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A MODAL SYSTEM PROPERLY INDEPENDENT OF BOTH THE BROUWERIAN SYSTEM AND S4

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Although proper subsystems of S5, it is well-known that the Brouwerian system (hereafter referred to as simply 'B') and S4 are independent of each other. This independence, however, is of a peculiar nature: if the proper axiom of either system is appended to the axiomatic basis of the other system, a system deductively equivalent to S5 results. We might say, to coin a new phrase, that these two systems are "properly independent of each other with respect to S5." This rather unusual sense of independence might perhaps lead us to speculate as to whether there exists another system properly independent of both B and S4 with respect to S5; that is, a system such that, if its proper axiom is appended to either the axiomatic basis of B or S4, a system deductively equivalent to S5 results. That there does indeed exist such a system will be shown in section 1. In section 2, we shall examine the modal structure of this system. We shall show that it, like S4, is characterized by possessing exactly fourteen distinct modalities. Finally, in the last section, a Kripke-style semantic interpretation for this system will be offered.

1 An elegant axiomatization of the Classical Propositional Calculus (**PC**) is afforded by the following three axioms

A1 $CpCqp$
A2 $CCpCqrCCpqCpr$
A3 $CCNpNqCqp$

together with the rules of uniform substitution and detachment. Of course the formation rules and the usual definitions of the other **PC** connectives are required, but they are familiar enough for them not to be explicitly formulated here. Now if we go on further to append the following two additional axioms

A4 $CLCpqCLpLq$
A5 $CLpp$

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along with the unrestricted rule of necessitation, viz.,

R1 $\vdash \alpha \rightarrow \vdash L\alpha$,

and the usual modal definitions and formation rules, we obtain a Lemmon-style axiomatization of modal system T. Three familiar derived rules of inference of T are the following:

R2 $\vdash C\alpha\beta \rightarrow \vdash CL\alpha L\beta$
R3 $\vdash C\alpha\beta \rightarrow \vdash CM\alpha M\beta$
R4 $\vdash CF\alpha G\alpha \rightarrow \vdash CG^*\alpha F^*\alpha$, where F^* and G^* are duals respectively of F and G (*cf.* [2], p. 164).

Some theorems of T which we shall employ in the subsequent discussion are:

T1 *ENMMNpLLp*
T2 *CAMpMqMApq*
T3 *ENLMMNpMLLp*
T4 *ENLLNpMMp*
T5 *CKLpLqLKpq*
T6 *ENLpMNp*
T7 *ENMLNpLMp*
T8 *ENMLNLMpLMLMp*
T9 *ENMpLNp*
T10 *ENLMNpMLp*
T11 *ENMLNpLMp*

Now if we append

B1 *CMLpp*

as an axiom to the axiomatic basis of T, we obtain modal system B. If, on the other hand, we add

B2 *CLpLLp*

to the basis of T, modal system S4 results. Adding

B3 *CMLpLp*

to the basis of T, however, gives modal system S5. Clearly, in order to show that modal systems B and S4 are properly independent of each other with respect to S5, we need only demonstrate that B1 and B2 jointly entail B3 in the field of T. Assume B1, B2 and the field of T, then

1	*CMLpp*	B1
2	*CLpLLp*	B2
3	*CMLLpLp*	1, *p*/*Lp*
4	*CMLpMLLp*	2, R3
B3	*CMLpLp*	3, 4, Syllogism

The above result, however, is well-known. What we are primarily concerned with is finding a modal system which is properly independent of both B and S4 with respect to S5. Such a system is axiomatized by simply appending

C1 *CMCMMpLMqCMMpLMq*

to the axiomatic basis of T. I call the resulting system, modal system X. Now let us assume B1, C1 and the field of T:

1	*CMLpp*	B1
2	*CMCMMpLMqCMMpLMq*	C1
3	*CMANMMpLMqANMMpLMq*	2, Implication
4	*CMANMMNpLMqANMMNpLMq*	3, *p/Np*
5	*ENMMNpLLp*	T1
6	*CMALLpLMqALLpLMq*	4, 5, Substitution of Equivalents
7	*CAMpMqMApq*	T2
8	*CAMLLpMLMqMALLpLMq*	7, *p/LLp*; *q/LMq*
9	*CAMLLpMLMqALLpLMq*	6, 8, Syllogism
10	*CCApqrKCprCqr*	PC
11	*CCAMLLpMLMqALLpLMqKCMLLpALLpLMqCMLMqALLpLMq*	10, *p/MLLp*; *q/MLMq*; *r/ALLpLMq*
12	*KCMLLpALLpLMqCMLMqALLpLMq*	9, 11, Detachment
13	*CMLMqALLpLMq*	12, Simplification
14	*CMLMqCNLLpLMq*	13, Implication
15	*CMLMqCNLLNqLMq*	14, *p/Nq*
16	*ENLLNqMMq*	T4
17	*CMLMqCMMqLMq*	15, 16, Substitution of Equivalents
18	*CMMqCMLMqLMq*	17, Permutation
19	*CLpp*	A5
20	*CLMqMq*	19, *p/Mq*
21	*CMLMqMMq*	20, R3
22	*CMLMqCMLMqLMq*	18, 21, Syllogism
23	*CMLMpCMLMpLMp*	22, *q/p*
24	*CKMLMpMLMpLMp*	23, Importation
25	*CpKpp*	PC
26	*CMLMpKMLMpMLMp*	25, *p/MLMp*
27	*CMLMpLMp*	24, 26, Syllogism
28	*CMLpLMLp*	27, R4
29	*CLMLpLp*	1, R2
B3	*CMLpLp*	28, 29, Syllogism

Clearly both B1 and C1 inferentially entail B3 in the field of T. Appending C1 then to the axiomatic basis of B yields S5 and, conversely, adding B1 to the basis of X also gives S5. Hence, modal systems B and X are properly independent of each other with respect to S5.

Now let us assume B2, C1 and the field of T:

1	*CLpLLp*	B2
2	*CMCMMpLMqCMMpLMq*	C1
3	*CMANMMpLMqANMMpLMq*	2, Implication

4	*CMANMMNpLMqANMMNpLMq*	3, *p/Np*
5	*ENMMNpLLp*	T1
6	*CMALLpLMqALLpLMq*	4, 5, Substitution of Equivalents
7	*CAMpMqMApq*	T2
8	*CAMLLpMLMqMALLpLMq*	7, *p/LLp*; *q/LMq*
9	*CAMLLpMLMqALLpLMq*	6, 8, Syllogism
10	*CCApqrKCprCqr*	PC
11	*CCAMLLpMLMqALLpLMqKCMLLpALLpLMqCMLMqALLpLMq*	10, *p/MLLp*; *q/MLMq*; *r/ALLpLMq*
12	*KCMLLpALLpLMqCMLMqALLpLMq*	9, 11, Detachment
13	*CMLLpALLpLMq*	12, Simplification
14	*CMLLpALMqLLp*	13, Commutation
15	*CMLLpCNLMqLLp*	14, Implication
16	*CKMLLpNLMqLLp*	15, Importation
17	*CKMLLpNLMMNpLLp*	16, *q/MNp*
18	*ENLMMNpMLLp*	T3
19	*CKMLLpMLLpLLp*	17, 18, Substitution of Equivalents
20	*CpKpp*	PC
21	*CMLLpKMLLpMLLp*	20, *p/MLLp*
22	*CMLLpLLp*	19, 21, Syllogism
23	*CMMpLMMp*	22, R4
24	*CMMpMp*	1, R4
25	*CLMMpLMp*	24, R2
26	*CMMpLMp*	23, 25, Syllogism
27	*CLpp*	A5
28	*CLLpLp*	27, *p/Lp*
29	*CMpMMp*	28, R4
30	*CMpLMp*	26, 29, Syllogism
B3	*CMLpLp*	30, R4

Clearly, modal system X is also properly independent of S4 with respect to S5.

It is easily demonstrated that modal system X is a subsystem of S5. This is accomplished by showing that B3 inferentially entails C1 in the field of T:

1	*CMLpLp*	B3
2	*CCpqCCrsCKprKqs*	PC
3	*CCMLpLpCCMqLMqCKMLpMqKLpLMq*	2, *p/MLp*; *q/Lp*; *r/Mq*; *s/LMq*
4	*CCMqLMqCKMLpMqKLpLMq*	1, 3, Detachment
5	*CMpLMp*	1, R4
6	*CMqLMq*	5, *p/q*
7	*CKMLpMqKLpLMq*	4, 6, Detachment
8	*CKLpLqLKpq*	T5
9	*CKLpLMqLKpMq*	8, *q/Mq*
10	*CKMLpMqLKpMq*	7, 9, Syllogism

11	*CNLKpMqNKMLpMq*	10, Transposition
12	*ENLpMNp*	T6
13	*ENLKpMqMNKpMq*	12, *p/KpMq*
14	*CMNKpMqNKMLpMq*	11, 13, Substitution of Equivalents
15	*CMANpNMqANMLpNMq*	14, DeMorgan
16	*CMANMqNpANMqNMLp*	15, Commutation
17	*CMCMqNpCMqNMLp*	16, Implication
18	*CMCMMpNNLMqCMMpNMLNLMq*	17, *q/Mp*; *p/NLMq*
19	*CMCMMpLMqCMMpNMLNLMq*	18, Double Negation
20	*ENMLNLMqLMLMq*	T8
21	*CMCMMpLMqCMMpLMLMq*	19, 20, Substitution of Equivalents
22	*CMMqMLMq*	6, R3
23	*CLpp*	A5
24	*CpMp*	23, R4
25	*CMqMMq*	24, *p/Mq*
26	*CMqMLMq*	22, 25, Syllogism
27	*CLMqLMLMq*	26, R2
28	*CLMLMqMLMq*	23, *p/MLMq*
29	*CMLqLMLq*	5, *p/Lq*
30	*CMLMqLMq*	29, R4
31	*CLMLMqLMq*	28, 30, Syllogism
32	*ELMLMqLMq*	27, 31, Definition *E*
C1	*CMCMMpLMqCMMpLMq*	21, 32, Substitution of Equivalents

In order to prove that modal system X is not only a subsystem of S5 but also a proper subsystem of S5, we employ the following matrix:

$$\mathfrak{P}1 \quad L(*12345678) = 18887888$$

This matrix verifies the entire axiomatic basis of modal system X, but rejects B3 for *p*/5: *CML*5*L*5 = *CM*77 = *C*17 = 7. (We, of course, assume that the reader is familiar with the usual eight-valued Boolean matrices for *C* and *N*.) Note, incidentally, as we would expect, this matrix also falsifies B2 for *p*/5: *CL*5*LL*5 = *C*7*L*7 = *C*78 = 2; and B1 for *p*/5: *CML*55 = *CM*75 = *C*15 = 5. Clearly then, modal system X is a proper extension of T, properly independent of both B and S4 with respect to S5, and a proper subsystem of S5.

Let us now derive some interesting theorems of X:

D1	*CMCMMpLMqCMMpLMq*	C1
D2	*CMANMMpLMqANMMpLMq*	D1, Implication
D3	*CMANMMNpLMqANMMNpLMq*	D2, *p/Np*
D4	*ENMMNpLLp*	T1
D5	*CMALLpLMqALLpLMq*	D3, D4, Substitution of Equivalents
D6	*CNALLpLMqNMALLpLMq*	D5, Transposition
D7	*ENMpLNp*	T9
D8	*ENMALLpLMqLNALLpLMq*	D7, *p/ALLpLMq*
D9	*CNALLpLMqLNALLpLMq*	D6, D8, Substitution of Equivalents

D10	*CKNLLpNLMqLKNLLpNLMq*	D9, De Morgan
D11	*CKNLLNpNLMNqLKNLLNpNLMNq*	D10, *p/Np*; *q/Nq*
D12	*ENLLNpMMp*	T4
D13	*CKMMpNLMNqLKMMpNLMNq*	D11, D12, Substitution of Equivalents
D14	*ENLMNqMLq*	T10
D15	*CKMMpMLqLKMMpMLq*	D13, D14, Substitution of Equivalents
D16	*CAMpMqMApq*	T2
D17	*CAMLLpMLMqMALLpLMq*	D16, *p/LLp*; *q/LMq*
D18	*CAMLLpMLMqALLpLMq*	D5, D17, Syllogism
D19	*CCApqrKCprCqr*	PC
D20	*CCAMLLpMLMqALLpLMqKCMLLpALLpLMqCMLMqALLpLMq*	D19, *p/MLLp*; *q/MLMq*; *r/ALLpLMq*
D21	*KCMLLpALLpLMqCMLMqALLpLMq*	D18, D20, Detachment
D22	*CMLLpALLpLMq*	D21, Simplification
D23	*CMLLpALMqLLp*	D22, Commutation
D24	*CMLLpCNLMqLLp*	D23, Implication
D25	*CKMLLpNLMqLLp*	D24, Importation
D26	*CKMLLpNLMMNpLLp*	D25, *q/MNp*
D27	*ENLMMNpMLLp*	T3
D28	*CKMLLpMLLpLLp*	D26, D27, Substitution of Equivalents
D29	*CpKpp*	PC
D30	*CMLLpKMLLpMLLp*	D29, *p/MLLp*
D31	*CMLLpLLp*	D28, D30, Syllogism
D32	*CMMpLMMp*	D31, R4

Being independent of both B and S4, we would naturally expect that there are formulae provable in X which are neither theses of B nor S4. Two such interesting formulae are D31 and D32.

D33	*CMLMqALLpLMq*	D21, Simplification
D34	*CMLMqCNLLpLMq*	D33, Implication
D35	*CMLMpCNLLNpLMp*	D34, *q/p*; *p/Np*
D36	*CMLMpCMMpLMp*	D12, D35, Substitution of Equivalents
D37	*CMMpCMLMpLMp*	D36, Permutation
D38	*CLpp*	A5
D39	*CLMpMp*	D38, *p/Mp*
D40	*CMLMpMMp*	D39, R3
D41	*CMLMpCMLMpLMp*	D37, D40, Syllogism
D42	*CMLMpKMLMpMLMp*	D29, *p/MLMp*
D43	*CKMLMpMLMpLMp*	D41, Importation
D44	*CMLMpLMp*	D42, D43, Syllogism
D45	*CMLpLMLp*	D44, R4

D44 and D45 are also theses of X provable in neither B nor S4.

D46	*CLLLpLLp*	D38, *p/LLp*
D47	*CpMp*	D38, R4
D48	*CMpMMp*	D47, *p/Mp*
D49	*CLpMLp*	D47, *p/Lp*

D50	*CMMpMMMp*	D47, *p/MMp*
D51	*CLLMpLMp*	D38, *p/LMp*
D52	*CLMpMLMp*	D47, *p/LMp*
D53	*CLLpMLLp*	D47, *p/LLp*
D54	*CLMMpMMp*	D38, *p/MMp*
D55	*CLMLpMLp*	D38, *p/MLp*
D56	*CLMLLpLLLp*	D31, R2
D57	*CMLLpLMLLp*	D45, *p/Lp*
D58	*CMLLpLLLp*	D56, D57, Syllogism
D59	*CLLpLLLp*	D53, D58, Syllogism
D60	*CMMMpMMp*	D59, R4

D59 and D60 are both provable in S4, but not in B.

D61	*CMLpMMLp*	D51, R4
D62	*CLMLMpLLMp*	D44, R2
D63	*CMLMpLMLMp*	D45, *p/Mp*
D64	*CMLMpLLMp*	D62, D63, Syllogism
D65	*CLMpLLMp*	D52, D64, Syllogism
D66	*CMMLpMLp*	D65, R4

D65 and D66 are also theses of S4 not provable in B.

D67	*CLLpLp*	D38, *p/Lp*
D68	*CMLpMp*	D38, R3
D69	*CLMLpLMp*	D68, R2
D70	*CMLpLMp*	D45, D69, Syllogism
D71	*CMLLpLp*	D31, D67, Syllogism

Finally, notice that D70 and D71 are provable in B, but not in S4.

There are several alternative ways for axiomatizing modal system X. We have already proved that

D5 *CMALLpLMqALLpLMq*

and

D15 *CKMMpMLqLKMMpMLq*

are theses of X. Actually either one of these two formulae may replace C1 in axiomatizing system X. In order to prove this, we need only show that D5 and D15 each entail C1 in the field of T. First, let us assume D5 and the field of T:

1	*CMALLpLMqALLpLMq*	D5
2	*CMCNLLpLMqCNLLpLMq*	1, Implication
3	*CMCNLLNpLMqCNLLNpLMq*	2, *p/Np*
4	*ENLLNpMMp*	T4
C1	*CMCMMpLMqCMMpLMq*	3, 4, Substitution of Equivalents

Now in order to show that D15 may also replace C1 in axiomatizing X, it will suffice to prove that D15 inferentially entails D5 (and hence C1) in the field of T:

1	*CKMMpMLqLKMMpMLq*	D15
2	*CNLKMMpMLqNKMMpMLq*	1, Transposition
3	*ENLpMNp*	T6
4	*ENLKMMpMLqMNKMMpMLq*	3, *p/KMMpMLq*
5	*CMNKMMpMLqNKMMpMLq*	2, 4, Substitution of Equivalents
6	*CMANMMpNMLqANMMpNMLq*	5, De Morgan
7	*CMANMMNpNMLNqANMMNpNMLNq*	6, *p/Np*; *q/Nq*
8	*ENMMNpLLp*	T1
9	*CMALLpNMLNqALLpNMLNq*	7, 8, Substitution of Equivalents
10	*ENMLNqLMq*	T11
D5	*CMALLpLMqALLpLMq*	9, 10, Substitution of Equivalents

Still another way of axiomatizing system X is by simply appending both

D32 *CMMpLMMp*

and

D45 *CMLpLMLp*

to the axiomatic basis of T. This is easily demonstrated by merely proving that both D32 and D45 inferentially entail D15 in the field of T:

1	*CMMqLMMq*	D32
2	*CMLpLMLp*	D45
3	*CCpqCCrsCKprKqs*	PC
4	*CCMLpLMLpCCMMqLMMqCKMLpMMqKLMLpLMMq*	3, *p/MLp*; *q/LMLp*; *r/MMq*; *s/LMMq*
5	*CCMMqLMMqCKMLpMMqKLMLpLMMq*	2, 4, Detachment
6	*CKMLpMMqKLMLpLMMq*	1, 5, Detachment
7	*CKLpLqLKpq*	T5
8	*CKLMLpLMMqLKMLpMMq*	7, *p/MLp*; *q/MMq*
9	*CKMLpMMqLKMLpMMq*	6, 8, Syllogism
10	*CKMLqMMpLKMLqMMp*	9, *p/q*; *q/p*
D15	*CKMMpMLqLKMMpMLq*	10, Commutation

2 Modal system X has fourteen distinct irreducible modalities; they are the following and their negations:

(a) α
(b) $L\alpha$
(c) $M\alpha$
(d) $LL\alpha$
(e) $MM\alpha$
(f) $ML\alpha$
(g) $LM\alpha$

The entailment relations which hold among these modalities are exhibited by the following diagram:

$$\begin{tikzcd}
L L\alpha \arrow[r] \arrow[ddd] & L\alpha \arrow[drr] & & \\
 & & ML\alpha \arrow[ul] \arrow[d] & \alpha \arrow[ddll] \\
 & & LM\alpha \arrow[dl] & \\
MM\alpha & M\alpha \arrow[l] & &
\end{tikzcd}$$

That these entailment relations among the modalities are as summarized in the above diagram are justified by the considerations that D38, D39, D47, D48, D49, D67, and D70 are all theses of X. An analogous diagram for the negative cases can be obtained by simply negating all of the formulae and reversing the direction of the arrows.

Before showing that there are no more than fourteen distinct modalities in X, we first take notice of some of the reduction laws provable in X:

D72	*ELMMpMMp*	D32, D54, Definition *E*
D73	*EMLLpLLp*	D31, D53, Definition *E*
D74	*ELLLpLLp*	D46, D59, Definition *E*
D75	*EMMMpMMp*	D50, D60, Definition *E*
D76	*ELMLpMLp*	D45, D55, Definition *E*
D77	*EMLMpLMp*	D44, D52, Definition *E*
D78	*ELLMpLMp*	D51, D65, Definition *E*
D79	*EMMLpMLp*	D61, D66, Definition *E*

We are now prepared to proceed with the proof.

If we add an *L* to (a) we obtain a modality equivalent to (b); adding an *M* to (a) gives a modality equivalent to (c). If we add an *L* to (b), a modality equivalent to (d) results; adding an *M* to (b) gives a modality equivalent to (f). If we add an *L* to (c), we obtain a modality equivalent to (g); adding an *M* to (c) results in a modality equivalent to (e). If we add an *L* to (d), then, in view of D74, we obtain a modality equivalent to (d) itself; adding an *M* to (d) again results in a modality equivalent to (d) itself because of D73. D72 assures us that adding an *L* to (e) results in a modality equivalent to (e) itself; if instead we add an *M* to (e), we again obtain a modality equivalent to (e) itself because of D75. Adding an *L* to (f), because of D76, results in a modality equivalent to (f) itself; adding an *M* to (f) still gives rise to a modality equivalent to (f) itself because of D79. Adding an *L* to (g) results in a modality equivalent to (g) itself because of D78; adding an *M*, on the other hand, still results in a modality equivalent to (g) itself because of D77.

Clearly the negative cases can be dealt with analogously; consequently, there are at most fourteen distinct modalities in X. Note, incidentally, that the above proof also entials that every iterated modality in X is reducible to an iterated modality containing no more than two modal operators; more specifically, to the two innermost modal operators.

In order to demonstrate that there are no fewer than fourteen distinct modalities in X, we will make use of matrix $\mathfrak{P}1$ of section 1.

(1) α fails to entail $L\alpha$ and $LL\alpha$ for $\alpha/2$, 3, 4, 5, 6, and 7; $ML\alpha$ for $\alpha/2$, 3, 4, 6, and 7; $LM\alpha$ for $\alpha/4$.
(2) $L\alpha$ fails to entail $LL\alpha$ for $\alpha/5$.
(3) $ML\alpha$ fails to entail α, $L\alpha$, and $LL\alpha$ for $\alpha/5$.
(4) $LM\alpha$ fails to entail α, $ML\alpha$, $L\alpha$, and $LL\alpha$ for $\alpha/2$, 3, 5, 6, and 7.
(5) $M\alpha$ fails to entail α, $L\alpha$, and $LL\alpha$ for $\alpha/2$, 3, 4, 5, 6, and 7; $LM\alpha$ for $\alpha/4$; $ML\alpha$ for $\alpha/2$, 3, 4, 6, and 7.
(6) $MM\alpha$ fails to entail $M\alpha$ and $LM\alpha$ for $\alpha/4$; $ML\alpha$ for $\alpha/2$, 3, 4, 6, and 7; α, $L\alpha$, and $LL\alpha$ for $\alpha/2$, 3, 4, 5, 6, and 7.

Again it is obvious that the negative cases can be dealt with in the same fashion; hence, we also conclude that there are no fewer than fourteen distinct modalities in X.

Modal system X then is similar to S4 in possessing exactly fourteen distinct modalities; however, four of the modalities are different. In S4, $LL\alpha$, $MM\alpha$, and their negations are not irreducible whereas $LML\alpha$, $MLM\alpha$, and their negations are. In X, on the other hand, the latter are reducible whereas the former are not.

3 In offering a semantic interpretation for modal system X, we shall employ the terminology, techniques, and lemmata of Hughes and Cresswell in [1]. Hughes and Cresswell define a semantic model for T as an ordered triple $\langle W, R, V\rangle$ where W is a set of objects (worlds), R is a reflexive relation defined over the members of W, and V is a value-assignment satisfying the conditions specified in [1], p. 73.

In constructing models for modal systems properly containing T, it quite often proves fruitful to impose additional requirements on the accessibility relation in a T-model. Hence, for example, a model for S4 results by imposing the additional requirement of transitivity, for B the additional requirement of symmetry, and for S5 both transitivity and symmetry. In constructing a model for X, however, we shall not proceed in this fashion. Rather than impose an additional requirement on the accessibility relation, we shall impose a stipulation upon the set W in a T-model. This stipulation will take the form of what I shall call, for the lack of a more imaginative phrase, the "iterated modality requirement." This requirement stipulates that if an iterated modality is true (or false) in any world in the model, then it is true (or false) in every world in the model.

More formally then we define an X-model as an ordered triple $\langle W, R, V\rangle$ where W is a set of objects (worlds) possessing the iterated modality requirement, R is a reflexive relation holding over the members of W, and V is a value-assignment satisfying the conditions specified in [1], p. 73. We now say that a wff, α, is X-logically true iff in every X-model $\langle W, R, V\rangle$ and for every $w_i \in W$, $V(\alpha, w_i) = 1$.

In section 1, we proved that modal system X may alternatively be axiomatized by appending both

D32 *CMMpLMMp*

and

D45 *CMLpLMLp*

to the axiomatic basis of T. Thus, in order to prove the soundness theorem for X, we need only show that both D32 and D45 are X-logically true. Let us begin with D32. Assume for the sake of reductio that D32 is not X-logically true; i.e., that $V(CMMpLMMp, w_i) = 0$. Clearly it follows that both

1 $$V(MMp, w_i) = 1$$

and

2 $$V(LMMp, w_i) = 0.$$

From 2 it follows that

3 $$V(MMp, w_j) = 0.$$

Hence, in view of the iterated modality requirement, it follows from 1 that

4 $$V(MMp, w_j) = 1$$

which contradicts 3. Consequently, $V(CMMpLMMp, w_i) = 1$.

Now let us consider D45. Assume for the sake of reductio that $V(CMLpLMLp, w_i) = 0$. Obviously we have

1 $$V(MLp, w_i) = 1$$

and

2 $$V(LMLp, w_i) = 0.$$

Thus it follows from 2 that

3 $$V(MLp, w_j) = 0.$$

But because of the iterated modality requirement it follows from 1 that

4 $$V(MLp, w_j) = 1$$

which is, of course, inconsistent with 3. Therefore, $V(CMLpLMLp, w_i) = 1$.

In order to prove the completeness theorem for X, we must show that the iterated modality requirement holds among maximal consistent sets. Let Γ be a whole system of such sets and let every $\Gamma_i \in \Gamma$ be maximal consistent with respect to modal system X. Let β also be any wff which is an iterated modality. Clearly what we must show is that if there exists a $\Gamma_j \in \Gamma$ such that $\beta \in \Gamma_j$, then β is in every $\Gamma_i \in \Gamma$. But Γ_j may possess either one of two characteristics; it may be such that (a) it has subordinates or subordinates$_*$ to it (*cf.* [1], pp. 157 and 158 for definitions of 'subordinate' and 'subordinate$_*$') or (b) it is itself a subordinate or subordinate$_*$ of any Γ_i. Let us begin with (a) first.

(a) Clearly what we must show here is that if β is in Γ_j, then β is not only in every subordinate of Γ_j, but also in every subordinate$_*$ of Γ_j. Let Γ_k be a subordinate of Γ_j and Γ_l a subordinate of Γ_k. More specifically then, we must show that if $\beta \epsilon \Gamma_j$, then $\beta \epsilon \Gamma_k$ and $\beta \epsilon \Gamma_l$. Now in section 2 we proved that every iterated modality in X is reducible to an iterated modality containing no more than two modal operators. But this means that every iterated modality is equivalent to any one of *LL*, *MM*, *ML*, or *LM* since these are the only irreducible iterated modalities in X. Consequently, if β is an iterated modality, it must be equivalent to any one of the following: $LL\gamma$, $MM\gamma$, $ML\gamma$, or $LM\gamma$. Now in order to prove (a) it will be required that we demonstrate that

(i) if $LL\gamma \epsilon \Gamma_j$, then $LL\gamma \epsilon \Gamma_k$ and $LL\gamma \epsilon \Gamma_l$;
(ii) if $MM\gamma \epsilon \Gamma_j$, then $MM\gamma \epsilon \Gamma_k$ and $MM\gamma \epsilon \Gamma_l$;
(iii) if $ML\gamma \epsilon \Gamma_j$, then $ML\gamma \epsilon \Gamma_k$ and $ML\gamma \epsilon \Gamma_l$;
(iv) if $LM\gamma \epsilon \Gamma_j$, then $LM\gamma \epsilon \Gamma_k$ and $LM\gamma \epsilon \Gamma_l$.

At this point we remind the reader that the lemmata employed are taken from Hughes and Cresswell in [1], pp. 152-154.

(i) If $LL\gamma \epsilon \Gamma_j$, then since $CLL\gamma LLL\gamma$ is a thesis of X (D59), we have $CLL\gamma LLL\gamma \epsilon \Gamma_j$ and so (by Lemma 3) $LLL\gamma \epsilon \Gamma_j$. Thus (by construction of Γ_k) $LL\gamma \epsilon \Gamma_k$. But $CLL\gamma LLL\gamma \epsilon \Gamma_k$ also, hence (again by Lemma 3) $LLL\gamma \epsilon \Gamma_k$ and so $LL\gamma \epsilon \Gamma_l$ (by construction of Γ_l). Now by induction on subordination, the result holds for any subordinate$_*$ of Γ_j.

(ii) If $MM\gamma \epsilon \Gamma_j$, then since $CMM\gamma LMM\gamma$ is a thesis of X (D32), we have $CMM\gamma LMM\gamma \epsilon \Gamma_j$ and so (by Lemma 3) $LMM\gamma \epsilon \Gamma_j$. Thus (by construction of Γ_k) $MM\gamma \epsilon \Gamma_k$. But $CMM\gamma LMM\gamma \epsilon \Gamma_k$ also, hence (again by Lemma 3) $LMM\gamma \epsilon \Gamma_k$ and so $MM\gamma \epsilon \Gamma_l$ (by construction of Γ_l). Now by induction on subordination, the result holds for any subordinate$_*$ of Γ_j.

Quite obviously steps (iii) and (iv) will proceed similarly using

D45 $CML\gamma LML\gamma$

and

D65 $CLM\gamma LLM\gamma$

respectively. Consequently, we leave proof of these steps to the reader.

(b) Taking Γ_j itself to be either a subordinate or a subordinate$_*$, we proceed as follows: let Γ_j be either Γ_m or Γ_n; also let Γ_m be subordinate to Γ_i and Γ_n subordinate to Γ_m. Where β is again any iterated modality of X, what we have to show is that if either $\beta \epsilon \Gamma_m$ or $\beta \epsilon \Gamma_n$, then $\beta \epsilon \Gamma_i$. We prove this by showing that if $\beta \notin \Gamma_i$, then both $\beta \notin \Gamma_m$ and $\beta \notin \Gamma_n$. Now for the same reason given above, β is of any of the four forms: $LL\gamma$, $MM\gamma$, $ML\gamma$, or $LM\gamma$. Hence what we now must show is

(i) if $LL\gamma \notin \Gamma_i$, then both $LL\gamma \notin \Gamma_m$ and $LL\gamma \notin \Gamma_n$;
(ii) if $MM\gamma \notin \Gamma_i$, then both $MM\gamma \notin \Gamma_m$ and $MM\gamma \notin \Gamma_n$;
(iii) if $ML\gamma \notin \Gamma_i$, then both $ML\gamma \notin \Gamma_m$ and $ML\gamma \notin \Gamma_n$;
(iv) if $LM\gamma \notin \Gamma_i$, then both $LM\gamma \notin \Gamma_m$ and $LM\gamma \notin \Gamma_n$.

(i) Suppose that $LL\gamma \notin \Gamma_i$. Then (by Lemma 2) $NLL\gamma \in \Gamma_i$, and hence, since $CNLL\gamma LNLL\gamma$ is a thesis of X (from D31 and transposition), we have (by Lemma 3) $LNLL\gamma \in \Gamma_i$. Thus (by construction of Γ_m) it follows that $NLL\gamma \in \Gamma_m$ and so (by Lemma 1) $LL\gamma \notin \Gamma_m$. But again because $CNLL\gamma LNLL\gamma$ is a thesis of X, we have $CNLL\gamma LNLL\gamma \in \Gamma_m$ and so (by Lemma 3) $LNLL\gamma \in \Gamma_m$. Hence (by construction of Γ_n) we have $NLL\gamma \in \Gamma_n$ and so $LL\gamma \notin \Gamma_n$ (by Lemma 1).

(ii) Assume that $MM\gamma \in \Gamma_i$. Then (by Lemma 2) $NMM\gamma \in \Gamma_i$, and hence, since $CNMM\gamma LNMM\gamma$ is a thesis of X (from D60 and transposition), we have (by Lemma 3) $LNMM\gamma \in \Gamma_i$. Now (by construction of Γ_m) we have $NMM\gamma \in \Gamma_m$ and so (by Lemma 1) $MM\gamma \notin \Gamma_m$. But again because $CNMM\gamma LNMM\gamma$ is a thesis of X, we have $CNMM\gamma LNMM\gamma \in \Gamma_m$ and, consequently, $LNMM\gamma \in \Gamma_m$ (by Lemma 3). Thus (by construction of Γ_n) we have $NMM\gamma \in \Gamma_n$ and so (by Lemma 1) $MM\gamma \notin \Gamma_n$.

Quite obviously steps (iii) and (iv) will proceed similarly using

D80 $CNML\gamma LNML\gamma$ (from D66 and transposition)

and

D81 $CNLM\gamma LNLM\gamma$ (from D44 and transposition)

respectively. Consequently, we consider the completeness theorem proved.

4 Before concluding this paper, we raise two open questions. First, do there exist other modal systems which are properly independent of both B and S4 with respect to S5? One way of answering this question affirmatively would be to determine that there are systems properly between X and S5; that is, that there exist extensions of X properly contained in S5. I must confess that I have been unable to determine this. In any event, it is clear that there do not exist non-Lewis extensions of X in the sense that there are non-Lewis extensions of S4; at least none which are axiomatized by appending

K1 $CLMpMLp$

to the axiomatic basis of X or any of its Lewis extensions (if there are any). To show this, assume **K1** and the field of X:

1	$CLMpMLp$	**K1**
2	$CMLLpLLp$	D31
3	$CMLpLMLp$	D45
4	$CLpp$	A5
5	$CLLpLp$	4, p/Lp
6	$CLMLpMLLp$	1, p/Lp

7	*CLMLpLLp*	2, 6, Syllogism
8	*CLMLpLp*	5, 7, Syllogism
9	*CMLpLp*	3, 8, Syllogism
10	*CMpLMp*	9, R4
11	*CLMpLp*	1, 9, Syllogism
12	*CMpLp*	10, 11, Syllogism
13	*CpMp*	4, R4
14	*CpLp*	12, 13, Syllogism

Clearly, appending **K1** as an axiom to the basis of X collapses it into the Classical Propositional Calculus.

Finally, the next question I would like to raise is this: does there exist a system which is properly independent of B, S4, and X with respect to S5?

REFERENCES

[1] Hughes, G. E., and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen, London (1968).

[2] Zeman, J. J., *Modal Logic: The Lewis-Modal Systems*, Oxford University Press, London (1973).

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