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A Highly Efficient "Transfinite Recursive Definitions" Axiom for Set Theory

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Introduction We will consider formal set theories with an axiom (schema) that is well-known as a theorem or principle of set theory, but to our knowledge has not been proposed previously as an axiom. Essentially, this new schema *RD* says that the epsilon relation is one on which functions can be defined by transfinite recursion. At first, one might suspect that this would be equivalent to the more familiar axiom schema of (transfinite) induction on epsilon. With enough other axioms present, this is true. At the same time, we find (as is sometimes the case) that recursion is in some ways a more powerful principle, and that this effect becomes more pronounced when the underlying logic is intuitionistic rather than classical.

Specifically, we will see that the technical situation is as follows:

1. Classically, even our strongest version of *RD* (out of three versions) is provable in *ZFC* set theory. So *RD* gives nothing new. Its main feature in this context is its great "efficiency": when added to just a few of the axioms of *ZF*, *RD* suffices to derive the full axioms of replacement, choice, and foundation in any form (including induction on epsilon). Thus *RD* can be used to give a very "short" axiomatization of *ZFC*.

2. Our use of *RD* arose while studying axiomatic set theories with the law of the excluded middle confined to bounded predicates–theories which we have called "partially intuitionistic". In this context, we find that the efficiency and strength of *RD* are even more striking. It is still the case that *RD*, plus a few basic axioms, proves replacement in any reasonable form, choice, and induction on epsilon (though certain other forms of foundation don't seem to follow). In the other direction, the two weaker forms of *RD* are still derivable from more standard axioms. However, full *RD* is not provable in any straightforward way even from very strong forms of these standard axioms. We

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conjecture that it is in fact independent of them. On the other hand, it turns out that the addition of *RD* to these more standard systems (even ones with a very limited form of replacement) does not increase their proof-theoretic strength.

Our definitions and notation are in Section 1. Section 2 contains the material outlined above concerning *RD* and its relationship to more familiar axioms. In Section 3 we discuss, informally and with no proofs, a related topic: the delicate relationship among different foundation axioms in weak (classical or partially intuitionistic) set theories.

1 ***Preliminaries*** We let $\mathcal{L}$ denote the usual first-order language (with equality) of set theory, with a symbol $\perp$ for absurdity rather than one for negation. We use the standard abbreviations $\sim\varphi$, $\varphi \longleftrightarrow \psi$, $\exists! x\varphi$, $x \subseteq y$, $\exists x \subseteq y\varphi$, $\forall x \subseteq y\varphi$, $\exists x \in y\varphi$, and $\forall x \in y\varphi$. The latter two will usually be further abbreviated to $\exists x_y\varphi$ and $\forall x_y\varphi$. We also write $\exists^{<2}x\varphi$ as an abbreviation for $\forall x, y(\varphi(x) \wedge \varphi(y) \rightarrow x = y)$. The symbol $\vdash$ will denote provability in the intuitionistic predicate calculus.

Definition A formula of $\mathcal{L}$ is called *bounded* (or *restricted*) iff all its quantifiers are of the form $\forall x \in y$ or $\exists x \in y$.

The set of bounded formulas is called Δ_0.

Definition A formula of $\mathcal{L}$ is called *$\mathcal{P}$-bounded* iff all its quantifiers are of the form $\forall x \in y$, $\exists x \in y$, $\forall x \subseteq y$, or $\exists x \subseteq y$.

The set of $\mathcal{P}$-bounded formulas is called $\Delta_0^{\mathcal{P}}$.

We now proceed to define various axioms in $\mathcal{L}$. First, we let *Ext, Pair, Sum, Pow, Inf, Reg,* and *AC* denote the usual forms of the axioms of extensionality, pairing, sum set, power set, infinity, regularity, and choice. We also use standard forms of the schemata of induction on epsilon, law of the excluded middle, separation, replacement, and collection:

$$\begin{aligned}
TI_\in^\varphi &=_{df} \forall x[\forall y_x\varphi(y) \rightarrow \varphi(x)] \rightarrow \forall x\varphi(x)\\
Sep_\varphi &=_{df} \forall a\exists b\forall x[x \in b \longleftrightarrow x \in a \wedge \varphi(x)]\\
LEM_\varphi &=_{df} \varphi \vee \sim\varphi\\
Rep_\varphi &=_{df} \forall x_a\exists! y\varphi(x, y) \rightarrow \exists b\forall y(y \in b \longleftrightarrow \exists x_a\varphi(x, y))\\
Coll_\varphi &=_{df} \forall x_a\exists y\varphi(x, y) \rightarrow \exists b\forall x_a\exists y_b\varphi(x, y).
\end{aligned}$$

(In all axiom schemata such as the above, unspecified predicates like φ may have free variables other than those shown.)

In the presence of classical logic plus a few basic axioms, replacement and collection imply each other and separation.[1] But without classical logic, replacement and collection do not imply full separation. In fact, in theories with $LEM_{x\in y}$, Sep_φ trivially implies LEM_φ. So our partially intuitionistic theories never have full *Sep*. Without classical logic, *Rep* and *Coll* also apparently do not imply each other. It is therefore natural to consider stronger forms of these axioms which imply both the standard versions. Here are two examples of this. The first, used by Tharp [2], is a concise axiom which seems to be strictly stronger than *Rep* and *Coll* combined. The second, which we call "iterated choice collection", is our idea of the strongest form of this type of axiom that one can reasonably concoct. It implies any natural form of replacement or collection, as well as AC and Sum (the variables f and t will always denote, respectively, a function and a transitive set):

$$Rep'_\varphi =_{df} \forall x_a \exists y \varphi(x, y) \rightarrow \exists b[\forall x_a \exists y_b \varphi(x, y) \wedge \forall y_b \exists x_a \varphi(x, y)]$$
$$ICC_\varphi =_{df} \forall x \exists y \varphi(x, y) \rightarrow \forall a \exists t, f[a \subseteq t \wedge Dom(f) = t \wedge Rng(f) \subseteq t \wedge \forall x_t \varphi(x, f(x))].$$

In the above, of course, $Dom(f)$ and $Rng(f)$ denote the domain and the range of f, respectively. Some other standard notation we will use is: $Rel(x)$ to mean x is a (binary) relation; the variable r for relations; $Fld(r)$ to denote $Dom(r) \cup Rng(r)$; $Fnc(x)$ to mean x is a function; $Trans(x)$ to mean x is transitive; and $\bar{x}$ or $TC(x)$ to denote the transitive closure of x. Less standardly, we will write x' for $\bar{x} \cup \{x\}$. (Though if x is transitive, in particular an ordinal, this corresponds to the usual definition of successor.)

The axiom ICC defined above was thought of as a sort of *sup* of two other strong forms of collection, called "choice collection" and "iterated collection":

$$CC_\varphi =_{df} \forall x_a \exists y \varphi(x, y) \rightarrow \exists f[Dom(f) = a \wedge \forall x_a \varphi(x, f(x))]$$
$$IC_\varphi =_{df} \forall x \exists y \varphi(x, y) \rightarrow \forall a \exists t[a \subseteq t \wedge \forall x_t \exists y_t \varphi(x, y)].$$

The technical connection among all these collection and replacement axioms is discussed at the end of the next section. One more somewhat standard axiom that will be useful is:

$WIO =_{df}$ "Every well-ordering is isomorphic to an ordinal."

We will now define our three axiom schemata of transfinite recursive definitions. First we need to define a predicate which we think of as saying, "f is a (recursively) 'good' function on the (transitive) domain t, with respect to the formula $\varphi(v_1, v_2, v_3)$":

$$GF_\varphi(f, t) =_{df} Fnc(f) \wedge Trans(t) \wedge Dom(f) = t \wedge \forall u_t \varphi(f \upharpoonright \bar{u}, f(u), u).$$

Definition For each φ as above (possibly with more free variables), RD_φ is the formalization of the following statement: If, for every x, every good function on $\bar{x}$ can be extended to one on x', *then* for every x there is a good function on $\bar{x}$.

As mentioned in the Introduction, we also define two weaker forms of this axiom. RD^U_φ (resp., RD^S_φ) is obtained from RD_φ by replacing the words "has an extension" with "has a unique extension" (resp., with "has a nonempty set of extensions"). So RD^U is the weakest of the three versions. (It would be natural to strengthen the conclusion of RD^U_φ (resp., RD^S_φ) by replacing the words "a good function" with "a unique good function" (resp., with "a nonempty set of good functions"). But even in our weakest theory IS_0, these stronger-looking versions of RD^U and RD^S are derivable from the original versions (see the remark following Corollary 2.8).)

The reader may wonder why we have chosen transitive sets as the natural domains of good functions. It would be possible to use arbitrary sets, but then certain details become somewhat messy. On the other hand, if we used ordinals, RD would be simpler to work with but at the same time less general. This latter form of RD will be discussed at the end of Section 2.

We are now ready to define several theories.

Definitions

$IS_0 =_{df} Ext + Pair + Pow + Inf + \Delta_0\text{-}LEM + \Delta_0\text{-}Sep$
$IS =_{df} IS_0 + Sum + Reg + \Delta_0\text{-}Coll$
$K_1 =_{df} IS + TI_\epsilon + WIO + \Delta_0^{\mathcal{P}}\text{-}Coll$
$K_2 =_{df} K_1 + LEM$
$K_3 =_{df} K_1 + RD$ ($= IS_0 + WIO + RD$, as we shall see)
$ZF =_{df} IS + LEM + Rep.$

So K_1 has both *LEM* and *Rep* restricted. K_2 adds full *LEM*, while K_3 adds full *RD* (which includes full *Rep*); and *ZF* adds both. One of the main results of [3] and [4] is that K_1, K_2, and K_3 are equiconsistent. (In fact, so are K_1, $K_2 + V = L$, and $K_3 + V = L$.)

*2 **Properties of the transfinite recursive definitions schemata*** In this section we investigate the relationship among the schemata RD^U, RD^S, RD, and more standard axioms. For the most part, these results are straightforward, but care is required when doing set-theoretic proofs without classical logic. Certain simple facts proved in [3] and/or [4] are not reproved here.

Let us first examine what it takes to prove each form of *RD*.

Lemma 2.1 $IS \vdash$

(a) $GF_\varphi(f, t_1) \wedge t_2 \subseteq t_1 \rightarrow GF_\varphi(f \restriction t_2, t_2)$
(b) $GF_\varphi(f, \bar{x}) \longleftrightarrow Dom(f) = \bar{x} \wedge \forall u_x GF_\varphi(f \restriction u', u')$.

Proof: Trivial.

Theorem 2.2 $IS + TI_\epsilon + Rep \vdash RD^U$.

Proof: Assume φ is given. Working in $IS + TI_\epsilon + Rep$, assume the hypothesis of RD^U_φ. We will prove the conclusion of it in the stronger form $\forall x \exists! f GF_\varphi(f, \bar{x})$. By TI_ϵ, it suffices to prove this for an arbitrary x, assuming it holds for each y in x. Now, by the hypothesis of RD^U_φ, a good function on $\bar{y}$ has a unique extension to one on y'. So we have $\forall y_x \exists! f GF_\varphi(f, y')$. By *Rep*, we can form the set of all these f's; then by *Sum* we can form the union of all these f's. Call it g. We will show $GF_\varphi(g, \bar{x})$. Trivially, g is a relation. The domain of g is $\bigcup_{y \epsilon x} y' = x \cup \bigcup_{y \epsilon x} \bar{y}$. Clearly, this set contains x, is transitive, and any other transitive set containing x must contain this set; i.e., $Dom(g) = \bar{x}$. (Note that we have just proved, in $IS + TI_\epsilon + Rep$, that every set has a transitive closure. This simple but vital fact can actually be proved in $IS + TI_\epsilon$.)

We next show g is a function. We just need to prove that the f's whose union is g are "compatible". So suppose we have $y_i \epsilon x$ and $GF_\varphi(f_i, y_i')$, for $i = 1, 2$. We need to show that $f_1 = f_2$ wherever both are defined. So say $w \epsilon y_1' \cap y_2'$. We want $f_1(w) = f_2(w)$; we'll prove the stronger statement $f_1 \restriction w' = f_2 \restriction w'$, using TI_ϵ. That is, we can assume $\forall v_w (f_1 \restriction v' = f_2 \restriction v')$. But note that $\bar{w} = \bigcup_{v \epsilon w} v'$. Thus $f_1 \restriction \bar{w} = f_2 \restriction \bar{w}$. But by Lemma 2.1(a), $f_1 \restriction \bar{w}$ is a good function on $\bar{w}$. So it has a unique extension to w', which thus must be the same for f_2.

So g is a function on $\bar{x}$. By Lemma 2.1(b), it is in fact a good function. Finally, we must prove g is the unique good function on $\bar{x}$. This proof is almost identical to the proof that g is a function, so we won't repeat it.

The reader may have noted that the above proof essentially follows classical reasoning. Recall that our partially intuitionistic theories all have Δ_0-*LEM*. In other words, classical logic may be used in them as long as a proof is "talking about" a set rather than the whole universe or a proper class. In simple results, this is often the case. The interested reader may wish to go through the previous proof and isolate the places in it where Δ_0-*LEM* is actually needed.

Lemma 2.3 *IS proves the equivalence of AC, Zorn's Lemma, the Multiplicative Axiom, the Hausdorff Maximal Principle, and the Well-Ordering Principle.*

Proof: In the usual classical proof of these equivalences, *LEM, Coll,* and *Sep* are needed only for bounded predicates. So the proof can be formalized in *IS*.

Theorem 2.4 $IS + TI_\epsilon + Rep + AC \vdash RD^S$.

Proof: Given φ, we work in $IS + TI_\epsilon + Rep + AC$ and assume the hypothesis of RD^S_φ. Consider the predicate $\varphi^*(x)$ which says: "for all y in x', the collection of good functions (with respect to φ) on $\bar{y}$ forms a nonempty set".

Using TI_ϵ, we will prove the conclusion of RD^S_φ in the stronger form $\forall x \varphi^*(x)$. (For note that $\varphi^*(x)$ implies in particular that the collection of good functions on $\bar{x}$ forms a nonempty set.)

So assume $\forall u_x \varphi^*(u)$. As in the proof of Theorem 2.2, we have that $\bar{x} = \bigcup_{u \epsilon x} u'$. So, given $y \epsilon \bar{x}$, $S_y = \{g\colon GF_\varphi(g, \bar{y})\}$ is a nonempty set. Also, by the hypothesis of RD^S_φ, for each $g \epsilon S_y$ there is a nonempty set of good extensions of g to y'. By *Rep* and *Sum*, we can form the set of all these extensions for all $g \epsilon S_y$. By Lemma 2.1(a), every good function on y' is an extension of one on $\bar{y}$, so we have the set $S^{(y)} = \{g\colon GF_\varphi(g, y')\} \neq \phi$.

Using *Rep* and *Sum* again, we next form the set $S = \bigcup_{y \epsilon \bar{x}} S^{(y)}$. We then define $R = \bigcup S$. Clearly, R is a relation with domain $\bar{x}$. R need not be a function, but by Lemma 2.1(b) every good function on $\bar{x}$ must be a subset of R. Also by that lemma,

$$\begin{aligned} GF_\varphi(g, \bar{x}) &\longleftrightarrow Dom(g) = \bar{x} \wedge \forall y_x GF_\varphi(g \restriction y', y') \\ &\longleftrightarrow Dom(g) = \bar{x} \wedge \forall y_x (g \restriction y' \epsilon S). \end{aligned}$$

Now, by Δ_0-*Sep* we can form the set $S_x = \{g \subseteq R\colon Dom(g) = \bar{x} \wedge \forall y_x (g \restriction y' \epsilon S)\}$. And, by the above, we have $S_x = \{g\colon GF_\varphi(g, \bar{x})\}$.

It remains to show that $S_x \neq \phi$. To do this, first use *AC* to define a well-ordering r on the range of R. Consider the predicate $\varphi^+(v_1, v_2, v_3)$ which says: "$v_3 \epsilon \bar{x}$ and v_2 is the r-least object such that $\varphi(v_1, v_2, v_3)$; or $v_3 \notin \bar{x}$ and $v_2 = \phi$." (The extra parameters of φ^+ are x, r, and those of φ.)

Now, the hypothesis of $RD^U_{\varphi^+}$ holds, as follows: say $GF_{\varphi^+}(g, t)$. If $t \notin \bar{x}$, then $g \cup \{\langle t, \phi\rangle\}$ is the unique φ^+-good extension of g to t'. If $t \epsilon \bar{x}$, then $t \subseteq \bar{x}$, whence it is clear that $GF_\varphi(g, t)$. So by the hypothesis of RD^S_φ, there is a nonempty *set* of φ-good extensions of g to t'; i.e., $\{v_2: \varphi(g, v_2, t)\}$ is a nonempty set. Also, this set is contained in the range of R. Hence this set must have an r-least member w. (Note that if the nonempty class $\{v_2: \varphi(g, v_2, t)\}$ were not a set, we could not make this conclusion.) Thus $g \cup \{\langle t, w\rangle\}$ is the unique φ^+-good extension of g to t'.

So by Theorem 2.2, the conclusion of $RD^U_{\varphi^+}$ holds. In particular, there is a φ^+-good function on $\bar{x}$. But such a function is also a φ-good function. So S_x is nonempty. This implies $\varphi^*(x)$, as desired.

The parenthetical remark near the end of the above proof points out why full RD is not provable in the same way. The next well-known fact illustrates this further.

Proposition 2.5 $ZFC \vdash RD$.

Proof: Given φ, assume the hypothesis of RD_φ. Define the new predicate $\varphi^+(v_1, v_2, v_3)$ which says, "v_2 is an object of least ordinal rank such that $\varphi(v_1, v_2, v_3)$". Then the hypothesis of $RD^S_{\varphi^+}$ is easily verified, so the conclusion of it holds by Theorem 2.4. This easily implies the conclusion of RD_φ.

Choosing the least rank in which an arbitrary predicate has a solution, as is required in the above proof, certainly requires classical logic.

We are now ready to go the other way and examine the consequences of the different forms of RD. The shortness of most of these proofs, in contrast to most of the above ones, should highlight further the strength of RD.

Theorem 2.6 *TI_ϵ, Reg, Sum, Rep, and Δ_0-Coll are all provable in $IS_0 + RD^U$.*

Proof: Given any $\varphi(x)$, let $\varphi^*(v_1, v_2, v_3)$ be simply $\varphi(v_3) \wedge v_2 = \phi$. Then the hypothesis of TI^φ_ϵ trivially implies the hypothesis of $RD^U_{\varphi^*}$, and the conclusion of $RD^U_{\varphi^*}$ trivially implies the conclusion of TI^φ_ϵ. So $IS + RD^U \vdash TI_\epsilon$.

Reg is a direct consequence of TI_ϵ and Δ_0-*LEM*.

Next, note that RD^U is "rigged" to imply that every set has a transitive closure; i.e., let φ be automatically true, and then RD^U_φ just says $\forall x$ ($\bar{x}$ exists). And once we have $\bar{x}$, $\bigcup x$ can be formed by Δ_0-*Sep*, since $\bigcup x = \{u \epsilon \bar{x}: \exists y_x (u \epsilon y)\}$. Thus $IS_0 + RD^U \vdash Sum$. (The reader who finds this procedure unesthetic is welcome to change the conclusion of all three forms of RD to $\forall t \exists g\, GF_\varphi(g, t)$, and add the axiom *Sum* to IS_0. All our results still hold under this revision.)

Now, given $\varphi(x, y)$, let $\varphi^*(v_1, v_2, v_3)$ be the predicate $[v_3 \epsilon a \wedge \varphi(v_3, v_2)] \vee [v_3 \notin a \wedge v_2 = v_3]$. Then Rep_φ follows immediately from $RD^U_{\varphi^*}$. So $IS_0 + RD^U \vdash Rep$.

Finally, we need to show that $IS_0 + RD^U \vdash \Delta_0$-*Coll*. We have already shown that $IS_0 + RD^U \vdash IS - \Delta_0\text{-}Coll + TI_\epsilon + Rep$, so we will prove Δ_0-*Coll* in the latter theory. So say $\varphi(x, y)$ is Δ_0, and assume $\forall x_a \exists y \varphi$. Now, it is not hard to show (in this theory) the basic facts about ordinals and the cumulative hierarchy. In particular, we can show that every set has an ordinal rank, $\rho(y)$ being, as usual, the least α such that $y \epsilon V_{\alpha+1}$. So we reason as follows: given x in a, pick a y_0 such that $\varphi(x, y_0)$, and let $\alpha = \rho(y_0)$. Then, by Δ_0-*Sep* we can form the set $\{\beta \epsilon \alpha': \exists y \epsilon V_{\beta+1}(\varphi(x, y))\}$. (The predicate to the right of the colon may not seem Δ_0, but it is with the function $\{\langle\beta, V_\beta\rangle: \beta \in \alpha'\}$ used as a parameter.) This set is a nonempty set of ordinals, so it has a unique least element. In other words, we can prove the existence of the least rank in which a Δ_0 predicate has a solution.

So we have $\forall x_a \exists! \beta[\beta$ is the least ordinal such that $\exists y(\rho(\beta) = y$ and $\varphi(x, y))]$. So by *Rep* we can form the set of all these β's. Then by *Sum* we can form their *sup*; call it α. Finally, set $b = V_{\alpha'}$, and we have $\forall x_a \exists y_b \varphi$.

Proposition 2.7 $\quad IS_0 + RD^S \vdash AC$.

Proof: Let $\varphi(v_1, v_2, v_3)$ be $(v_2 = v_3 = \phi) \vee (v_2 \in v_3)$. Then the hypothesis of RD^S_φ is trivial, and the conclusion of it is $\forall x \exists f[Dom(f) = \bar{x} \wedge \forall y \in \bar{x}(f(y) = y = \phi \vee f(y) \in y)]$, which becomes precisely AC if we simply restrict f to x instead of $\bar{x}$.

We now summarize our results concerning the "efficiency" of the various forms of *RD*. When we call two or more theories equal, we mean that they prove the same theorems.

Corollary 2.8

(a) $IS_0 + RD^U = IS + TI_\epsilon + Rep$
(b) $IS_0 + RD^S = IS + TI_\epsilon + Rep + AC$
(c) $IS_0 + RD^U + LEM = ZF$
(d) $IS_0 + RD^S + LEM = IS_0 + RD + LEM = ZFC.$

Proof: (a) follows from Theorems 2.2 and 2.6. (b) follows from Theorems 2.4 and 2.6, and Corollary 2.7. (c) is immediate from (a) and the definition of *ZF*. Finally, (d) comes from (c) and Theorem 2.5.

So, even without classical logic, we have standard axioms which are exactly equivalent to RD^U and to RD^S; and with classical logic the same holds for *RD*. As we have already mentioned, this does not seem to hold for *RD* in a partially intuitionistic context.

Theorem 2.9 $\quad IS_0 + RD \vdash ICC.$

Proof: Assume $\forall x \exists y \varphi(x, y)$. Working in $IS_0 + RD$, we first claim $\forall a \exists f(Dom(f) = \bar{a} \wedge \forall x_{\bar{a}} \varphi(x, f(x)))$. (This will already show $IS_0 + RD \vdash CC$.) Let $\varphi^*(v_1, v_2, v_3)$ be $\varphi(v_3, v_2)$. Then the hypothesis of RD_{φ^*} is trivially established, and the conclusion is precisely $\forall a \exists f(Dom(f) = \bar{a} \wedge \forall x_{\bar{a}} \varphi(x, f(x)))$.

However, given a, we do not just want a φ^*-good function on $\bar{a}$ (that is, a function f with $\forall x \in \bar{a} \varphi(x, f(x)))$. We want one on some t containing a, with $Rng(f) \subseteq t$. Intuitively, such an f should be constructed by starting with one on $\bar{a}$, and then extending it infinitely many times, letting the domain of each successive function be the transitive closure of the field of the previous one.

To accomplish this formally, we use *RD* again. Let $\varphi'(v_1, v_2, v_3)$ be the predicate which says: $v_3 \notin \omega$ and $v_2 = \phi$; or $v_3 = 0$ and v_2 is a φ^*-good function on $\bar{a}$; or $v_3 = n'$ for some n in ω, v_1 is a function with domain n', and v_2 is a φ^*-good function on $TC(Fld(v_1(n)))$.

We prove the hypothesis of $RD_{\varphi'}$: say g is a φ'-good function on $\bar{y}$. We need to extend it to one on y', which means finding v_2 which satisfies

$\varphi'(g, v_2, y)$. But we have already shown that every transitive set has a φ^*-good function on it. Since the desired v_2 is always ϕ or a φ^*-good function on a specific transitive set, it must exist.

So we have the conclusion of RD_φ. Given any set a, let g be a φ'-good function on ω. Note that each $g(n)$ is then a φ^*-good function. We define a function f, with domain $\bigcup_{n \in \omega} Dom(g(n))$, by the rule $f(x) = [g(n)](x)$, where n is the least number such that $x \in Dom(g(n))$. Let t be the domain of f. It is then completely straightforward to verify that f and t satisfy the conclusion of ICC_φ, and we omit the details.

Corollary 2.10 *In the theory IS_0, the following implications hold (i.e., each arrow $A \to B$ actually means $IS_0 + A \vdash B$):*

$$\begin{tikzcd}
 & & & AC & \\
RD \arrow[r] & ICC \arrow[r] & CC \arrow[ur] \arrow[dr] & & Rep \\
 & & & Rep' \arrow[ur] \arrow[dr] & \\
 & & & & Coll
\end{tikzcd}$$

Proof: The first arrow is Theorem 2.9. All the others are trivial.

Recall that the implications $RD \to TI_\epsilon \to Reg$ and $RD \to Sum$ could also be added to the above chart. However, for the moment we want to focus on the relationship of RD to collection and replacement axioms.

We do not know how to reverse any of the above arrows, even in the presence of a theory like $IS + TI_\epsilon + WIO + AC$. In fact, we conjecture that none of them is reversible. (This includes the conjecture that Rep' does not follow from $Rep + Coll$.)

If these conjectures are correct, they highlight further the strength of RD: it stands at the top of this five-level hierarchy of axioms in which full collection and replacement (usually thought of as strong axioms) share the bottom rung. (However, we repeat that this hierarchy collapses when we talk about proof-theoretic strength.)

For the sake of readability, we have left the schema IC out of the above chart. The following implications are immediate: $ICC \to IC$, $IC \to Coll$, and $IC + CC \to ICC$. We conjecture that CC does not imply IC, and IC does not imply Rep (even in the theory $IS + TI_\epsilon + WIO + AC$).

We conclude this section with a brief discussion of the schema RD with the domains of good functions restricted to ordinals, as mentioned near the end of Section 1.

Definition For each formula $\varphi(v_1, v_2, v_3)$, we let RDO_φ be the same predicate as RD_φ, except that the variable x in RD_φ gets replaced by an ordinal variable. Also, let RDN_φ be the same predicate with the ordinals further restricted to be no greater than ω.

(I.e., RDN_φ essentially says: if, for every natural number n, every φ-good function on n can be extended to one on $n + 1$, then there is a φ-good function on ω.)

Clearly, RDO is weaker than RD, but one may suspect that it is not much weaker. Theorem 2.13 will bear this out, showing that RDO implies RD when added to the theory K_1. K_1 does not contain *AC* nor much *Rep* or *Coll*, but it does contain the nontrivial foundation axioms TI_ϵ and *WIO* (as well as $\Delta_0^{\mathcal{P}}$-*Coll*), which make it much stronger than *IS*. Loosely, we can therefore say that while *RD* combines foundation, *Rep, Coll,* and *AC, RDO* only combines *Rep, Coll,* and *AC* (as does *ICC*, but in a different way).

After Corollary 2.10, we conjectured that *RD* is independent of $IS + TI_\epsilon + WIO + ICC$. By Theorem 2.13, the conjecture that *RDO* is independent of these axioms is no stronger. However, we also make the stronger conjecture that *RDN* is independent of these axioms.

Now to our results about *RDO*.

Lemma 2.11 *IS + WIO + RDO* $\vdash$ *Every set is in one-to-one correspondence with some ordinal.*

Proof: Given any set x, let φ be the predicate:

$$(v_2 \in x - Rng(v_1)) \vee (v_2 = x - Rng(v_1) = \phi)$$

(x is an extra parameter in φ). Trivially, the hypothesis of RDO_φ must always hold, since such a v_2 must exist. So, by *RDO*, we have that there is a φ-good function on every ordinal.

Note that, if $GF_\varphi(f, \alpha)$, then either $f\colon \alpha \xrightarrow{1-1} x$ or there is some $\beta < \alpha$ such that $f \restriction \beta\colon \beta \xrightarrow[\text{onto}]{1-1} x$.

Now, in *IS + WIO*, we can prove Hartogs' Theorem in the usual way: given any set y, we can form the set of all well-orderings on subsets of y. (Using an appropriately large parameter, this requires only Δ_0-*Sep*.) Then, by *WIO* and Σ_1-*Rep* (which holds in *IS*), we can form the set of all the ordinals which are isomorphic to these well-orderings. Finally, in *IS* it can be proved that the *sup* of any set of ordinals is an ordinal. So let α be the successor of the *sup* of this set of ordinals. Clearly, α cannot be mapped one-to-one into y, as desired.

So, for our given x, let α be an ordinal as guaranteed by Hartogs' Theorem, and let f be a φ-good function on α. Since f cannot be one-to-one, some restriction of f to a smaller ordinal must be a bijection.

Corollary 2.12 *IS + WIO + RDO* $\vdash$ *AC*.

Proof: By the above, this theory proves that every set can be well-ordered, which (in *IS*) implies *AC*.

Clearly, the two results above still hold using the weaker RDO^S (defined analogously to RD^S) instead of *RDO*.

It may seem strange that an axiom like *WIO* is needed to prove something as basic as Hartogs' Theorem, but as far as we know this is the case. As we will mention in Theorem 3.1, *WIO*, Hartogs' Theorem, and several other predicates are interchangeable in the presence of $IS + TI_\epsilon$.

In the remarks following Theorem 3.2, we will conjecture that *WIO* does not follow from the full schema *RD*. Some readers may wonder why, in the theory *IS + RD*, one could not apply the method of the proof of Lemma 2.11 in reverse, to prove that every set can be mapped one-to-one to an ordinal (which in turn would immediately imply Hartogs' Theorem and *WIO*). The reason is this: when the predicate φ says that $f(u)$ is not in $Rng(f \restriction \bar{u})$, then every φ-good function on an *ordinal* must be one-to-one, as in Lemma 2.11 (up to the point where the desired range x is exhausted). The same conclusion fails, however, for φ-good functions on arbitrary transitive sets.

Theorem 2.13 $K_1 + RDO \vdash RD$.

Proof: Working in $K_1 + RDO$, assume that the hypothesis of RD_φ holds. In K_1 (or even $K_1 - WIO$), it is easy to prove the standard facts about the cumulative hierarchy, including that every set is contained in some V_α. So, by Lemma 2.1(a), it will suffice to prove that there is a φ-good function on every V_α.

Let φ^* be the predicate: $GF_\varphi(v_2, V_{v_3+1}) \wedge \bigcup (Rng(v_1)) \subseteq v_2$.

We claim that the hypothesis of RDO_{φ^*} holds. For say f is a φ^*-good function on α. So for each β in α, $\varphi^*(f \upharpoonright \beta, f(\beta), \beta)$ holds. This says that $f(\beta)$ is a φ-good function on $V_{\beta+1}$ which contains every $f(\gamma)$ for $\gamma < \beta$. In other words, f defines an increasing α-sequence of φ-good functions on the sets $V_{\beta+1}$, for $\beta < \alpha$. So let $g = \bigcup (Rng(f))$. $\bigcup\limits_{\beta<\alpha} V_{\beta+1} = V_\alpha$, so g is clearly a function on V_α, and by Lemma 2.1(b) it is in fact a φ-good function.

To prove the hypothesis of RDO_{φ^*}, we must show that f can be extended to a φ^*-good function on $\alpha + 1$, which simply means that g can be extended to a φ-good function on $V_{\alpha+1}$. By the hypothesis of RD_φ, we know that g has a φ-good extension to $V_\alpha \cup \{y\}$, for each $y \in V_{\alpha+1} - V_\alpha$. By Lemma 2.11, define a bijection between some ordinal and $V_{\alpha+1} - V_\alpha$. Using this bijection and an obvious instance of RDO, we can use these one-set-at-a-time extensions of g to show that there is a φ-good extension of g to all of $V_{\alpha+1}$.

Thus, the hypotheses of RDO_{φ^*} holds. So, by RDO, there is a φ^*-good function on every ordinal. And we have already seen that this implies there is a φ-good function on every V_α, as desired.

*3 **Different forms of the axiom of foundation*** We have shown that the schema RD, and its weaker versions, combine aspects of the axioms of choice, replacement (and collection), and foundation. We have seen that, in a partially intuitionistic context, the various classically equivalent forms of choice are still equivalent, but we have conjectured that this is markedly not the case for several classically equivalent forms of replacement and collection. For the sake of completeness, we will now discuss the relationships among various foundation axioms. Some versions which are classically equivalent will even create theories of different proof-theoretic strengths in our partially intuitionistic setting (in contrast to the situation with replacement and collection axioms).

When we refer to foundation axioms, we are actually talking about two different types of statements. One type, the more usual one, consists of statements which express in some way the well-foundedness of the epsilon relation. The other type consists of statements which assert properties of other well-founded relations.

There are three standard versions of the first type of foundation: regularity, TI_ϵ, and the "least number principle" in the form of the schema:

$$LNP_\epsilon^\varphi =_{df} \exists x\varphi(x) \rightarrow \exists x[\varphi(x) \wedge \forall y_x \sim\varphi(y)].$$

Classically, TI_ϵ and LNP_ϵ are trivially equivalent. But, by a simple and well-known argument, LNP_ϵ implies full LEM (assuming we have $LEM_{x=y}$), and so it cannot be considered as an axiom for a partially intuitionistic theory. The same limitation holds for LNP on other well-founded relations. (These difficulties with LNP were mentioned after the proofs of Theorem 2.4 and Proposition 2.5. In our theories K_1 and K_3, LNP_ϵ^φ cannot be proved in general even for decidable φ, but it can be proved for bounded and even for $\mathcal{P}$-bounded φ.)

As previously mentioned, in the presence of Δ_0-LEM, Reg follows immediately from TI_ϵ. In theories without Δ_0-LEM, this implication does not normally hold, and TI_ϵ is usually the more appropriate axiom. Depending on what other axioms are present, Reg may imply LEM_ϵ.

The other direction is less trivial—the usual proof of TI_ϵ from Reg requires full separation. Replacing Reg by TI_ϵ in a weak set theory can increase strength, as we will see in Theorem 3.2.

We now move on to the other type of foundation axiom. There are a greater variety of this type, most of which are equivalent to each other. The axiom WIO is the one example of this sort of foundation axiom which we have already defined. Another example is the statement that every well-founded relation has an (ordinal-valued) rank function. (A rank function on a binary relation r is a function f such that, for all y in the field of r, $f(y) = sup\{f(x)': \langle x, y\rangle \epsilon\, r\}$. Also, when we say that r is well-founded, we mean that every nonempty set has an r-minimal element.) Another important axiom of this type is transfinite induction for well-founded relations. This is actually a schema, since transfinite induction is a schema:

$$TI^\varphi(r) =_{df} \forall x[\forall y(\langle y, x\rangle \epsilon\, r \rightarrow \varphi(y)) \rightarrow \varphi(x)] \rightarrow \forall x \varphi(x).$$

We now summarize the main equivalences for this type of foundation axiom. Proofs of them may be found in [3] (Theorem 2.4.9); they are all pretty straightforward.

Theorem 3.1 *In the theory $IS + TI_\epsilon$, each of the following proves all the others:*

(i) *WIO*
(ii) *Every well-founded relation has a rank function*
(iii) *Every well-founded partial ordering has a rank function*
(iv) *Every well-founded relation satisfying extensionality is isomorphic to the epsilon relation on some transitive set*
(v) *Hartogs' Theorem: for every set there is an ordinal which cannot be mapped one-to-one into it*
(vi) *Transfinite induction for well-founded relations*
(vii) *Transfinite induction for well-founded partial orderings*
(viii) *Transfinite induction for well-orderings.*

In the notation of [3], the schema (vii) above is called SBI, which stands for "set bar induction".

The equivalence of statements (i), (ii), (iii), and (v) actually holds in the weaker theory IS. On the other hand, any of the schemata (vi)-(viii), plus Reg, proves TI_ϵ. (So *Reg* + *SBI* is an exact equivalent of TI_ϵ + *WIO*.) This implies that the entire equivalence of the above theorem does not hold in *IS*, since we will see below that $IS + WIO \nvdash TI_\epsilon$.

We remark that all of the axioms of Theorem 3.1 follow immediately from *Reg* + (full) *Sep*, or from TI_ϵ plus a single instance of *Sep*.

We now consider the proof-theoretic strength of these different foundation axioms. Loosely, we can say that *Reg* is weaker than TI_ϵ, which in turn is weaker than *Reg* + *WIO*, which in turn is weaker than TI_ϵ + *WIO*. More precisely, we have the following theorem. Again, we omit the proof for the sake of brevity, since it requires a great deal of technical machinery. Most of our assertions about strengths of theories are taken from Appendix 2 of [1] or pages 27-28 of [3].

When we write $T_1 < T_2$ or say that T_2 is stronger than T_1, we mean that T_2 proves the consistency of T_1. Also, when we say two theories are equiconsistent or have the same strength, we mean that their equiconsistency can be proved in formal number theory.

Theorem 3.2 *Let T be the theory IS +* $\Delta_0^{\mathcal{P}}$*-Rep, or that theory plus AC and/or full LEM (so there are four possibilities for T). Then*

$$T < T + TI_\epsilon < T + WIO < T + SBI < T + Sep.$$

We remark that the four theories denoted by T, Classical Type Theory, and *IS-Coll* all have the same strength. But $T + TI_\epsilon$ is already stronger than Zermelo set theory.

The four theories T + *SBI* are essentially K_1, $K_1 + AC$, K_2, and $K_2 + AC$. So, as mentioned in Section 1, these theories and the theory K_3 all have the same strength.

In Section 2, we showed that many statements are derivable in $IS_0 + RD$. We believe that *WIO* is an exception; in fact, we conjecture that $IS + RD < K_3$.

We conclude this section with two generalizations of the schema *SBI*. Let $\sigma(x, y)$ be any predicate (possibly with extra parameters). Then we can think of σ as defining a "class binary relation". Let WF_σ be the statement that $\sigma(x, y)$ defines a decidable, well-founded partial ordering, i.e.,

$$\begin{aligned} WF_\sigma =_{df} &\forall x, y(\sigma \vee \sim\sigma) \wedge \forall x, y, z(\sigma(x, y) \wedge \sigma(y, z) \rightarrow \sigma(x, z)) \\ &\wedge \forall u[u \neq \phi \rightarrow \exists y_u \forall x_u \sim\sigma(x, y)]. \end{aligned}$$

Also, there is no problem stating the principle of transfinite induction on a class binary relation; we let TI_σ^φ be the same predicate as $TI^\varphi(r)$, with $\sigma(x, y)$ replacing $\langle x, y\rangle \epsilon r$. (So $TI^\varphi(r)$ and TI_ϵ^φ are both special cases of TI_σ^φ.)

Now, define *BI* (full bar induction) to be the double schema $WF_\sigma \rightarrow TI_\sigma^\varphi$. Friedman [1] uses *Reg* + *BI* as the foundation axioms in the partially intuitionistic theory $ZFC^{1/2}$. Also, on page A2.3 of [1], he defines an axiom which is between *SBI* and *BI* in content. He calls it *F*, but we prefer to call it *RBI*, "restricted bar induction". It is the double schema

$$[WF_\sigma \wedge \exists b \forall x, y(\sigma(x, y) \rightarrow x \epsilon b \wedge y \epsilon b)] \rightarrow TI_\sigma^\varphi.$$

It is clear that *RBI*, like *SBI* and the other axioms listed in Theorem 3.1, follows from *Reg* + *Sep*. Also, in the presence of separation for decidable predicates (i.e., the schema $\forall x(\varphi \vee \sim\varphi) \rightarrow Sep_\varphi$), *RBI* becomes equivalent to *SBI*. So *RBI* is provable in $K_1 + Rep$, hence in K_3. But we believe *RBI* to be independent of the theories $T + SBI$ (in the notation of Theorem 3.2). On the other hand, Friedman points out that $T + RBI$ has the same strength as $T + SBI$.

The full schema *BI* is more intriguing. It is the only axiom we know of that is provable in classical *ZF* and seems plausible in a partially intuitionistic context, but which we do not know how to prove in K_3. Also, the interpretation used in [3] to interpret K_3 into K_1 does not work for *BI*. In fact, we conjecture that $ZFC^{1/2}$ (which is contained in $K_3 + BI$) is stronger than K_3. On the other hand, Theorem 6.4 of [1] shows that $ZFC^{1/2}$ can be interpreted in a classical theory which is not much stronger than K_2.

There are some interesting parallels between *BI* and *RD*. Each is the strongest-looking version of a certain type of axiom. Each is provable in classical set theory (*RD* requires choice, *BI* does not) but for each one the usual proof requires full replacement and "least rank picking" of the sort that uses full classical logic (recall Proposition 2.5). Hence, neither one seems provable in natural theories obtained from *ZFC* by restricting *Rep* and/or *LEM*. Nor does either one seem to imply the other. It would be interesting to formulate an axiom that would combine *BI* and *RD* in a natural way. It would presumably state that functions can be defined by transfinite recursion on well-founded class relations. This "bar recursion" principle for set theory would require the language of Von Neumann-Bernays set theory to even state. It is not clear whether it would be provable in any reasonable version of that theory, though it would probably at least be consistent if stated properly.

NOTE

1. The "few basic axioms" we have in mind here are those of the theory *IS*, defined at the end of this section. These axioms may be "basic", but they are not trivial. I.e. to prove *Coll* from *Rep* requires *Pow*, and the reverse implication requires Δ_0-*Sep*.

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