Notre Dame Journal of Formal Logic
Volume 21, Number 3, July 1980

Equivalence Relations and S5

G. E. HUGHES

1 An equivalence relation is commonly defined as one which is reflexive, symmetrical, and transitive. This paper* starts from the problem of finding a pair of conditions on a dyadic relation which together yield equivalence but neither of which by itself yields either reflexiveness or symmetry or transitivity. It will be shown that there are infinitely many such pairs of conditions.

There is a parallel problem in modal logic, that of finding a pair of formulas which, if added to the minimal normal modal logic K, yield precisely $S5$, but neither of which, when added to K, yields either $Lp \supset p$ or $p \supset LMp$ or $Lp \supset LLp$ as a theorem. It will be shown that there are infinitely many such pairs of formulas.

2 One solution to the second problem is provided by the following formulas:

$A \qquad LMLp \supset p$
$B \qquad MLp \supset LMLLp.$

Since in $S5$ an affirmative modality is equivalent to its last member, it is clear that A and B are theorems of $S5$ and hence that $S5$ contains $K + A + B$. For the converse it is sufficient to derive $MLp \supset Lp$ and $Lp \supset p$. We first note that A is interdeducible in the field of K with its dual:

$A' \qquad p \supset MLMp.$

We then have:

$MLp \supset Lp \qquad [B, A(Lp/p) \times \text{Syll}]$
$Lp \supset p \qquad [A'(Lp/p), B(MLp/p), A(LMLp/p), A \times \text{Syll}]$

*I acknowledge with gratitude the help given to me by Dr. R. L. Epstein in preparing this paper. In particular, he is responsible for the generalizations of conditions Y and Z in Section 4.

Received September 7, 1979

That neither $Lp \supset p$ nor $p \supset LMp$ nor $Lp \supset LLp$ is a theorem of $K + A$ or of $K + B$ can be shown as follows. The frames of Figures 1 and 2 are frames for $K + A$ and $K + B$ respectively. Yet since neither frame is reflexive or symmetrical or transitive, the three formulas mentioned can be falsified in each of them.

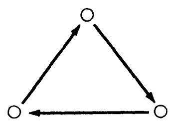

Fig. 1

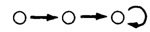

Fig. 2

For good measure we can also show that neither A nor B is a theorem of $K + Lp \supset p$ (T) or of $K + p \supset LMp$ (B^0) or of $K + Lp \supset LLp$ ($S4^0$). These systems are known to be characterized by the classes of all reflexive, symmetrical, and transitive frames respectively. However,

(i) In the model on the reflexive transitive frame of Figure 3 in which $V(p) = \{y\}$, A is false at x

(ii) In the model on the symmetrical frame of Figure 4 in which $V(p) = \emptyset$, A is false at x

(iii) In the model on the reflexive transitive frame of Figure 5 in which $V(p) = \{y\}$, B is false at x

(iv) In the model on the symmetrical frame of Figure 6 in which $V(p) = \{y\}$, B is false at y.

Fig. 3

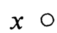

Fig. 4

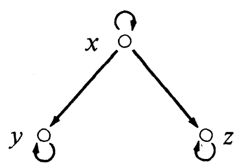

Fig. 5

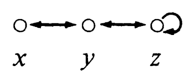

Fig. 6

3 $K + A$ and $K + B$ are characterized respectively by the classes of all frames satisfying the following conditions Y and Z:

Y $\forall x \exists z(xRz \wedge \forall w(zRw \supset wRx))$

Z $(xRy \wedge xRz) \supset \exists w(zRw \wedge \forall v(wR^2v \supset yRv))$.

The proofs of soundness are left to the reader. We prove completeness by the method of canonical models.

3.1 Completeness of $K + A$ We have to show that in the canonical model $\langle W, R, V\rangle$ for $K + A$, R satisfies Y. Let x be any point in W. What is needed is to show that there is some point $z \in W$ such that: (i) xRz, and (ii) every point to which z is related is related to x. It is sufficient to prove that

$$\Gamma = \{\alpha: L\alpha \in x\} \cup \{LM\beta: \beta \in x\}$$

is consistent, for: (a) if Γ is consistent there will be some $z \in W$ such that $\Gamma \subseteq z$; (b) since $\{\alpha: L\alpha \in x\} \subseteq z$, we shall have xRz; and (c) since $\{LM\beta: \beta \in x\} \subseteq z$, then if zRw we shall have $\{M\beta: \beta \in x\} \subseteq w$, and hence wRx.

Suppose that Γ is inconsistent. Then for some wff's $L\alpha, \beta_1, \ldots, \beta_n \in x$,

$$\vdash \alpha \supset \sim(LM\beta_1 \wedge \ldots \wedge LM\beta_n).$$

Hence by K,

$$\vdash L\alpha \supset L\sim(LM\beta_1 \wedge \ldots \wedge LM\beta_n).$$

Hence, since $L\alpha \in x$, we have $L\sim(LM\beta_1 \wedge \ldots \wedge LM\beta_n) \in x$, and thus

(*) $\quad \sim M(LM\beta_1 \wedge \ldots \wedge LM\beta_n) \in x$.

But since $\beta_1, \ldots, \beta_n \in x$, we have, by A',

$$MLM(\beta_1 \wedge \ldots \wedge \beta_n) \in x$$

and so, by K, $M(LM\beta_1 \wedge \ldots \wedge LM\beta_n) \in x$.
But this contradicts (*); therefore Γ is consistent, as required.

3.2 ***Completeness of $K + B$*** Let $\langle W, R, V\rangle$ be the canonical model for $K + B$, and let x, y, z be any points in W such that xRy and xRz. We have to prove that there is some $w \in W$ such that: (i) zRw and (ii) for every v such that wR^2v, we have yRv. It is sufficient to show that

$$\Gamma = \{\alpha\colon L\alpha \in z\} \cup \{LL\beta\colon L\beta \in y\}$$

is consistent; for suppose some $w \in W$ includes Γ, then: (a) since $\{\alpha\colon L\alpha \in z\} \subseteq w$, we have zRw, and (b) since $\{LL\beta\colon L\beta \in y\} \in w$, then for any v such that wR^2v we have $\{\beta\colon L\beta \in y\} \subseteq v$, and so yRv.

Suppose that Γ is inconsistent. Then for some $L\alpha \in z$ and some $L\beta \in y$,

$$\vdash \alpha \supset \sim LL\beta.$$

Hence by K,

$$\vdash ML\alpha \supset ML\sim LL\beta.$$

Now $L\alpha \in z$ and xRz; so $ML\alpha \in x$, and therefore

$$ML\sim LL\beta \in x.$$

Hence by K,

(**) $\quad \sim LMLL\beta \in x$.

But since $L\beta \in y$ and xRy, we have $ML\beta \in x$, and hence by B:

$$LMLL\beta \in x$$

which contradicts (**). Therefore Γ is consistent as required.

4 The results of Sections 2 and 3 amount to an indirect proof that conditions Y and Z together yield equivalence, and thus provide one solution to the first problem of Section 1. (We shall give a direct proof in a moment.)

Conditions Y and Z can be generalized as follows. For each $n \in$ Nat ($\geqslant 1$), we define

$Y_n \quad \forall x \exists z(xRz \wedge \forall w(zR^n w \supset wRx))$
$Z_n \quad (xRy \wedge xRz) \supset \exists w(zRw \wedge \forall v(wR^{n+1}v \supset yR^n v))$.

We shall show that if n is odd, then if R satisfies Y_n and Z_n then R is symmetrical, reflexive, and transitive, i.e., is an equivalence relation. Since the original Y and Z are simply Y_1 and Z_1 respectively, the proof will clearly cover them.

We note that each Y_n explicitly includes seriality, i.e., the condition that $\forall x \exists z(xRz)$.

Proof that R is symmetrical: Suppose aRb. Then by Y_n there is some c such that

(1) aRc
(2) $\forall x(cR^n x \supset xRa)$.

Since $aRc \wedge aRc$ (by (1)), then by Z_n there is some d such that

(3) cRd
(4) $\forall x(dR^{n+1}x \supset cR^n x)$.

Now by seriality there is some e such that

(5) $dR^{n-1}e$.

(This holds even if $n = 1$, for then $e = d$.) Then from (3) and (5) we have $cR^n e$. Hence by (2),

(6) eRa.

Now from (5), (6), and aRb we have $dR^{n+1}b$; hence by (4) we have $cR^n b$ and so, by (2), bRa.

Note that this result holds whether n is odd or even.

Proof that R is reflexive: Let a be any element. Then by Y_n there is some b such that

(1) aRb
(2) $\forall x(bR^n x \supset xRa)$.

From (1), by symmetry, we have bRa and, therefore, if n is odd, $bR^n a$. Hence by (2), we have aRa.

Proof that R is transitive: Suppose aRb and bRc. We can assume symmetry and reflexiveness. By Y_n there is some d such that

(1) cRd
(2) $\forall x(dR^n x \supset xRc)$.

Moreover, since aRa and aRb, there is (by Z_n) some e such that

(3) bRe
(4) $\forall x(eR^{n+1}x \supset aR^n x)$.

We note that by symmetry, (3) gives eRb. Suppose now that $n = 1$. Then by eRb and bRc (given) we have eR^2c; so by (4) we have aRc. Suppose now that $n > 1$. Then by eRb, bRc, (1), and reflexiveness we have $eR^{n+1}d$. Hence by (4) we have $aR^n d$; hence by symmetry we have $dR^n a$, and so by (2) we again have aRc.

We note that the oddness of n is essential to the proof of reflexiveness, but that, *given reflexiveness,* transitivity follows whether n is odd or even. We also note without proof that if n is even, R satisfies the condition that if xR^3y then xRy.

That neither reflexiveness nor symmetry nor transitivity follows from any Y_n or Z_n by itself can be shown from the fact that in Figures 7 and 8 we have models for Y_n and Z_n respectively. Clearly neither is reflexive or symmetrical or transitive.

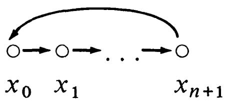

Fig. 7

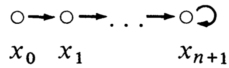

Fig. 8

5 Corresponding to the conditions Y_n and Z_n respectively are the modal formulas (for each $n \geqslant 1$)

A_n $\quad LM^nLp \supset p$
B_n $\quad ML^np \supset LML^{n+1}p$.

Clearly the A and B of Section 2 are A_1 and B_1 respectively. Since in *S5* every affirmative modality is equivalent to its last member, each A_n and each B_n is a theorem of *S5*; so *S5* always contains $K + A_n + B_n$. We now show that if n is odd, $K + A_n + B_n$ contains *S5*. To do so it is sufficient to derive $p \supset LMp$, $Lp \supset p$ and $Lp \supset LLp$.

We note that each A_n is interdeducible in the field of K with its dual

A'_n $\quad p \supset ML^nMp$

(1)	$ML^nM(p \supset p)$	$[A'_n(p \supset p/p)$, PC]
(2)	$ML^nM(p \supset p) \supset M(p \supset p)$	$[K]$
(3)	$M(p \supset p)$	[(1), (2) $\times$ MP]
(4)	$Lp \supset Mp$	[(3) $\times$ K].

All subsequent theorems will be of the form $\alpha \supset Xp$, where X is an affirmative modality. Clearly (4) enables us to replace L by M anywhere in the consequent of such a theorem.

(5)	$ML^nMp \supset LML^{n+1}Mp$	$[B_n(Mp/p)]$
(6)	$ML^nMp \supset LM^nLLMp$	[(5) $\times$ (4)]
(7)	$LM^nLLMp \supset LMp$	$[A_n(LMp/p)]$
(8)	$p \supset LMp$	$[A'_n$, (6), (7) $\times$ Syll]
(9)	$MLp \supset p$	[(8) $\times$ Duality]
(10)	$LLp \supset p$	[(4)(Lp/p), (9) $\times$ Syll].

(9) and (10) enable us to delete ML and any even number of consecutive L's in the consequent of a theorem.

(11)	$Lp \supset ML^nMLp$	$[A'_n(Lp/p)]$
(12)	$Lp \supset ML^np$	[(11) $\times$ (9)].

Now since n is odd, $n - 1$ is even. Hence:

(13)	$Lp \supset p$	[(12) X (9) X (10)]
(14)	$Lp \supset LMLL^n p$	[(12), B_n X Syll]
(15)	$Lp \supset L^{n+1}p$	[(14) X (9)].

Now if $n = 1$, (15) = $Lp \supset LLp$; and if $n > 1$ we have

(16) $Lp \supset LLp$ [(15), (13) X Syll (as often as required)].

(8), (13), and (16) are the required theorems.

None of these three is a theorem either of $K + A_n$ or of $K + B_n$ (for any n). This is proved by the fact the frames illustrated in Figures 7 and 8 are frames for $K + A_n$ and $K + B_n$ respectively, but all three formulas can be falsified in each.

Completeness proofs for $K + A_n$ and $K + B_n$ relative to the classes of frames satisfying Y_n and Z_n respectively (for any $n \geqslant 1$) can be obtained by straightforward generalizations of the completeness proofs given in Section 3.

6 For any even n ($\geqslant 2$), $K + A_n + B_n$ yields a system, weaker than *S5*, which is characterized by the class of frames in which R is serial, symmetrical, and such that if xR^3y then xRy. It is equivalent to the system obtained by adding to K the axioms $Lp \supset Mp$, $p \supset LMp$, and $Lp \supset LLLp$. The proof is left to the reader.

The system in question does not appear to be equivalent to any of the standard ones in the literature.

7 We turn now to the relations of the Y_n's and the Z_n's among themselves.

(a) If $m > n$, Z_n entails Z_m.

Proof: $B_n(L^{m-n}p/p) = B_m$.

(b) If $m < n$, Z_n does not entail Z_m.

Proof: The model of Figure 8 in Section 4 satisfies Z_n but not Z_m for $m < n$.

Thus the Z_n's form a sequence in descending order of strength. The situation with respect to the Y_n's, however, is more complex.

(c) If $m > 1$, then Y_1 and Y_m are independent.

Proof: (i) The model of Figure 9 is a model for Y_1, as is easy to check. But this model does not satisfy Y_m for $m > 1$.

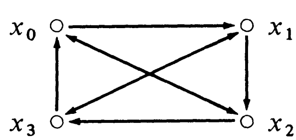

Fig. 9

For consider x_0. The only points to which it is related are x_1 and x_2. Now from each of these we can reach x_0 (or for that matter x_1) in m steps, for any $m > 1$. But x_0 is not related to itself (nor is x_1 related to x_0). Hence Y_1 does not entail Y_m. (ii) That Y_m does not entail Y_1 for $m > 1$ is a special case of the next result, (d).

(d) If m is not of the form $n + r(n + 2)$, then Y_n does not entail Y_m.

Proof: The model of Figure 7 satisfies Y_n. Clearly it contains $n + 2$ points, and the only point to which x_0 is related is x_1. Now it is evident that n steps will take us from x_1 to x_{n+1}, and also that any multiple of $n + 2$ *further* steps will again take us to x_{n+1}; but any other number of steps will take us to some point other than x_{n+1}, and no such other point is related to x_0. Hence Y_m is not satisfied if m is not of the form $n + r(n + 2)$.

(e) If $n > 1$ and m is of the form $n + r(n + 2)$, then Y_n entails Y_m.

Proof: We can prove this by showing how to derive $A_{n+r(n+2)}$ (for arbitrary r) as a theorem of $K + A_n$ ($n > 1$). The key step in the proof is the derivation of the perhaps surprising theorem $MLp \supset LLp$.

Assume K and

A_n $\quad LM^nLp \supset p \ (n > 1)$.

We note as before that the dual of A_n is

A'_n $\quad p \supset ML^nMp$

and that as in Section 5 we can derive (1) $Lp \supset Mp$. We then have:

(2)	$LMp \supset ML^nMLMp$	$[A'_n(LMp/p)]$
(3)	$\supset ML(p \supset L^{n-1}MLMp)$	$[K]$
(4)	$\sim LMp \supset ML\sim p$	$[K]$
(5)	$\supset ML(p \supset L^{n-1}MLMp)$	$[K]$
(6)	$ML(p \supset L^{n-1}MLMp)$	[(3), (5) $\times$ PC]
(7)	$LM^nL(p \supset L^{n-1}MLMp)$	[(6), K, (1)]
(8)	$p \supset L^{n-1}MLMp$	[(7), A_n $\times$ MP]
(9)	$LMMp \supset LMML^{n-1}MLMp$	[(8), K]
(10)	$\supset LM^nLMLMp$	[$\times$ (1)]
(11)	$LMMp \supset MLMp$	[(10), A_n]
(12)	$LMLp \supset MLLp$	[(11), Duality].

We note that if we have a theorem of the form $\alpha \supset X\beta$, where X is an affirmative modality, then (12) enables us to replace LML by MLL anywhere in X.

(13)	$LMLMLp \supset MLLMLp$	[(12)(MLp/p)]
(14)	$\supset MLMLLp$	[$\times$ (12)]
(15)	$\supset MMLLLp$	[$\times$ (12)]
(16)	$\supset MML(MLp \supset LLp)$	$[K]$
(17)	$\sim LMLMLp \supset MLMLM\sim p$	$[K]$
(18)	$\supset MMLLM\sim p$	[$\times$ (12)]
(19)	$\supset MML(MLp \supset LLp)$	$[K]$
(20)	$MML(MLp \supset LLp)$	[(16), (19) $\times$ PC]
(21)	$LM^nL(MLp \supset LLp)$	[(20), K, (1)]
(22)	$MLp \supset LLp$	[(21), A_n $\times$ MP].

(22) enables us to replace ML by LL anywhere in X in a theorem of the form $\alpha \supset X\beta$. Now let $m = n + r(n + 2)$. Clearly A'_n, $A'_n(ML^nMp/p)$ $\times$ Syll yields

(23) $p \supset ML^nMML^nMp$

and hence by repetition we have

(24) $p \supset ML^nMML^nM \dots ML^nMp$

where ML^nM occurs $r + 1$ times.
It is easy to see that there are $n + 2 + r(n + 2)$ operators in the modality in (24). We can now use (22) to replace each M except the first and the last by L, thus giving ourselves $n + r(n + 2)$ L's. We thus have

(25) $p \supset ML^{n+r(n+2)}Mp,$

which is A_m.

The upshot of (c)-(e) is that if $n \neq m$ then Y_n and Y_m are independent *except* when $n > 1$ and $m \equiv n \bmod (n + 2)$, in which case Y_n entails Y_m, but not conversely.

We note the following corollary of the proof in (e):

$T + A_n = S5$ for any $n > 1$.

Proof: T gives the theorem $LLp \supset Lp$, and this with (22) yields $MLp \supset Lp$, and hence a standard basis for *S5*.

8 The results obtained in Section 7 enable us to generalize the results of Sections 4 and 5 even further. For we can now prove the following:

For any odd $n > 1$ and *any* k (even or odd) $\geqslant 1$, $K + A_n + B_k = S5$, and Y_n and Z_k yield equivalence.

Proof: If n is odd then for every even r, $n + r(n + 2)$ is also odd. Hence no matter how large k may be, there will always be some odd $m > k$ such that $m = n + r(n + 2)$ for some r. By (e) in Section 7, A_m is a theorem of $K + A_n$; by (a), B_m is a theorem of $K + B_k$; hence $K + A_n + B_k$ contains $K + A_m + B_m$, and by Section 5 the latter yields *S5*. Similarly, by Section 7(e) and (a), Y_n and Z_k entail Y_m and Z_m, respectively, and by Section 4 these together yield equivalence.

Department of Philosophy
Victoria University of Wellington
Wellington, New Zealand