vee [\mathbf{T}_n(\ulcorner A \urcorner) \circ \mathbf{T}_n(\ulcorner B \urcorner)]$ $(\circ \equiv \wedge, \vee, \rightarrow)$

$$\mathbf{T}_{n+1}(\ulcorner QxAx \urcorner) \leftrightarrow \mathbf{T}_n(\ulcorner QxAx \urcorner) \vee Qx\mathbf{T}_n(\ulcorner A\overline{x} \urcorner) \quad (Q \equiv \exists, \forall)$$

7.2.2 Theorem *For all sentences A such that* $\mathrm{d}(A) \leq n$

$$\mathbf{T}_n(\ulcorner A\overline{x} \urcorner) \leftrightarrow Ax.$$

Proof: By induction on n. As to the basis $n = 0$, the conclusion follows from the definition of $\mathbf{T}_0$ and the properties of Val

$$\mathrm{Val}(t_1) = \mathrm{Val}(t_2) \leftrightarrow t_1 = t_2.$$

The proof of the step is easily given by cases, according to the main logical constant in A.

7.3 *Truth definition of sentences and normal ω-derivations*

7.3.1 Let $\Gamma \Rightarrow A$ symbolize: "A is deducible from Γ"; Γ is always finite in ω-derivations, we put $\Gamma = \{A_1, \ldots, A_m\}$; $\mathbf{T}_n(\ulcorner \Gamma \urcorner)$ is a shorthand for $\mathbf{T}_n(\ulcorner A_1 \urcorner) \wedge \ldots \wedge \mathbf{T}_n(\ulcorner A_m \urcorner)$.

7.3.2 Theorem *Let $\Gamma \Rightarrow A$; $\mathrm{d}(A_i) \leq n(i = 1, \ldots, m)$; $\mathrm{d}(A) \leq n$.*

$$\textit{If } \Gamma \vdash_{\omega} A \textit{ is normal, then } \mathbf{T}_n(\ulcorner \Gamma \urcorner) \rightarrow \mathbf{T}_n(\ulcorner A \urcorner).$$

Proof: The whole proof would be taken from (the formalization of) the subformula property. We give only the part of the proof which make use of **6.1.2**, by induction on the length of a path. The basis is obvious; the steps are given by cases. Let P be the formula preceding A in the path.

Case 1. A is in the **E-part** of the path. By induction hypothesis

$$\mathbf{T}_n(\ulcorner \Gamma \urcorner) \rightarrow \mathbf{T}_n(\ulcorner P \urcorner).$$

A is subformula of P, hence by definition of $\mathbf{T}_n$

$$\mathbf{T}_n(\ulcorner P \urcorner) \rightarrow \mathbf{T}_n(\ulcorner A \urcorner).$$

Case 2. A is in the minimum part of the path.

Subcase 2.a. A is the first formula in the minimum part: do as in Case 1.
Subcase 2.b. A is consequence of the Λ-rule. By induction hypothesis

$$\mathbf{T}_n(\ulcorner \Gamma \urcorner) \rightarrow \mathbf{T}_n(\Lambda).$$

The conclusion is false; hence, for some i, $\mathbf{T}_n(\ulcorner A_i \urcorner)$ is false, and the Theorem follows by vacuous implication.

Subcase 2.c. A is consequence of a basic rule. Then Val preserve $\mathbf{T}_0$. In the case the conclusion of the rule is false, it is also false for the premiss: then do as in Subcase 2.b.

Case 3. A is in the **I-part** of the path. Induction hypothesis

$$\mathbf{T}_n(\ulcorner \Gamma \urcorner) \to \mathbf{T}_n(\ulcorner P_i \urcorner)$$

(for all i; P_i the i-th premiss of A in the i-th path through A). All P_i are subformulas of A, hence their degree is $k < n$. Then by the definition of $\mathbf{T}_n$

if for all i it is $\mathbf{T}_k(\ulcorner P_i \urcorner)$, then $\mathbf{T}_n(\ulcorner A \urcorner)$.

7.3.3 Corollary *For a sentence A such that* $\mathrm{d}(A) \leqslant n$

$$\mathrm{Norm}(\ulcorner A \urcorner) \to \mathbf{T}_n(\ulcorner A \urcorner).$$

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