Notre Dame Journal of Formal Logic
Volume 22, Number 4, October 1981

The n-adic First-Order Undefinability of the Geach Formula

R. E. JENNINGS, P. K. SCHOTCH, and D. K. JOHNSTON*

By adopting natural generalizations of relational frame and relational model we showed in [2] that the deontic law D, $\Box p \rightarrow \Diamond p$, is not universally first-order definable. In effect, we showed there that for $n > 2$, there is no n-adic first-order sentence β such that for every n-ary frame F, $F \models D$ iff $F \models \beta$ in the first-order sense.

The notion of n-ary frame and model employed there may be summarized as follows: $F = \langle U, R\rangle$ is an n-ary relational frame iff U is a nonempty set and R is an n-ary relation on U. Valuations on F are classical for PC formulas. For modal formulas,

$$V(\Box\alpha) = \{x \mid \forall y_1 \ldots y_{n-1}, xRy_1 \ldots y_{n-1} \Rightarrow y_1 \in V(\alpha) \text{ or } \ldots \text{ or } y_n \in V(\alpha)\}.$$

We say that α is valid on F or F is a frame for $\alpha(F \models \alpha)$ iff $V(\alpha) = U$ for every valuation V on F. That this is the correct generalization of frame and model is argued at some length in [3].

Corresponding to the modal notion of a frame is the first-order notion of a model. If F is an n-ary frame and α^* is a sentence in the first-order theory of a single n-adic predicate, then we say that F is a first-order model for $\alpha^*(F \models \alpha^*)$ iff $V(\alpha^*) = 1$ for every assignment of individual variables to objects in F. Taking these notions of frame and first-order model we arrive at the notion of n-adic first-order definability. If there is an n-adic first-order sentence α^* such that for every n-ary frame F, $F \models \alpha^*$ (in the first-order sense) iff $F \models \alpha$, we say that α is *n-adically first-order definable*. If α is n-adically first-order definable for every n, then we say that α is *universally first-order definable*.

*Partially supported by National Research Council Grants A4523 and A4085.

Received July 19, 1978; revised July 9, 1980

The result that D is not universally first-order definable extends the results for the McKinsey formula $\Box\Diamond p \rightarrow \Diamond\Box p$ of Goldblatt and is achieved by the ultraproduct technique introduced by that author in [1]. We present a further result of this nature for the converse of the McKinsey formula usually called the Geach formula G: $\Diamond\Box p \rightarrow \Box\Diamond p$. Under a restriction to a binary first-order language, G is characterized by pointwise strong convergence. Its defining first-order sentence is: $\forall x, y, z,\ xRy\ \&\ xRz \Rightarrow \exists w: yRw\ \&\ zRw$. We show here that if the restriction to a first-order language with one *binary* predicate is dropped then G is not characterized by any first-order relational property.

Theorem *If $n > 2$, then G is not n-adically first-order definable.*

Proof: For each natural number i we define the ternary frame $F_i = U_i, R_i$ as follows:

$U_i = \{x, y_1 \ldots y_{2i+1}, z\}$
xR_iy_j, y_{j+1} for each $j < 2i + 1$
xR_iy_{2i+1}, y_1
y_iR_iz, z for each $j < 2i + 1$
zR_ix, x
R_i contains no other triples.

We illustrate the first two frames in Diagrams 1 and 2.

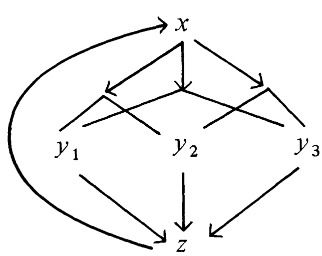

Diagram 1. F_1

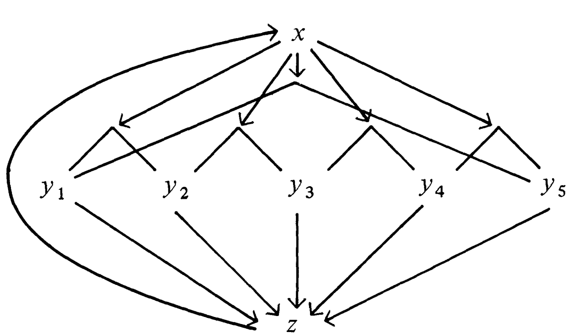

Diagram 2. F_2

It is easily seen that G cannot fail on any of the F_i's. Assume that G fails at x. Then there will be a y_j where $\Box p$ holds and a y_j where $\Box\neg p$ holds, and so $p \wedge \neg p$ will hold at z. Similarly, if G fails at some y_j, then $\Box p$ and $\Box\neg p$ must hold at z, and so $p \wedge \neg p$ will hold at x. Now suppose that G fails at z. Then $\Box p$ and $\Box\neg p$ will hold at x. But this too is impossible.

The ultraproduct F_G of the F_i's over a nonprincipal ultrafilter G will have the structure shown in Diagram 3

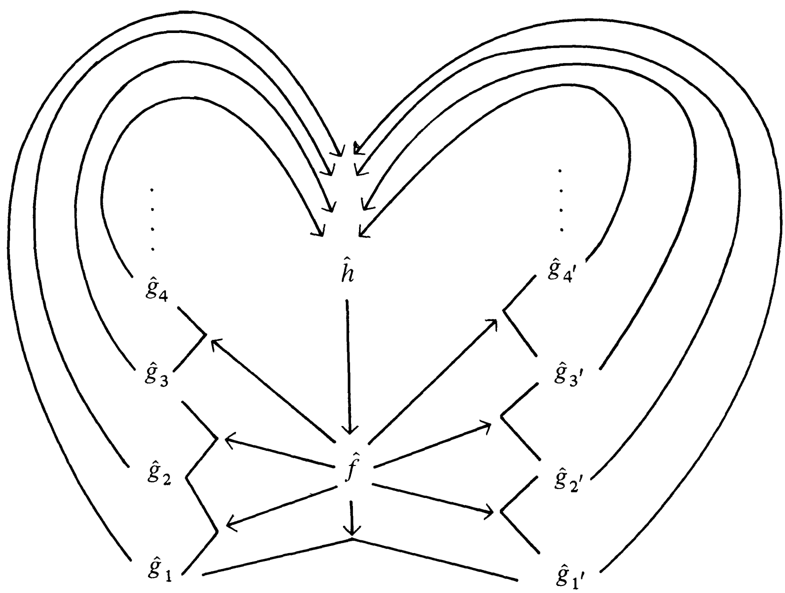

Diagram 3. F_G

where V is a valuation for which

$V(p) = \{\hat{g}_j : j \text{ odd}\}$
$V(\neg p) = \{\hat{g}_j : j \text{ even}\}$.

G will fail at $\hat{h}$. Thus the class of ternary G frames is not closed under ultraproducts. Since an $n + 1$-ary frame can be generated from an n-ary frame by defining $R'(n_1, \ldots, n_n n_n) \Leftrightarrow R(n_1, \ldots, n_n)$, we may assert that for $n > 2$, the class of n-ary G frames is not closed under ultraproducts. This proves the theorem.

Corollary *If M^m and N^m are any m-membered sequences of $\Box$'s and $\Diamond$'s then G_m: $M^m\Diamond\Box p \rightarrow N^m\Box\Diamond p$ is not n-adically first-order definable for $n > 2$.*

Proof: The adaptation of the G frame sequence to obtain this result is straightforward. We simply interpose m points between z and x in each frame. The first frame in the sequence is illustrated in Diagram 4.

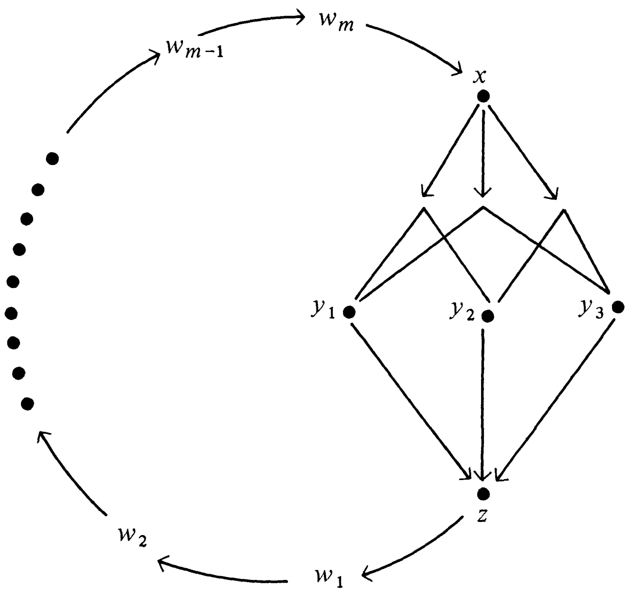

Diagram 4. F_1'

REFERENCES

[1] Goldblatt, R. I., "First order definability in modal logic," *The Journal of Symbolic Logic*, vol. 40 (1975), pp. 35-40.

[2] Jennings, R. E., P. K. Schotch, and D. K. Johnston, "Universal first order definability in modal logic," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 26 (1980), pp. 327-330.

[3] Schotch, P. K. and R. E. Jennings, "Modal logic and the theory of modal aggregation," *Philosophia*, vol. 9 (1980), pp. 265-278.

R. E. Jennings and D. K. Johnston

Department of Philosophy
Simon Fraser University
Burnaby, British Columbia
Canada

R. E. Jennings and P. K. Schotch

Department of Philosophy
Dalhousie University
Halifax, Nova Scotia B3H 3J5
Canada