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A NOTE ON THREE-VALUED MODAL LOGIC

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One of the advantages of the strong completeness result in [1] is that it allows one to extend the usual apparatus for proving completeness of two-valued modal propositional logics to three-valued logics. In the sequel[1] we carry out this programme for two logics, the modal part of which closely resembles the fundamental two-valued normal logic usually called (unfortunately) **K**.[2] The non-modal part of the logic is, in both cases, the Łukasiewicz three-valued logic (which we call $Ł_3$) as axiomatized by Wajsberg.

Of course there have been other attempts (by Łukasiewicz e.g.) to construct many-valued modal logics, but almost all of these involve taking a truth-functional view of the modal operators. In the case of a three-valued base logic, this course is almost guaranteed to result in certain theses which upon interpretation are inconsistent with any intuitive reading of the modal operators. This should not be the occasion of despair, however, since precisely the same thing happens if one tries to take a truth-functional approach to modality on a two-valued base. The way out of these difficulties which seems to have enjoyed the best reception in the latter case, has been to abandon truth-functionality and to employ "possible worlds" semantics.

Given the success of this strategy, it seems very natural to use it again to do modal logic in a three-valued setting. We must expect some differences, but these turn out to be not so substantial as might be anticipated.[3] We employ the terminology of [1], except for some trivial

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2. Unfortunately, because this terminology conflicts with that of Sobociński and others who use **K** in the names of a family of extensions of S4.

3. Especially by those who affect to find three-valued logic impossibly clumsy and lacking in the all-around "niceness" of its two-valued competitor. In this connection see [3], p. 153.

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changes which are largely stylistic. Familiarity on the part of the reader with the latter work is assumed.

Axioms The two logics considered in this study will have as axioms certain wffs drawn from the following:

[W] the Wajsberg axioms for Ł$_3$[4]
[Ł$_3$K] $\Box(p \to q) \to (\Box p \to \Box q)$
[Ł$_3$M$_2$] $\neg(\Box p \leftrightarrow \neg\Box p)$
[Ł$_3$M$_3$] $\neg\Box p \to (\Box(\neg p \to p) \to \Box p)$
[Ł$_3$S] $\overline{\neg\Box p} \to \Box\overline{\neg p}$

The logic **Ł$_3$M$_2$** has as axioms [W], [Ł$_3$K], and [Ł$_3$M$_2$]. The logic **Ł$_3$M$_3$** has as axioms [W], [Ł$_3$K], [Ł$_3$M$_3$] and [Ł$_3$S].

Rules Both **Ł$_3$M$_2$** and **Ł$_3$M$_3$** employ the same rules of inference viz. *Modus ponens* for "$\to$", and uniform substitution for propositional variables (i.e., the usual rules for **Ł$_3$**), together with:

[RRŁ$_3$] $\vdash \alpha \to \beta \Rightarrow \vdash \Box\alpha \to \Box\beta$
[RNŁ$_3$] $\vdash \alpha \Rightarrow \vdash \Box\alpha$

The concept of a wff being provable from a set of wffs and being provable *simpliciter* is taken over from [1], p. 325, the obvious changes being made in the latter concept in order to accommodate the above two modal rules.

Basic semantics By a *frame* we understand, as usual, a pair $(\mathbf{U}, R)$ where $\mathbf{U}$ is a non-empty set and $R \subseteq \mathbf{U} \times \mathbf{U}$ is a binary relation. Members of $\mathbf{U}$ are called points (although $\mathbf{U}$ is often understood intuitively as the set of possible worlds) and R is called the accessibility or relative possibility relation of the frame. By a *three-valued model*, we understand a frame together with a *valuation*. The latter is a function V: $\mathsf{Nat} \to 3^U$ which assigns to every propositional variable p_n a three-valued "set". We may think of V intuitively, as associating with every propositional variable the fuzzy atomic proposition expressed by that variable according to the model. For ease of exposition it is convenient to think of V in its equivalent form: V: $\mathsf{Nat} \times \mathbf{U} \to 3$, i.e., as a function which assigns to every variable and every point the truth-value taken on by that variable at that point.

The general concept of truth-value at a point is defined as follows (where $/\alpha/u$ denotes the truth-value taken on by α at the point u):

$$/\alpha/u = V(n, u) \text{ if } \alpha = p_n$$

$$\left.\begin{array}{l} /\alpha \to \beta/u \\ /\neg\alpha/u \end{array}\right\} \begin{array}{l} \text{as given in the tables in [1], p. 326,} \\ \text{suitably relativized to the point } u. \end{array}$$

Ł$_3$M$_2$ We begin conservatively by considering a modal logic on a three-valued base in which the modal operators take only (the) two (classical)

4. See [1], p. 325.

truth-values. There are reasons for advancing the position that modal discourse is essentially two-valued (i.e., classically two-valued) which might be thought compelling. A possible position on this matter is that, while for epistemological or metaphysical reasons we may not be in a position to say whether or not some sentence α is true, we are always in a position to adjudicate the weaker claim that α is possibly true. We take the semantics of $\text{Ł}_3\mathbf{M}_2$ to be based on the basic semantics given above, with the general concept of truth-value at a point expanded as follows:

$$/\Box\alpha/u = \begin{cases} 1 \text{ if } (\forall v)(uRv \Rightarrow /\alpha/v = 1) \\ 0 \text{ otherwise}^5 \end{cases}$$

Soundness That $\text{Ł}_3\mathbf{M}_2$ is sound with respect to the semantics given above may be verified by routine calculation.

Completeness To show that $\text{Ł}_3\mathbf{M}_2$ is strongly complete with respect to our semantics we employ the same proof technique as used in [1]. That is, we show that every syntactically consistent set of wffs is also semantically consistent. Strong completeness then follows by the argument on p. 329 of [1]. By the $\text{Ł}_3\mathbf{M}_2$ *canonical model*, we understand that model

$$\mathfrak{M}_{\text{Ł}_3\mathrm{M}_2} = \langle(\mathbf{U}_{\text{Ł}_3\mathrm{M}_2}, R_{\text{Ł}_3\mathrm{M}_2}), V_{\text{Ł}_3\mathrm{M}_2}\rangle$$

where $\mathbf{U}_{\text{Ł}_3\mathrm{M}_2}$, the canonical domain, is the set of $\text{Ł}_3\mathbf{M}_2$ maximal consistent sets (as defined in [1]) and $R_{\text{Ł}_3\mathrm{M}_2}$ is defined as follows:

$$\text{For all } u \text{ and } v \in \mathbf{U}_{\text{Ł}_3\mathrm{M}_2}\text{: } uR_{\text{Ł}_3\mathrm{M}_2}v \text{ iff for all } \alpha\text{: } u \vdash \Box\alpha \Rightarrow v \vdash \alpha$$

The canonical valuation $V_{\text{Ł}_3\mathrm{M}_2}: \mathrm{Nat} \times \mathbf{U}_{\text{Ł}_3\mathrm{M}_2} \to 3$ is defined:

$$V_{\text{Ł}_3\mathrm{M}_2}(n, u) = \begin{cases} 1 \text{ if } u \vdash p_n \\ 0 \text{ if } u \vdash \neg p_n \\ \frac{1}{2} \text{ if } \nvdash p_n \text{ and } \nvdash \neg p_n \end{cases}$$

In order that syntactical consistency imply semantical consistency, it is clearly sufficient that the following result obtain:

Fundamental theorem for $\mathfrak{M}_{\text{Ł}_3\mathrm{M}_2}$ *For all points u in the canonical domain, and all wffs* α:

$$/\alpha/u = \begin{cases} 1 \text{ if } u \vdash \alpha \\ 0 \text{ if } u \vdash \neg\alpha \\ \frac{1}{2} \text{ if } u \nvdash \alpha \text{ and } u \nvdash \neg\alpha \end{cases}$$

Proof: The proof of the fundamental theorem is by induction on the length of α, with the basis step proved by appeal to the definition of the canonical valuation. For the induction step, we need consider only the case in which α is of the form $\Box\beta$ where the result holds for β.

Case 1: Suppose that $/\Box\beta/u \neq 1$. It follows from this that $(\mathrm{E}v)(uR_{\text{Ł}_3\mathrm{M}_2}v$ & $/\beta/v \neq 1)$. By the hypothesis of induction (hereafter **HI**) $v \nvdash \beta$. By definition of $R_{\text{Ł}_3\mathrm{M}_2}$, it follows that $u \nvdash \Box\beta$. Thus $/\Box\beta/u = 1$ if $u \vdash \Box\beta$.

5. This sort of truth-condition is investigated briefly for infinite valued logic in [2].

Case 2: Suppose that $/\Box\beta/u \neq 0$. It follows that $(\forall v)(uR_{Ł_3M_2}v \Rightarrow v \vdash \beta)$. We now show that $u \vdash \neg\Box\beta \Rightarrow (\mathrm{E}v)(uR_{Ł_3M_2}v \ \&\ v \nvdash \beta)$. Let $\mathrm{N}(u)$ be $\{\gamma\colon u \vdash \Box\gamma\}$. For $u \vdash \neg\Box\beta$ it follows that $\mathrm{N}(u) \nvdash \beta$, since suppose otherwise: By [1] Lemma 1 (b) there is some $n \in \mathrm{Nat}$ such that: $\gamma_1, \ldots, \gamma_n \vdash \beta$, $\gamma_i \in \mathrm{N}(u)$ $1 \leq i \leq n$. From this and [1] Lemma 1 (q) it follows that:

$$\vdash \gamma_1 \rightarrow (\gamma_1 \rightarrow \ldots \rightarrow (\gamma_n \rightarrow (\gamma_n \rightarrow \beta)) \ldots)$$

By [RRŁ$_3$]:

$$\vdash \Box\gamma_1 \rightarrow \Box(\gamma_1 \rightarrow \ldots \rightarrow (\gamma_n \rightarrow (\gamma_n \rightarrow \beta)) \ldots)$$

By [1] Lemma 1 (d):

$$u \vdash \Box\gamma_1 \rightarrow \Box(\gamma_1 \rightarrow \ldots \rightarrow (\gamma_n \rightarrow (\gamma_n \rightarrow \beta)) \ldots)$$

By construction of $\mathrm{N}(u)$ and an obvious use of [Ł$_3$K] and [1] Lemma 1 (d) again, repeated as many times as necessary, we finally obtain: $u \vdash \Box\beta$, contrary to hypothesis. There are now two possibilities to consider. Either $\mathrm{N}(u) \vdash \neg\beta$, in which case we simply "blow up" $\mathrm{N}(u)$ into a maximal consistent set which will clearly serve as the point v required above, or neither $\mathrm{N}(u) \vdash \beta$ nor $\mathrm{N}(u) \vdash \neg\beta$. In the latter case, we simply add to $\mathrm{N}(u)$ both $\overline{\beta}$ and $\overline{\neg\beta}$. In view of [1] Lemma 1 (i), the resulting set must be syntactically consistent, and upon maximization will also serve as our v. Since $\neg(\mathrm{E}v)(uR_{Ł_3M_2}v \ \&\ v \nvdash \beta)$, it follows that $u \nvdash \neg\Box\beta$.

Case 3: We now show that $u \nvdash \neg\Box\beta \Rightarrow u \vdash \Box\beta$, so that in view of cases 1 and 2 above, $/\Box\beta/u = \frac{1}{2}$ cannot obtain. Suppose $u \nvdash \neg\Box\alpha$, it follows that $u \cup \{\Box\alpha\}$ is syntactically consistent. For if not, then by [1] Lemma 1 (t), $u \vdash \overline{\Box\alpha}$, i.e., $u \vdash \Box\alpha \rightarrow \neg\Box\alpha$. It cannot also be the case that $u \vdash \overline{\neg\Box\alpha}$, since this would imply that $u \vdash \Box\alpha \leftrightarrow \neg\Box\alpha$ and in view of [Ł$_3$M$_2$] u would be syntactically inconsistent, contrary to hypothesis. But if $u \nvdash \overline{\neg\Box\alpha}$, then $u \cup \{\neg\Box\alpha\}$ is syntactically consistent, and by definition of maximality $u \vdash \neg\Box\alpha$. Again by definition of maximality, if $u \cup \{\Box\alpha\}$ is syntactically consistent, then $u \vdash \Box\alpha$.

This completes the proof of the fundamental theorem for $\mathfrak{M}_{Ł_3M_2}$.

Ł$_3$M$_3$ We next investigate a logic in which the modal operators are allowed to take on all three of our truth-values. This kind of logic is of interest since there clearly are things to be said against the view that our intuitive notion of the concepts of possibility and necessity must always be two-valued. One difficulty in particular with our previous truth-conditions is that although our account of "$\Box$" may be acceptable, the derived truth-condition for "$\Diamond$" makes that operator too weak. Thus using:

$$/\Diamond\alpha/u = \begin{cases} 1 \text{ if } (\mathrm{E}v)(uRv \ \&\ /\alpha/v \neq 0) \\ 0 \text{ otherwise} \end{cases}$$

allows $\Diamond\alpha$ to be true at some point even though the best α can do at any related point is $\frac{1}{2}$. A genuine possibility operator, it could be argued, would not be as flabby as this. If we try to remedy the situation by employing the stronger truth-condition:

$$/\Diamond\alpha/u = \begin{cases} 1 \text{ if } (\mathrm{E}v)(uRv \ \& \ /\alpha/v = 1) \\ 0 \text{ otherwise,} \end{cases}$$

precisely the same problem comes back to us, this time with respect to the truth-condition for "$\Box$". We might choose to abandon the inter-definability of "$\Box$" and "$\Diamond$" at this point, but an alternative is to use a different truth-condition on which the above difficulty will not arise. In order to do the semantics of $\text{Ł}_3\mathbf{M}_3$, we employ the following truth-condition due to S. K. Thomason[6]:

$$/\Box\alpha/u = \underset{v:uRv}{\mathrm{Min}}[/\alpha/v]$$

or less compactly:

$$/\Box\alpha/u = \begin{cases} 1 \text{ if } (\forall v)(uRv \Rightarrow /\alpha/v = 1) \\ 0 \text{ if } (\mathrm{E}v)(uRv \ \& \ /\alpha/v = 0) \\ \frac{1}{2} \text{ if } (\mathrm{E}v)(uRv \ \& \ /\alpha/v = \frac{1}{2}) \ \& \ (\forall w)(uRw \Rightarrow /\alpha/w \neq 0) \end{cases}$$

On this account the derived truth-condition for "$\Diamond$" comes out:

$$/\Diamond\alpha/u = \underset{v:uRv}{\mathrm{Max}}[/\alpha/v]$$

Soundness The proof of soundness which is again omitted is entirely routine.

Completeness We proceed as before, the notion of the $\text{Ł}_3\mathbf{M}_3$ canonical model being taken over *mutatis mutandis* from the last section.

Case 1: This case is handled in precisely the same fashion as before.

Case 2: Suppose $/\Box\beta/u \neq 0$. It follows on the truth condition that: $(\forall v)(uR_{\text{Ł}_3\mathrm{M}_3}v \Rightarrow /\beta/v \neq 0)$, and thus by **HI** that $(\forall v)(uR_{\text{Ł}_3\mathrm{M}_3}v \Rightarrow v \nvdash \neg\beta)$. We show now that $u \vdash \neg\Box\beta \Rightarrow (\mathrm{E}v)(uR_{\text{Ł}_3\mathrm{M}_3}v \ \& \ v \vdash \neg\beta)$. Suppose that $u \vdash \neg\Box\beta$, then $\mathrm{N}(u) \cup \{\neg\beta\}$ is syntactically consistent by the following argument: If $\mathrm{N}(u) \cup \{\neg\beta\}$ is syntactically inconsistent, then $\mathrm{N}(u) \vdash \overline{\neg\beta}$ and by the argument of case 2 of the last section: $u \vdash \Box(\overline{\neg\beta})$. By [$\text{Ł}_3\mathbf{M}_3$] and [1] Lemma 1 (a) $u \vdash \neg\Box\beta \rightarrow (\Box(\overline{\neg\beta}) \rightarrow \Box\beta)$. Two uses of [1] Lemma 1 (d) give: $u \vdash \Box\beta$, which means that u is syntactically inconsistent contrary to hypothesis. Since $\mathrm{N}(u) \cup \{\neg\beta\}$ is syntactically consistent, it follows that there is some v such that $uR_{\text{Ł}_3\mathrm{M}_3}v$ and $v \vdash \neg\beta$. Thus $u \nvdash \neg\Box\beta$.

Case 3: Suppose that $u \nvdash \Box\beta$ and $u \nvdash \neg\Box\beta$. It follows that $\mathrm{N}(u) \nvdash \beta$, by the argument of case 2 of the last section. Also $\mathrm{N}(u) \cup \{\neg\beta\}$ is syntactically inconsistent, for suppose not: then $\mathrm{N}(u) \nvdash \overline{\neg\beta}$ and in view of [ŁS] $u \nvdash \overline{\neg\Box\beta}$. But this means that $u \cup \{\neg\Box\beta\}$ is syntactically consistent and by definition of maximality $u \vdash \neg\Box\beta$, contrary to hypothesis. It follows from these two facts and **HI** that: $(\mathrm{E}v)(uR_{\text{Ł}_3\mathrm{M}_3}v \ \& \ /\beta/v = \frac{1}{2})$ and $(\forall w)(uR_{\text{Ł}_3\mathrm{M}_3}w \Rightarrow /\beta/w \neq 0)$. These two imply that $/\Box\beta/u = \frac{1}{2}$.

This completes the proof of the fundamental theorem for $\mathfrak{M}_{\text{Ł}_3\mathrm{M}_3}$.

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6. See [4].

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