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Congruences in Lemmon's S0.5

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1 The status of S0.5 The system *S0.5*, defined by Lemmon [4], has a special status among the modal propositional calculi. It is the weakest system which has the following properties:

(i) The set of theses is closed under substitution and material detachment
(ii) The theses of modal degree at most one are the same as in *S5*.

Indeed, *S1-S5* and Feys' *T* all have the same theses of modal degree at most one and, among the systems which have this property, *S5* is the strongest, and *S0.5* is the weakest.

It has been proved by Hughes and Cresswell ([2], p. 288) that the rule of replacement of strict equivalents does not hold in *S0.5*.

Now no particular connective can be said to have a "birthright" to be present in a replacement theorem. In the normal modal systems (in the sense of Kripke [3] or in the sense of Lemmon-Scott [5]), the classical double implication, $\longleftrightarrow$, has this property. In the nonnormal Lewis systems (*S1, S2, S3*) there is no replacement theorem for $\longleftrightarrow$, but there is one for "strict equivalence", defined as $L(x \rightarrow y) \wedge L(y \rightarrow x)$ (it could be defined as $L(x \longleftrightarrow y)$).

Can we define in *S0.5* another connective which has the replacement property? I will study this problem using general notions and results about "congruences" in propositional calculi, which are recalled in the next section (some of these definitions and results have been published, in a somewhat different form, in [9]).

The results will be that such connectives $C(x, y)$ (indeed infinitely many such connectives) exist, but they characterize only identity between formulas, i.e., $\vdash C(x, y)$ iff $x = y$. A few consequences of this fact are discussed in Section 4.

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2 *Congruences in propositional calculi*

Definition 2.1 A *congruence, E,* is an equivalence relation between formulas, compatible with the set of theses, for which the replacement property holds; i.e.:

a. For all formulas x, y; if xEy and $\vdash x$ then $\vdash y$.
b. If $a(x)$ is a formula containing x as a subformula, and $a(y)$ is like $a(x)$ except containing y in place of an occurrence of the subformula x, then: if xEy then $a(x)Ea(y)$.

Definition 2.2 A congruence is *formula-definable* if there is a formula $f(x, y)$ containing x and y as subformulas, such that

$$xEy \text{ iff } \vdash f(x, y).$$

In the classical propositional calculus the classical "equivalence", E, is a congruence, and it is formula-definable by

$$xEy \text{ iff } \vdash x \longleftrightarrow y.$$

But there are other congruences, for instance identity, and these other congruences are not formula-definable (see Theorem 2.2 below).

Congruences are relations, and can therefore be ordered by inclusion.

Theorem 2.1 *The set of all congruences of a propositional calculus, ordered by inclusion, constitutes a complete lattice.*

Proof: Let $\mathcal{C}$ be any set of congruences. The greatest lower bound of $\mathcal{C}$ is the intersection of all the relations in $\mathcal{C}$, and the least upper bound is the transitive closure of the union of all the relations in $\mathcal{C}$.

In particular, the set of all the congruences of a propositional calculus contains a minimum element, the identity, and a maximum element.

Theorem 2.2 *If a congruence is formula-definable, it is the maximum congruence.*

Proof: Let us suppose E is definable by the formula f, and let E' be another congruence. Then we have:

(1)	$xE'y$	[hypothesis]
(2)	xEx	[reflexivity of E]
(3)	$\vdash f(x, x)$	[by (2) and the definability of E]
(4)	$f(x, x)E'f(x, y)$	[by (1) and the replacement property for E' (Def. 2.1b) with $a(z) = f(x, z)$]
(5)	$\vdash f(x, y)$	[by (3), (4) and Def. 2.1a]
(6)	xEy	[by (5) and the definability of E]

Thus $xE'y$ implies xEy for all formulas x, y and every congruence E'.

Corollary *If identity is formula-definable, it is the sole congruence.* (This corollary is one of the chief ways to use Theorem 2.2.)

Remarks (i) Only one congruence can be formula-definable, but the same congruence (the maximal one) can be characterized by various formulas, even by infinitely many formulas. For example, in the classical proposition calculus,

$$xEy \text{ iff } \vdash x \longleftrightarrow y$$

or

$$xEy \text{ iff } \vdash \neg x \longleftrightarrow \neg y$$

or

$$xEy \text{ iff } \vdash \neg\neg x \longleftrightarrow \neg\neg y,$$

and so on.

(ii) Definition 2.2 can be extended by allowing a "formula-definable" congruence to be characterized by a *set* (either finite or infinite) of formulas $\{f_1, f_2, \ldots\}$:

$$xEy \text{ iff } \vdash f_1(x, y) \text{ and } \vdash f_2(x, y) \text{ and } \ldots.$$

Then nothing need be changed in Theorem 2.1, the Corollary, or in Remark (i). For instance, in the classical implicational calculus, the maximal congruence may be characterized by

$$xEy \text{ iff } \vdash x \rightarrow y \text{ and } \vdash y \rightarrow x.$$

3 Congruences in Lemmon's S0.5 In this section *S0.5* is defined using $\neg$ (negation), $\rightarrow$ (classical implication), and L (necessity) as primitives.

An axiomatization consists of three axiom schemas and a rule:

$$\begin{array}{ll} \vdash Lt & \text{if } t \text{ is a tautology} \\ \left.\begin{array}{l} \vdash Lx \rightarrow x \\ \vdash L(x \rightarrow y) \rightarrow (Lx \rightarrow Ly) \\ x, x \rightarrow y \,/\, y \end{array}\right\} & \text{for all formulas } x, y \end{array}$$

(It follows at once that *S0.5* contains the classical propositional calculus.)

A decision method, based on a special nonnormal Kripke-style semantics, has been discovered by Cresswell [1] (see also Hughes and Cresswell [2], Ch. 15). It will be used to prove:

Theorem 3.1 *In S0.5 identity is formula-definable, and is therefore the sole congruence.*

Proof: Let us consider the formula

$$L(Lx \rightarrow Ly)$$

where x and y are arbitrary formulas. If $x = y$, this formula becomes $L(Lx \rightarrow Lx)$, which is a substitution instance of $L(p \rightarrow p)$ (where p is a propositional variable), and hence an *S0.5*-thesis. But if $x \neq y$, Cresswell's method leads to the examination of the formula $Lx \rightarrow Ly$ in a nonnormal world ([2], pp. 286-287), where x and y can receive different values; whence a *S0.5*-model which falsifies the formula. Then

$$x = y \quad \text{iff} \quad \vdash L(Lx \rightarrow Ly),$$

and the result follows by the Corollary.

By the same decision procedure we see that

$$\begin{aligned} x = y &\text{ iff } L(Lx \longleftrightarrow Ly) \\ x = y &\text{ iff } LLx \to LLy \\ x = y &\text{ iff } LLx \longleftrightarrow LLy \\ x = y &\text{ iff } L(LLx \to LLy), \end{aligned}$$

and so on.

Then, if one tries, for instance, to define a "super-strict equivalence" by

$$xEy \text{ iff } \vdash L(Lx \longleftrightarrow Ly)$$

this new connective has the replacement property, but E is simply identity!

Remarks (i) The five formulas just examined are particular cases of a general feature of *S0.5* which can be proved by the decision method, namely: A formula of modal order greater than one is a thesis of *S0.5* if and only if it is a substitution instance of a first-order thesis. The first-order theses of *S0.5* are exactly the theses of the "Basic Modal Logic" of Pollock [7]. This proves a conjecture of Lemmon (see [4], p. 181, footnote).

(ii) A formula such as $LL(x \longleftrightarrow y)$ cannot characterize a congruence, for no formula of the form LLz can be a thesis of *S0.5*.

4 Consequences

They are severe!

If *S0.5* had been defined with all the classical connectives ($\neg, \to, \vee, \wedge, \longleftrightarrow$) none of them could be "defined" by means of the others. For instance $p \vee q$ and $(p \to q) \to q$ would not be "synonymous", in the sense of Smiley [11], for

$$\vdash LL(p \vee q) \to LL(p \vee q)$$

while

$$LL(p \vee q) \to LL((p \to q) \to q)$$

is not a thesis.

Similarly, if we consider the formulas

(1) $LLLp \to LL\neg M\neg p$
(2) $LLMp \to LL\neg L\neg p$

and apply the decision procedure given in [2], then:

- if L is primitive and M defined (as $\neg L\neg$), then (2) is a thesis but (1) is not
- if M is primitive and L defined (as $\neg M\neg$), then (1) is a thesis but (2) is not
- if L and M are both primitive, neither (1) nor (2) is a thesis.

This last fact had already been noted by Milberger [6].

5 Other weak modal logics One may conjecture that identity is the sole congruence in certain very weak modal propositional systems.

In [8] and [10] various modal systems are constructed systematically, starting from a very weak system S_a and using two operations, ν and ρ. In that construction $\rho S_a = S0.5$, $\nu\rho S_a = T$, and $\nu\rho\nu S_a = S4$. It has just been proved that identity is formula-definable in ρS_a. It follows that the same result holds for the weaker systems (S_a and S_b). The maximal congruence may be defined by "strict equivalence" (as in the Lewis systems), and the congruence does not reduce to identity, in seven systems. It remains to examine the case of the systems νS_a and $\rho\nu S_a$, which can be considered as "by-products" of the construction. It will presently be proved that identity is the sole congruence in these systems.

The system $\rho\nu S_a$ differs from $\rho S_a = S0.5$ by a supplementary axiom schema:

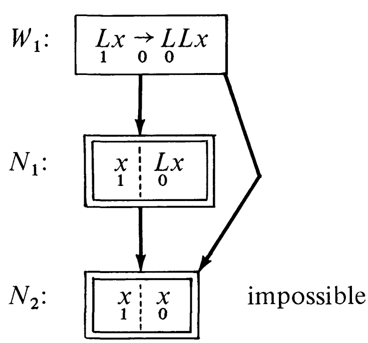
$$\vdash Lx \to LLx \qquad \text{for every formula } x.$$

This suggests the following modifications of Cresswell's semantics for $S0.5$ (see [1] or [2], Ch. 15). It will be postulated, for a new class of nonnormal frames, that:

(i) In a nonnormal world, N, any formula of the form Lx can be true, but it can be false only if x is false in another world accessible to N (then there are worlds accessible to a nonnormal one).
(ii) The accessibility relation is transitive.

Validity is defined as in the Kripke-style semantics with nonnormal worlds (see [2], Ch. 15).

The proof of validity of the axioms and rules of $S0.5$ is similar to that in Hughes and Cresswell ([2], p. 286). The supplementary axiom schema, $Lx \to LLx$, is proved valid as in $S4$, as shown by the following chart in the style of Hughes and Cresswell, where the W's are normal worlds and the N's are nonnormal.



Now let us consider the formula

$$L(L(Lx \to Lp) \to L(Ly \to Lp)),$$

where p is a variable which occurs in neither x nor y. It is valid if $x = y$, for in this case it has the form $L(z \to z)$. It is not valid if $x \neq y$, as shown by the following chart:

W_1: $\boxed{\underset{0}{L}(L(Lx \to Lp) \to L(Ly \to Lp))}$

$\downarrow$

N_1: $\boxed{\boxed{\underset{1}{L}(Lx \to Lp) \underset{0}{\to} \underset{0}{L}(Ly \to Lp)}}$

$\downarrow$

N_2: $\boxed{\boxed{\underset{1}{Ly} \underset{0}{\to} \underset{0}{Lp}}}$

$\downarrow$

N_3: $\boxed{\boxed{\underset{0}{p}}}$

No step contradicts propositional logic, for $x \neq y$ and $p \neq y$.

Let us abbreviate that formula as $F(x, y, p)$. If $p_1, p_2, \ldots, p_n, \ldots$ are all the propositional variables, we have

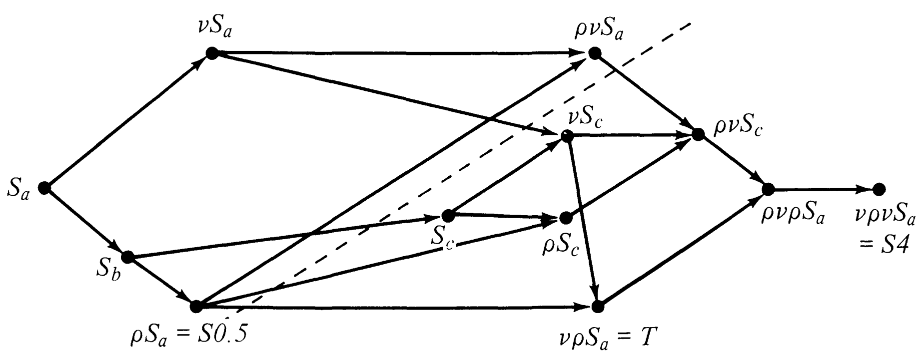
$$xEy \text{ iff } F(x, y, p_1) \text{ and } F(x, y, p_2) \ldots \text{ and } F(x, y, p_n) \ldots.$$

By Theorem 2.2 (and its Corollary and Remarks) it follows that identity is the sole congruence in the logic defined by validity in the above class of frames. But as this logic is at least as strong as $\rho\nu S_a$, we have:

Theorem 5.1 *In $\rho\nu S_a$ identity is the sole congruence.*

It follows that identity is also the sole congruence in all systems weaker than $\rho\nu S_a$, such as νS_a, $\rho S_a = S0.5$, etc., since $\vdash L(z \to z)$ even in S_a.

In the following chart (see [8] or [10]), where the arrows show relative strength of the systems constructed from S_a, the systems at the left of the broken line have identity as sole congruence.



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