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Does IPC Have a Binary Indigenous Sheffer Function?

HERBERT E. HENDRY

The question is whether there is a binary function $*$ such that: (1) each of the intuitionist functions $\sim$, &, $\vee$, and $\supset$ is definable in terms of $*$ (i.e., $*$ is a Sheffer function for $\{\sim, \&, \vee, \supset\}$), and (2) $*$ is definable in terms of $\sim$, &, $\vee$, and $\supset$ (i.e., $*$ is indigenous to $\{\sim, \&, \vee, \supset\}$).[1] The answer is: No.

The proof that follows will make reference to the Gödelian three-valued system G_3. G_3 is determined by the following tables[2]:

	&			$\vee$			$\supset$			$\sim$
	T	I	F	T	I	F	T	I	F	
T	T	I	F	T	T	T	T	I	F	F
I	I	I	F	T	I	I	T	T	F	F
F	F	F	F	T	I	F	T	T	T	T

with T as the designated value. It is easily verified that all theorems of the intuitionist propositional calculus (*IPC*) are tautologies of G_3. (The converse does not hold.)

Assume for a contradiction that $*$ is an indigenous Sheffer function for *IPC*. Then, there is some formula D containing no connectives other than $\sim$, &, $\vee$, and $\supset$ such that $(p * q) \equiv D$ is a theorem[3] of *IPC*. It follows that $(p * q) \equiv D$ is a tautology of G_3. Thus $*$ is an indigenous Sheffer function for G_3. Consider the matrix that defines $*$:

$*$	T	I	F
T	α_1	γ_2	α_2
I	γ_1	β	δ_2
F	α_3	δ_1	α_4

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$\{T, F\}$ is closed under the G_3-functions. So α_1, α_2, α_3, and α_4 must each be classical. When only classical values are involved $\sim$ and & behave exactly as do their classical counterparts. But $\{\sim, \&\}$ is functionally complete in classical logic. Therefore, α_1, α_2, α_3, and α_4 must agree with the values of one of the two Sheffer functions $\downarrow$ and $|$ for classical logic. So the matrix for $*$ must be one of the following:

$*$	T	I	F
T	F	γ_2	F
I	γ_1	β	δ_2
F	F	δ_1	T

$*$	T	I	F
T	F	γ_2	T
I	γ_1	β	δ_2
F	T	δ_1	T

G_3 has only six singulary functions:

p	$\sim p$	$\sim\sim p$	$(p \,\&\, \sim p)$	$\sim(p \,\&\, \sim p)$	$(p \vee \sim p)$
T	F	T	F	T	T
I	F	T	F	T	I
F	T	F	F	T	T

This can be verified by observing that the result of applying any one of $\sim$, &, $\vee$, and $\supset$ to these six functions is itself one of the six functions. It follows that $\beta = F$. For otherwise $(p * p)$ would not be a G_3-function. So the matrix for $*$ must be one of the following:

$*$	T	I	F
T	F	γ_2	F
I	γ_1	F	δ_2
F	F	δ_1	T

M_1

$*$	T	I	F
T	F	γ_2	T
I	γ_1	F	δ_2
F	T	δ_1	T

M_2

In either event $\sim p = (p * p)$. Assume that M_1 is the matrix for $*$. Then no one of γ_1, γ_2, δ_1, and δ_2 can be I. For, if $\gamma_1 = I$, $(p * \sim\sim p)$ is not a G_3-function. If $\gamma_2 = I$, $(\sim\sim p * p)$ is not a G_3-function. If $\delta_1 = I$, $(\sim p * p)$ is not a G_3-function. And, if $\delta_2 = I$, $(p * \sim p)$ is not a G_3-function. Thus γ_1, γ_2, δ_1, and δ_2 are classical, and $*$ never assumes the value I. It follows that $*$ cannot be a Sheffer function for G_3. Thus M_2 must be the matrix for $*$. Then, $\gamma_1 = \gamma_2 = F$. For, if γ_1 is either T or I, $(p * \sim\sim p)$ is not a G_3-function; and if γ_2 is either T or I, $(\sim\sim p * p)$ is not a G_3-function. Neither δ_1 nor δ_2 can be F. For, if $\delta_1 = F$, $(\sim p * p)$ is not a G_3-function; and if $\delta_2 = F$, $(p * \sim p)$ is not a G_3-function. Thus the matrix for $*$ is narrowed down to:

$*$	T	I	F
T	F	F	T
I	F	F	δ_2
F	T	δ_1	T

where δ_1 and δ_2 are either T or I. δ_1 and δ_2 can't both be T. Otherwise $*$ would never assume the value I. This leaves just three alternatives: (1) $\delta_1 = T$ and $\delta_2 = I$, (2) $\delta_1 = I = \delta_2$, or (3) $\delta_1 = I$ and $\delta_2 = T$. Consider now two rows of the truth table for $(p \supset q)$:

p	q	$(p \supset q)$
T	I	I
I	I	T

It can easily be verified that no one of the remaining three alternatives is sufficient to define a function that agrees with $(p \supset q)$ in these two rows. More specifically it can be verified that the only functions definable in terms of the remaining candidates must agree with the values of one of p, q, $\sim p$, or $\sim\sim p$ in these rows. Thus $(p \supset q)$ cannot be defined in terms of $*$, and, contrary to our assumption, $*$ is not a Sheffer function for $\{\sim, \&, \vee, \supset\}$. QED

The relationship between the Gödelian systems and *IPC* is the following:

$$IPC \subset \ldots \subset G_n \subset G_{n-1} \subset \ldots \subset G_3 \subset G_2 \subset G_1$$

where G_2 is classical two-valued logic, and G_1 is the "system" having all well-formed formulas as tautologies. The only feature of *IPC* that was appealed to in the above proof was that $IPC \subset G_3$. So the same argument shows that where $n > 3$ there is no indigenous binary Sheffer function for G_n.

Even though *IPC* has no indigenous binary Sheffer function, the question of how $\{\sim, \&, \vee, \supset\}$ might be replaced by a more economical set of primitives still arises. McKinsey [4] has proved that $\{\sim, \&, \vee, \supset\}$ is not redundant, i.e., that no one of its members can be defined in terms of the others.[4] Thus economy cannot be obtained by mere deletion. Still, some economies are possible, for & and $\supset$ can be replaced by $\equiv$. The proof is as follows: $(p \supset q) \equiv [q \equiv (p \vee q)]$ and $(p \,\&\, q) \equiv [(p \equiv q) \equiv (p \vee q)]$ are both theorems of *IPC*. Thus $\{\sim, \&, \vee, \supset\}$ may be replaced by $\{\sim, \vee, \equiv\}$. What further economies are available is an open question.

NOTES

1. See [3] for more on the concept of an indigenous (vs alien) Sheffer function.
2. T, I, and F are used rather than 1, 2, and 3 in order to facilitate comparison with classical two-valued logic. G_3 is the third system in the Gödelian sequence G_n where the elements of G_n are $1, \ldots, n$ with 1 designated and the operations $\sim$, &, $\vee$, and $\supset$ are so defined that: $\sim i = n$ if $i \neq n$; $\sim i = 1$ if $i = n$; $(i \,\&\, j) = \max(i, j)$; $(i \vee j) = \min(i, j)$; $(i \supset j) = 1$ if $i \geqslant j$; and $(i \supset j) = j$ if $i < j$. See [1].
3. $(p \equiv q) = [(p \supset q) \,\&\, (q \supset p)]$.
4. Although there can be little doubt concerning the soundness of McKinsey's proof, his characterization of that proof is defective. See [2].

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Department of Philosophy
Michigan State University
East Lansing, Michigan 48824