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INVESTIGATIONS INTO THE SENTENTIAL CALCULUS WITH IDENTITY

STEPHEN L. BLOOM and ROMAN SUSZKO

The sentential calculus with identity (**SCI**) is obtained from the classical sentential calculus by adding a new "identity connective" $\equiv$ and axioms which say "$p \equiv q$" means "p is identical to q". The second author was led to a study of this calculus by a desire to formalize part of the Ontology of Wittgenstein's *Tractatus* (see [7], [8]).[1] Aside from this somewhat uncommon beginning, we think that there are independent reasons for studying the **SCI**. Firstly, it seems to be as general as a sentential logic can get: both classical and modal theories may be developed in it and (by weakening an axiom) intuitionist theories as well. Furthermore, the study of its interpretations leads to interesting mathematical problems, (e.g. concerning topological Boolean algebras) and sheds light on why the classical sentential calculus is so well-behaved.

Some people, upon discovering that the identity connective was not truth-functional, have thought that **SCI** is an *intensional* logic. We emphatically deny this. The essence of intensionality is that the rule "equals may be replaced by equals" fails. However, this rule *does* hold in the **SCI** (see the remarks following **1.3**).

The paper is divided into four sections. The first is a collection of most of the basic definitions and theorems. The second and third sections discuss the questions of decidability and adequacy. The last section presents a particular theory built in the logic of the **SCI**. We have omitted most proofs in **1** to keep the size of the paper within reasonable bounds.

1. *Definitions and Elementary Results*. The *formulas* Fm of a language $\mathfrak{L}$ of the sentential calculus with identity are generated in the usual way from an infinite set VAR of *sentential* variables by the standard connectives $\neg$ (negation) and $\rightarrow$ (material implication) as well as the binary *identity connective* $\equiv$. Considered as an abstract algebra, $\mathfrak{L} = \langle Fm, \neg, \rightarrow, \equiv \rangle$ is free in

1. The relationship between the **SCI** and papers on the identity connective by other authors (M. J. Cresswell, H. Greniewski, A. N. Prior) is discussed in [8].

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the class of all algebras similar to $\mathfrak{L}$; i.e. those algebras $\mathfrak{A} = \langle A, \dot{\neg}, \dot{\rightarrow}, \dot{\equiv}\rangle$, where A is a set, $\dot{\neg}$ a unary function from A into A, and both $\dot{\rightarrow}$ and $\dot{\equiv}$ are binary functions from $A \times A$ into A. Any algebra similar to $\mathfrak{L}$ is called an SCI-*algebra*, and $\mathfrak{L}$ is called an SCI-*language*. An SCI-language (considered as an algebra) is determined up to isomorphism by the cardinality of its set of sentential variables. The other truth-functional connectives $\vee$ (disjunction), $\wedge$ (conjunction), $\leftrightarrow$ (material equivalence) are to be construed as the usual abbreviations.[2]

Throughout this paper the letters φ, ψ, and θ (sometimes with subscripts) will be used only to denote formulas of an SCI-language $\mathfrak{L}$; the letters Φ and Γ will always denote sets of formulas; "p" (sometimes with subscripts) will denote a variable. (Of course, any variable is at the same time a formula).

Any function F from the power set of Fm into itself having the properties C1, C2 and C3 will be called a *consequence operation* on $\mathfrak{L}$. (See [9] and [3]).

C1. $\Phi \subseteq F(\Phi)$, all $\Phi \subseteq Fm$.
C2. If $\Phi \subseteq \Gamma$, $F(\Phi) \subseteq F(\Gamma)$; all Φ, $\Gamma \subseteq Fm$.
C3. $F(F(\Phi)) = F(\Phi)$, all $\Phi \subseteq Fm$.

We shall be concerned with two kinds of consequence operations on $\mathfrak{L}$: a "syntactical" one (Cn) defined from axioms and a rule of inference, and a semantical one, C_M.

The logical axioms for $\mathfrak{L}$ are defined from the two sets of schema *TFA* (truth-functional axioms) and *IDA* (identity axioms) below. The axioms in *TFA* are sufficient to derive (using just modus ponens) all truth-functional tautologies. The axioms *IDA* say that $\equiv$ is a congruence on $\mathfrak{L}$ at least as strong as material equivalence.

1.1 *Definition*. *TFA* is the set of all formulas of $\mathfrak{L}$ having the form (a), (b), (c_1) or (c_2) below; *IDA* consists of those formulas of the form (d) - (h) below.

(a) $\varphi \rightarrow (\psi \rightarrow \varphi)$
(b) $(\varphi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \theta))$
(c_1) $\neg\varphi \rightarrow (\varphi \rightarrow \psi)$
(c_2) $(\varphi \rightarrow \psi) \rightarrow ((\neg\varphi \rightarrow \psi) \rightarrow \psi)$
(d) $\varphi \equiv \varphi$
(e) $\varphi \equiv \psi \rightarrow \neg\varphi \equiv \neg\psi$
(f) $\varphi_1 \equiv \psi_1 \rightarrow (\varphi_2 \equiv \psi_2 \rightarrow (\varphi_1 \rightarrow \varphi_2) \equiv (\psi_1 \rightarrow \psi_2))$
(g) $\varphi_1 \equiv \psi_1 \rightarrow (\varphi_2 \equiv \psi_2 \rightarrow (\varphi_1 \equiv \varphi_2) \equiv (\psi_1 \equiv \psi_2))$
(h) $\varphi \equiv \psi \rightarrow (\varphi \rightarrow \psi)$.

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2. It makes a difference which truth functional connectives are taken as primitive; see footnote 6. We have chosen the two $\neg$ and $\rightarrow$ for ease of exposition only. The reader will be able to see how to modify all of our subsequent definitions and theorems to conform with any other choice of the primitive truth-functional connectives.

The *only* rule of inference is modus ponens (from φ and $\varphi \to \psi$ infer ψ). The consequence operation Cn is defined via the above axioms and rule as follows.

1.2 *Definition.* $\mathrm{Cn}(\Phi)$ is the smallest set of formulas closed under the rule modus ponens, which contains *TFA*, *IDA*, and the set Φ.

1.3 Proposition. Cn *has the properties* C1-C3. *Furthermore,*

(a) $\varphi \in \mathrm{Cn}(\Phi \cup \{\psi\}) \Longleftrightarrow \psi \to \varphi \in \mathrm{Cn}(\Phi)$.
(b) $\varphi \in \mathrm{Cn}(\Phi) \Longleftrightarrow \varphi \in \mathrm{Cn}(\Gamma)$, *where* Γ *is some finite subset of* Φ.[3]

The members of $\mathrm{Cn}(\emptyset)$ ($\emptyset$ is the empty set) are called the *logical theorems*. It may be easily checked that all formulas of the form

$$\text{(RL)} \qquad (\varphi \equiv \psi) \to (\mathrm{G}(\varphi) \equiv \mathrm{G}(\psi))$$

are logical theorems, where $\mathrm{G}(\varphi)$ is any formula containing φ as a part, and $\mathrm{G}(\psi)$ is the result of replacing some occurrences of φ by ψ in $\mathrm{G}(\varphi)$.

Since $\mathfrak{L}$, considered as an algebra, is free, any function from *VAR* into an SCI-algebra $\mathfrak{A}$ may be extended uniquely to a homomorphism of *Fm* into $\mathfrak{A}$. When $\mathfrak{A}$ is the language $\mathfrak{L}$ itself, such a homomorphism is called a *substitution* on $\mathfrak{L}$.

1.4 *Definition.* The set Φ of formulas is
(a) a *theory* if $\mathrm{Cn}(\Phi) = \Phi$;
(b) *consistent* if $\mathrm{Cn}(\Phi) \neq Fm$.
(c) *maximal consistent*, (or *complete*) if Φ is consistent, but is not a proper subset of any consistent set.
(d) *invariant* if $h(\Phi) \subseteq \Phi$, for every substitution h on $\mathfrak{L}$.

1.5 *Proposition.*
(a) *Any consistent set is contained in a maximal consistent set.*
(b) *Any maximal consistent set is a theory.*
(c) *If* $\varphi \notin \mathrm{Cn}(\Phi)$, *there is a maximal consistent superset* Γ *of* Φ *which does not contain* φ.
(d) *If* $\varphi \in \mathrm{Cn}(\Phi)$ *then* $h(\varphi) \in \mathrm{Cn}(h(\Phi))$ *for any substitution* h *on* $\mathfrak{L}$.[4]

We will interpret $\mathfrak{L}$ using the "matrix method" of Tarski [4]. The idea is to specify a set $A(\neq \emptyset)$ over which the variables range, and to interpret each connective as an operation on A. This involves an SCI-algebra $\mathfrak{A}$. We then pick a subset of A to be thought of as the "true" or "designated" elements. This idea will be clarified by the definitions below.

1.6 *Definition.* Let $\mathfrak{A} = \langle A, \dot{\neg}, \dot{\to}, \dot{\equiv}\rangle$ be an SCI-algebra, and let B be a subset of A.

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3. Both "iff" and "$\Longleftrightarrow$" are abbreviations of 'if, and only if". We also use "$\Longrightarrow$" as an abbreviation of "implies". The symbol "$\Longrightarrow$" should not be confused with the symbol "$\to$" which denotes a connective of $\mathfrak{L}$.

4. A consequence operation having this property was called *structural* in [3].

(a) a *valuation* of $\mathfrak{A}$ is a homomorphism h from $\mathfrak{L}$ into $\mathfrak{A}$ (i.e., $h(\neg\varphi) = \dot\neg h(\varphi)$, $h(\varphi \equiv \psi) = (h(\varphi) \dot\equiv h(\psi))$, etc.)
(b) B is *closed* if, for each a, $b \in A$, whenever a and $a \dot\to b$ are in B, so is b.
(c) B is *proper* if $B \neq A$.
(d) B is *admissible* if, for every valuation h of $\mathfrak{A}$, and every φ in TFA or IDA, $h(\varphi) \in B$.
(e) B is *prime* if $a \in B$ or $\dot\neg a \in B$, for all $a \in A$.
(f) B is *normal* if $a \dot\equiv b \in B \Leftrightarrow a = b$, for all a, $b \in A$.
(g) B is a *filter* if B is proper, closed and admissible.

An **SCI**-*matrix* M is a pair $\langle \mathfrak{A}, B \rangle$ consisting of an **SCI**-algebra $\mathfrak{A}$ and a filter B in $\mathfrak{A}$. An important example of an **SCI**-matrix (a *canonical* **SCI**-*matrix*) is $\langle \mathfrak{L}, \mathrm{Cn}(\Phi) \rangle$, where Φ is any consistent set of formulas. If B is a prime, normal filter, then $\langle \mathfrak{A}, B \rangle$ is called a *model*. The interpretations of $\mathfrak{L}$ are the **SCI**-matrices (see definition **1.7**), and the *intended* interpretations are the models, since in any model the interpretation of the logical connectives is the desired one: if $\langle \mathfrak{A}, B \rangle$ is a model and h is a valuation of $\mathfrak{A}$, then

$$h(\neg\varphi) \in B \Leftrightarrow h(\varphi) \notin B;$$
$$h(\varphi \to \psi) \in B \Leftrightarrow h(\varphi) \notin B \text{ or } h(\psi) \in B$$
$$h(\varphi \equiv \psi) \in B \Leftrightarrow h(\varphi) = h(\psi).$$

1.7 *Definition.* Let $\langle \mathfrak{A}, B \rangle$ be an **SCI**-matrix and let h be a valuation of $\mathfrak{A}$.
(a) *h satisfies φ in* $\langle \mathfrak{A}, B \rangle$ (in symbols, $\varphi \in \mathrm{Sat}_h(\mathfrak{A}, B)) \Leftrightarrow h(\varphi) \in B$;
(b) *φ is true in* $\langle \mathfrak{A}, B \rangle$ (in symbols, $\varphi \in \mathrm{TR}(\mathfrak{A}, B)) \Leftrightarrow \varphi \in \mathrm{Sat}_h(\mathfrak{A}, B)$, for every valuation h of $\mathfrak{A}$.
(c) *φ is valid in* $\mathfrak{A} \Leftrightarrow \varphi \in \mathrm{TR}(\mathfrak{A}, B)$, for every filter B in $\mathfrak{A}$.
(d) *φ is valid* $\Leftrightarrow \varphi$ is valid in $\mathfrak{A}$, for every **SCI**-algebra $\mathfrak{A}$.

For example, every formula in $TFA \cup IDA$ is valid, as may be easily verified. This fact is a special case of theorem **1.9** below. Any **SCI**-matrix $M = \langle \mathfrak{A}, B \rangle$ determines a consequence operation C_M on $\mathfrak{L}$ as follows:[5]

1.8 *Definition.* $\varphi \in \mathrm{C}_M(\Phi) \Leftrightarrow$ for all valuations h of $\mathfrak{A}$, if $h(\Phi) \subseteq B$, then $h(\varphi) \in B$.

It may be easily verified that C_M satisfies C1 - C3 above. Note that $\mathrm{TR}(\mathfrak{A}, B)$ is $\mathrm{C}_M(\emptyset)$. Hence a formula φ is valid iff $\varphi \in \mathrm{C}_M(\emptyset)$ for every M. The relation between the syntactic consequence Cn and the semantic consequence operations C_M is given by the following completeness theorem. It may be proved using proposition **1.5**(c) and **1.12**, **1.13** below.

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5. This semantic consequence operation was first studied in [3]. In most applications of the "matrix method", especially in [6], it proved sufficient to restrict the set of designated elements (here, the *filters*) to be a unit set. In interpretations of the **SCI** this is impossible, since, for example $p \to p$ and $p \equiv p$ must both be true and $\neg((p \to p) \equiv (p \equiv p))$ is consistent.

1.9 *Theorem* (Completeness Theorem).

(a) Φ is *consistent* $\Leftrightarrow$ *there is a model* $M = \langle \mathfrak{A}, B\rangle$ *and a valuation* h *of* $\mathfrak{A}$ *such that* $\Phi \subseteq h^{-1}(B)$.
(b) $\varphi \in \mathrm{Cn}(\Phi) \Leftrightarrow$ *for every* **SCI**-*matrix* M, $\varphi \in \mathrm{C}_M(\Phi)$
(c) $\varphi \in \mathrm{Cn}(\Phi) \Leftrightarrow$ *for every model* M, $\varphi \in \mathrm{C}_M(\Phi)$
(d) φ *is valid* $\Leftrightarrow \varphi$ *is a logical theorem.*

1.10 *Remark.* Two important observations follow from **1.9**. Firstly, by constructing an appropriate model, (see **4.6**) one can show that the formula

$$\text{(Fr)} \qquad (p_1 \rightarrow p_2) \rightarrow [(p_2 \rightarrow p_1) \rightarrow (p_1 \equiv p_2)]$$

is not valid, and thus not a logical theorem. If the formula (Fr) were provable, then the identity connective $\equiv$ would be only another notation for $\leftrightarrow$ (material equivalence). Hence we may justly say that the sentential calculus with identity is a refinement of the classical sentential calculus (**SC**). **SC** is obtainable from **SCI** by adding all substitution instances of (Fr) above as axioms. In [8] (Fr) was called the "Fregean axiom".

Secondly, it can now be seen that if Cn° is the consequence operation defined from the rule modus ponens and the axioms *TFA only*, and if φ, Φ do not contain an occurrence of the identity connective, then $\varphi \in \mathrm{Cn}(\Phi) \Leftrightarrow \varphi \in \mathrm{Cn}^\circ(\Phi)$. So in this sense, the **SCI** is a conservative extension of **SC**.[6]

In **3**, the question of whether there is a fixed matrix or model M° such that $\mathrm{C}_{M^\circ} = \mathrm{Cn}$ is answered affirmatively. Any matrix having this property is called *adequate for* Cn. If M has only the weaker property that $\mathrm{C}_M(\emptyset) = \mathrm{Cn}(\emptyset)$, then M is called *weakly-adequate for* Cn. (This notion of adequacy corresponds to that given in [3]). Note that $B_0 = \langle\{0, 1\}, \{1\}\rangle$ is adequate for the classical sentential calculus consequence, where $\{0, 1\}$ is the two element Boolean algebra, and $\{1\}$ is the prime filter. The following propositions deal with the question of when $\mathrm{C}_M = \mathrm{C}_{M'}$, for different **SCI**-matrices M, M'.

1.11 *Definition.* Let $M = \langle \mathfrak{A}, B\rangle$ and $M' = \langle \mathfrak{A}', B'\rangle$ be **SCI** matrices. A function h: $\mathfrak{A} \mapsto \mathfrak{A}'$ is a *matrix-homomorphism* (*matrix-isomorphism*) from M into M' if h is an algebraic homomorphism (isomorphism) from $\mathfrak{A}$ into $\mathfrak{A}'$ *and* $h^{-1}(B') = B$.

1.12 *Proposition. If h is a matrix homomorphism from M into M' which maps $\mathfrak{A}$ onto $\mathfrak{A}'$, then* $\mathrm{C}_M = \mathrm{C}_{M'}$.

A standard application of **1.12** is in the formation of "quotient-matrices". If $M = \langle \mathfrak{A}, B\rangle$ is any **SCI**-matrix, define the binary relation $\approx_B$ on

6. The completeness theorem may be used to show that if $\vee$ (disjunction) is included in the set of primitive connectives then e.g. $\neg(p \vee q \equiv \neg p \rightarrow q)$ is consistent. Hence to construe $p \vee q$ as an *abbreviation* of $\neg p \rightarrow q$ is, in effect, to adopt $p \vee q \equiv \neg p \rightarrow q$ as an axiom. Similar remarks may be made about any abbreviation.

$\mathfrak{A}$ by

$$a \approx_B b \Longleftrightarrow a \doteq b \in B, \text{ for all } a, b \in \mathfrak{A}.$$

(When the filter B in question is clear from context, we will write $\approx$ instead of $\approx_B$). From definitions **1.1** and **1.6** it follows that $\approx$ is a congruence on $\mathfrak{A}$. Let $|a|$ denote the congruence class of the element a. Then $\mathfrak{A}/\approx$, the set of congruence classes of elements of $\mathfrak{A}$, becomes an **SCI**-algebra with the following definitions:

$$\dot{\neg}|a| = |\dot{\neg}a|;\ |a| \dot{\rightarrow} |b| = |a \dot{\rightarrow} b|;\ |a| \doteq |b| = |a \doteq b|\ .$$

Let $B/\approx$ be the set $\{|b| : b \in B\}$. Clearly $B/\approx$ is a filter in $\mathfrak{A}/\approx$, so that $M/\approx\ = \langle \mathfrak{A}/\approx, B/\approx \rangle$ is an **SCI**-matrix.

1.13 *Proposition. $B/\approx$ is a normal filter in $\mathfrak{A}/\approx$, and $B/\approx$ is prime iff B is prime in $\mathfrak{A}$. Furthermore, the natural map $a \mapsto |a|$ is a matrix homomorphism of M onto $M/\approx$, so that $\mathsf{C}_M = \mathsf{C}_{M/\approx}$.*

Those **SCI**-matrices $\langle \mathfrak{A}, B \rangle$ in which B is a normal filter are characterized by **1.14**.

1.14 *Proposition. Suppose B is a normal filter in $\mathfrak{A}$. Then there is an **SCI**-language $\mathfrak{L}_0$, a consistent set Φ of formulas of $\mathfrak{L}_0$ and a matrix isomorphism from $\langle \mathfrak{L}/\approx, \mathsf{Cn}(\Phi)/\approx \rangle$ onto $\langle \mathfrak{A}, B \rangle$ (where $\approx$ is $\approx_\Phi$).*

Proof: Let V be any set having the same cardinality as $\mathfrak{A}$, and let $\mathfrak{L}_0$ be the **SCI**-language having $VAR = V$. Let h be any 1-1 function from VAR onto $\mathfrak{A}$ and extend h to a homomorphism of $\mathfrak{L}_0$ onto $\mathfrak{A}$. Let Φ be $h^{-1}(B)$ (i.e., $\Phi = \mathsf{Sat}_h(\mathfrak{A}, B)$). Then $\mathsf{Cn}(\Phi) = \Phi$ and by the completeness theorem Φ is consistent. The normality of B implies that h is well-defined in $\mathfrak{L}_0/\approx$, and is both 1-1 and onto.

We end this section by giving a useful description of prime normal filters.

1.15 *Proposition. Let $\mathfrak{A} = \langle A, \dot{\neg}, \dot{\rightarrow}, \doteq, \rangle$ be an **SCI**-algebra and suppose $B \subseteq A$. B is a prime filter in $\mathfrak{A}$ iff* (a), (b) and (c) *below hold; B is a prime, normal filter iff* (d) *holds as well. For all a, b in A,*

(a) $a \dot{\rightarrow} b \notin B \Longleftrightarrow a \in B$ *and* $b \notin B$
(b) $a \notin B \Longleftrightarrow \dot{\neg} a \in B$
(c) $\approx_B$ *is a congruence on* $\mathfrak{A}$, *where* $a \approx_B b \Longleftrightarrow a \doteq b \in B$.
(d) $\approx_B$ *is the identity; i.e.,* $a \approx_B b \Longleftrightarrow a = b$.

2. *Finite models and the decidability of* $\mathsf{Cn}(\emptyset)$. A finite model $M = \langle \mathfrak{A}, B \rangle$ is a model in which the number of elements in the **SCI**-algebra $\mathfrak{A}$ is finite. We let $|M|$ denote this number. For each pair of natural numbers $\langle n, t \rangle$ where $2 \leq n$, $1 \leq t < n$, one may construct a finite model $M = \langle \mathfrak{A}, B \rangle$ such that $|M| = n$ and $|B|$, the number of elements of B, is t. Indeed, let $A = \{1, 2, \ldots, n\}$, $B = \{2, \ldots, t\}$. Using **1.15**, we see there may be many ways to define the operations $\dot{\neg}$, $\dot{\rightarrow}$, $\doteq$, on A; the *only* restriction is that the "multiplication tables" have the following form:

	$\dot{\neg}$
b	a'
a	b'

$\dot{\to}$	b	a
b	b'	a'
a	b''	b'''

$\dot{\equiv}$	b	a
b	b'	a''
a	a'	b''

where a, a', a'' are in $A - B$, and b, b', b'', b''' are in B. That is, (to repeat **1.15**) for any c, d in A: $\dot{\neg}c \notin B$ iff $c \in B$; $c \dot{\to} d \notin B$ iff $c \in B$ and $d \notin B$; $c \equiv d \notin B$ iff $c \neq d$. These observations may be extended to prove the following.

2.1 Theorem. *For any positive integer n, there is a finite* SCI*-algebra $\mathfrak{A} = \langle A, \dot{\neg}, \dot{\to}, \dot{\equiv}\rangle$ containing n prime, normal filters $B_1, \ldots, B_n$ such that when $i \neq j$,* $\mathrm{TR}(\mathfrak{A}, B_i) \neq \mathrm{TR}(\mathfrak{A}, B_j)$, *and hence* $\mathrm{C}_{\langle\mathfrak{A}, B_i\rangle} \neq \mathrm{C}_{\langle\mathfrak{A}, B_j\rangle}$.

Proof: Let $A_0 = \{0, 1\}^n$, the set of n-tuples of 0's and 1's, and define A as the union of A_0 and the $n(n-1)/2$ elements $\langle 0, 2, 0, \ldots, 0\rangle$; $\langle 0, 0, 2, 0, \ldots, 0\rangle$, $\langle 0, 0, 3, 0, \ldots, 0\rangle; \ldots; \langle 0, \ldots, 0, 2\rangle$, $\langle 0, \ldots 0, 3\rangle, \ldots, \langle 0, \ldots, 0, n\rangle$. *For a in A*, $(a)_i$ will denote the i^{th}-coordinate of a. The functions $\dot{\neg}$, $\dot{\to}$ and $\dot{\equiv}$ are defined as follows:

$$(\dot{\neg}a)_i = \begin{cases} 0 \text{ if } (a)_i \neq 0 \\ 1 \text{ otherwise} \end{cases}$$

$$(a \to b)_i = \begin{cases} 0 \text{ if } (a)_i \neq 0 \text{ and } (b)_i = 0; \\ 1 \text{ otherwise} \end{cases}$$

$$a \equiv b = \begin{cases} \langle 0, 0, \ldots, 0\rangle \text{ if } a \neq b \\ \langle 1, 1, \ldots, 1\rangle \text{ otherwise.} \end{cases}$$

The subsets $B_1, \ldots, B_n$ are defined by:

$$B_i = \{a \in A : (a)_i \neq 0\}$$

From **1.15** and the above definitions it follows that each set B_i is a prime, normal filter in A. In order to show $\mathrm{TR}(\mathfrak{A}, B_i) \neq \mathrm{TR}(\mathfrak{A}, B_j)$ when $i \neq j$ notice that $|B_1| < |B_2| < \ldots < |B_n|$. (This was the purpose of adding the additional elements to A_0). If $i < j$ and $|A - B_j| = r$, then the formula

$$[\neg(p_1 \equiv p_2) \wedge \neg(p_1 \equiv p_3) \wedge \ldots \wedge \neg(p_r \equiv p_{r+1})] \to [p_1 \vee p_2 \vee \ldots \vee p_{r+1}]$$

is in $\mathrm{TR}(\mathfrak{A}, B_j)$ but not in $\mathrm{TR}(\mathfrak{A}, B_i)$. This concludes the proof.

As was remarked in **1**, the two element Boolean algebra (with unit filter) is adequate for the classical sentential calculus. In contrast, we have the following result.

2.2 Theorem. *There is no finite model weakly adequate for* Cn, *and hence no adequate finite model.*

Proof: Let M be any finite model, and let $n = |M|$. Then the "Gödel formula" φ, where φ is

$$p_1 \equiv p_2 \vee p_1 \equiv p_3 \vee \ldots \vee p_n \equiv p_{n+1}$$

is true in M but is not true in any model M', with $|M'| > n$. It follows from **1.9** (d) that φ is not in $\mathrm{Cn}(\emptyset)$. Hence $\mathrm{TR}(M') \neq \mathrm{Cn}(\emptyset)$.

The next theorem shows that Cn has the "finite model property". It is used to prove the recursive decidability of $\mathrm{Cn}(\emptyset)$. For any formula φ, let $\mathrm{DES}(\varphi)$ be the number of subformulas of φ.

2.3 Theorem. *If φ is satisfiable in some model, then φ is satisfiable in a finite model M with $|M| \leq \mathrm{DES}(\varphi) + 1$.*

Proof: Suppose φ is satisfiable in the model $N = \langle \mathfrak{A}, B\rangle$ by the valuation h. Suppose $\mathfrak{A} = \langle A, \dot{\neg}, \dot{\rightarrow}, \dot{\equiv}\rangle$, and let $\varphi_1, \varphi_2, \ldots, \varphi_n = \varphi$ be all subformulas of φ (so $n = \mathrm{DES}(\varphi)$). Let C_0 be the set $\{h(\varphi_1), \ldots, h(\varphi_n)\}$. Note that $h(\varphi_n) = h(\varphi) \in B$, since h satisfies φ by hypothesis. If C_0 contains an element of $A - B$, let $C = C_0$. Otherwise, let $C = C_0 \cup \{O\}$, where O is any element of $A - B$. Denote $h(\varphi_n)$ by 1. We let $D = C \cap B$, and we will so define the operations $\mathring{\neg}$, $\overset{\circ}{\rightarrow}$, $\overset{\circ}{\equiv}$ on C such that D becomes a prime, normal filter in C. For a, b in C, define (where $\#$ is $\rightarrow$ or $\equiv$):

$$\mathring{\neg} a = \begin{cases} \dot{\neg} a & \text{if } \dot{\neg} a \in C; \\ 1 & \text{if } \dot{\neg} a \in B - C; \\ O & \text{otherwise} \end{cases}$$

$$a \overset{\circ}{\#} b = \begin{cases} a \overset{\circ}{\#} b & \text{if } a \overset{\bullet}{\#} b \in C; \\ 1 & \text{if } a \overset{\bullet}{\#} b \in B - C; \\ O & \text{otherwise} \end{cases}$$

That D is a prime, normal filter in the SCI-algebra $\mathfrak{C} = \langle C, \mathring{\neg}, \overset{\circ}{\rightarrow}, \overset{\circ}{\equiv}\rangle$ follows from the facts that N is a model (i.e. B is a prime, normal filter in $\mathfrak{A}$) *and* (for $a, b \in C$)

$$\begin{aligned} \mathring{\neg} a \in D &\Longleftrightarrow \dot{\neg} a \in B; \\ a \overset{\circ}{\rightarrow} b \in D &\Longleftrightarrow a \dot{\rightarrow} b \in B; \\ a \overset{\circ}{\equiv} b \in D &\Longleftrightarrow a \dot{\equiv} b \in B. \end{aligned}$$

Hence $M = \langle \mathfrak{C}, D\rangle$ is a finite model, and $|M| \leq 1 + \mathrm{DES}(\varphi)$.
Finally, let h' be the valuation of $\mathfrak{C}$ defined by

$$h'(p) = \begin{cases} h(p) & \text{if } p \text{ is a subformula of } \varphi \\ 1 & \text{otherwise} \end{cases}$$

From the way the functions in $\mathfrak{C}$ are defined, it is easily checked that $h'(\varphi) = h(\varphi) = 1$, so that h' satisfies φ in M, q.e.d..

Trivially this estimate is best possible since the smallest model in which the formula consisting of a single variable p is satisfiable has two elements; i.e. $1 + \mathrm{DES}(p)$ elements. But this does not tell the whole story: e.g. if φ contains no negation signs, then φ is satisfiable in the two element model. We do not discuss these matters further here. Clearly from **2.3**, we have:

2.4 *Corollary. There is an effective procedure to determine, given a formula φ, whether φ is a logical theorem.*

The decidability of the classical sentential calculus may be proved using the so-called truth table method, which may be considered as an application of the following well-known theorem:

For every mapping $f: VAR \mapsto \{0, 1\}$ there exists a unique classical complete set of truth functional formulas Φ such that the characteristic function of Φ is an extension of f.

This theorem can be generalized to the **SCI**. Consequently, there is a generalization of the truth table method for **SCI** formulas which may also be used to prove the decidability of $\mathsf{Cn}(\emptyset)$.

2.5 *Definition.* A mapping $t: Fm \mapsto \{0, 1\}$ is called a *truth valuation* of the **SCI** formulas if t is the characteristic function of some maximal consistent theory.

Using the two operations $\dot{\neg}, \dot{\rightarrow}$ on the set $\{0, 1\}$: $(\dot{\neg}1) = 0$, $(\dot{\neg}0) = 1$, $x \dot{\rightarrow} y = 0$ iff $x = 1$ and $y = 0$, one easily proves the following.

2.6 *Proposition. A mapping $t: Fm \mapsto \{0, 1\}$ is a truth valuation iff the following conditions hold for all $\varphi, \psi, \alpha, \beta \in Fm$:*

(a) $t(\neg\varphi) = \dot{\neg} t(\varphi)$

(b) $t(\varphi \rightarrow \psi) = t(\varphi) \dot{\rightarrow} t(\psi)$

(c) $t(\varphi \equiv \varphi) = 1$;

(d) *if* $t(\varphi \equiv \psi) = 1$, *then* $t(\varphi) = t(\psi)$

(e) *if* $t(\varphi \equiv \psi) = 1$, *then* $t(\neg\varphi \equiv \neg\psi) = 1$

(f) *if* $t(\varphi \equiv \psi) = t(\alpha \equiv \beta) = 1$ *then* $t((\varphi \rightarrow \alpha) \equiv (\psi \rightarrow \beta)) = t((\varphi \equiv \alpha) \equiv (\psi \equiv \beta)) = 1$.

Let Eq be the set of all equations. We define the formulas and equations of degree at most $k = 0, 1, 2, \ldots$.

2.7 *Definition.*

(a_1) $Fm_0 = VAR$

(a_2) $Fm_{k+1} = Fm_k \cup \{\neg\varphi, \varphi \rightarrow \psi, \varphi \equiv \psi : \varphi, \psi \in Fm_k\}$.

(b) $Eq_0 = \emptyset$ and $Eq_{k+1} = Eq \cap (Fm_{k+1} - Fm_k)$

2.8 *Definition.* Every mapping $f: VAR \cup Eq_1 \mapsto \{0, 1\}$ is called *elementary truth-valuation* if it satisfies the following conditions for $i, j, k = 1, 2, \ldots$:

(a) $f(p_k \equiv p_k) = 1$

(b) if $f(p_i \equiv p_j) = 1$ then $f(p_i) = f(p_j)$

(c) if $f(p_i \equiv p_k) = f(p_j \equiv p_k) = 1$ then $f(p_i \equiv p_j) = 1$.

The restriction t' to $VAR \cup Eq_1$ of any truth valuation t clearly is an elementary truth valuation. Conversely, we have the following.

2.9 Theorem. *Every elementary truth valuation can be extended to a truth valuation.*

Before giving the proof, let us first discuss some properties of (elementary) truth valuations. Observe that every elementary truth valuation f is a unique union $h \cup g$ of two disjoint zero-one valued functions h, g such that $VAR = Fm_0$ = domain of h and Eq_1 = domain of g. Moreover, g is the characteristic function of an equivalence relation $\vdash\!\dashv$ on VAR, satisfying the condition

$$(*)\ \textit{if } p_i \vdash\!\dashv p_j \textit{ then } h(p_i) = h(p_j)$$

Hence, elementary truth valuations may be considered as pairs $\langle h, \vdash\!\dashv\rangle$ where h is any mapping of VAR into $\{0, 1\}$ and $\vdash\!\dashv$ is any equivalence relation on VAR such that (*) holds. We are going to show that the (full) truth-valuations may be represented in an analogous manner.

2.10 *Definition.* An infinite sequence of pairs $\{\langle h_k, \underset{k}{\vdash\!\dashv}\rangle\}_{k\in N}$ is called a sequence of *partial truth valuations* if the following conditions are satisfied for each $k = 0, 1, \ldots$

(a) h_k maps Fm_k into $\{0, 1\}$.

(b) $h_{k+1}(\varphi) = h_k(\varphi)$, if $\varphi \in Fm_k$; i.e. h_{k+1} is an extension of h_k;

(c) $h_{k+1}(\neg\varphi) = \dot{\neg} h_k(\varphi)$ if $\varphi \in Fm_k$,

(d) $h_{k+1}(\varphi \to \psi) = h_k(\varphi) \dot{\to} h_k(\psi)$ if $\varphi, \psi \in Fm_k$;

(e) $\underset{k}{\vdash\!\dashv}$ is an equivalence relation on Fm_k,

(f) $\varphi \underset{k+1}{\vdash\!\dashv} \psi \Longleftrightarrow \varphi \underset{k}{\vdash\!\dashv} \psi$ if $\varphi, \psi \in Fm_k$; i.e. $\underset{k+1}{\vdash\!\dashv}$ is an extension of $\underset{k}{\vdash\!\dashv}$,

(g) if $\varphi \underset{k}{\vdash\!\dashv} \psi$ then $h_k(\varphi) = h_k(\psi)$;

(h) $h_{k+1}(\varphi \equiv \psi) = 1 \Longleftrightarrow \varphi \underset{k}{\vdash\!\dashv} \psi$, if $\varphi, \psi \in Fm_k$;

(i) if $\varphi \underset{k}{\vdash\!\dashv} \psi$ and $\alpha \underset{k}{\vdash\!\dashv} \beta$ then $\neg\varphi \underset{k+1}{\vdash\!\dashv} \neg\psi$, $(\varphi \to \alpha) \underset{k+1}{\vdash\!\dashv} (\psi \to \beta)$ and $(\varphi \equiv \alpha) \underset{k+1}{\vdash\!\dashv} (\psi \equiv \beta)$.

2.11 *Proposition.* (1) *If t is a truth valuation and $t_k(\varphi) = t(\varphi)$ and $\varphi \overset{t}{\underset{k}{\vdash\!\dashv}} \psi \Longleftrightarrow t_k(\varphi \equiv \psi) = 1$ for all $\varphi, \psi \in Fm_k$, $k = 0, 1, \ldots$ then $\{\langle t_k \overset{t}{\underset{k}{\vdash\!\dashv}}\rangle\}$ is a sequence of partial truth valuations; obviously $t = \bigcup_k t_k$.*

(2) *On the other hand, if $\{\langle h_k, \underset{k}{\vdash\!\dashv}\rangle\}$ is a sequence of partial truth valuations then the union $t = \bigcup_k h_k$ is a truth valuation such that $\{\langle h_k, \underset{k}{\vdash\!\dashv}\rangle\}$ is just the corresponding sequence of partial truth valuations; i.e. $h_k = t_k$ and $\underset{k}{\vdash\!\dashv} = \overset{t}{\underset{k}{\vdash\!\dashv}}$ for all $k = 0, 1, \ldots$.*

Remark. According to the last proposition there is a one-one correspondence between truth valuations and sequences of partial truth valuations $\{\langle h_k, \underset{k}{\vdash\!\dashv}\rangle\}$. Observe that the mapping h_k assigns a truth value 0 or 1 to every formula in Fm_k. On the other hand, the equivalence relation $\underset{k}{\vdash\!\dashv}$ determines, according to definition **2.10**(h), the truth value of all equations in Eq_{k+1}, that is, all "new" equations in Fm_{k+1}. The truth values of all other formulas in $Fm_{k+1} - Fm_k$ are uniquely determined (according to (c) and (d) in definition **2.10**) by the truth values of formulas in Fm_k.

Proof of **2.9**: Suppose now f is an elementary truth valuation represented by $\langle h, \vdash\!\dashv\rangle$. Define:

(a) $h_0 = h$ and $\underset{0}{\vdash\!\dashv} = \vdash\!\dashv$,

(b) $h_1(p_i \equiv p_j) = 1$ iff $p_i \underset{0}{\vdash\!\dashv} p_j$

(c) $h_{k+1}(\varphi \equiv \psi) = 0$ and non $\varphi \underset{k}{\vdash\!\dashv} \psi$, iff $\varphi, \psi \in Fm_k$ and distinct,

(d) $h_{k+1}(\neg\varphi) = \dot{\neg} h_k(\varphi)$, if $\varphi \in Fm_k$,

(e) $h_{k+1}(\varphi \to \psi) = h_k(\varphi) \dot{\to} h(\psi)$, if $\varphi, \psi \in Fm_k$.

One may easily see that $\{\langle h_k, \underset{k}{\vdash\!\dashv}\rangle\}_{k\in N}$ is a sequence of partial truth valuations and the truth valuation $t = \bigcup_{k=0}^{\infty} h_k$ is an extension of f. Thus, theorem **2.9** is proved.

Remark. If we replace the condition (c) by the following two:

(c_1^*) $h_{k+1}(\varphi \equiv \psi) = 1 \Longleftrightarrow h_k(\varphi) = h_k(\psi)$, if $\varphi, \psi \in Fm_k$
(c_2^*) $\varphi \underset{k}{\vdash\!\dashv} \psi \Longleftrightarrow h_k(\varphi) = h_k(\psi)$, if $\varphi, \psi \in Fm_k$

then the above procedure defines another truth valuation t^* which also is an extension of f. According to t all non-trivial equations are false. On the other hand the truth valuation t^* makes all equations behave like biconditionals.

2.12 Corollary. *The set of those variables and elementary equations which are true under any elementary truth valuation, is consistent.*

Remark. Truth valuations provide another proof of **2.4**:
Proof. Let $Fm_k^{(n)}$ be the set of all those formulas in Fm_k which contain at most the variables $p_1, \ldots, p_n$. The restrictions of (elementary) truth valuations to $Fm_k^{(n)}$ may be called (elementary) n/k-truth-valuations. Since the set $Fm_k^{(n)}$ is finite, there are finitely many elementary n/k-truth-valuations and finitely many n/k-truth-valuations. Each of them may be effectively presented as a zero-one-valued assignment table for all formulas in $Fm_k^{(n)}$. Moreover, given an elementary n/k-truth-valuation f there is an effective although not univocal procedure of step wise extending f over $Fm_1^{(n)}, \ldots, Fm_k^{(n)}$ to any n/k-truth-valuation which is an extension of f. Since, in general, $\varphi \in \mathsf{Cn}(\emptyset)$ iff $t(\varphi) = 1$ for every truth valuation t, it follows that for any φ in $Fm_k^{(n)}$:

$$\varphi \in \mathsf{Cn}(\emptyset) \text{ iff } t(\varphi) = 1 \text{ for every } n/k\text{-truth-valuation } t.$$

Thus, given any formula φ, one has to find the smallest set $Fm_k^{(n)}$ containing φ. Then, by inspection of all n/k-truth-valuations, one may tell whether or not $\varphi \in \mathsf{Cn}(\emptyset)$.

2.13 *Corollary. An equation $\varphi \equiv \psi$ is in* $\mathsf{Cn}(\emptyset)$ *if and only if it is trivial, i.e. φ and ψ are the same.*

3. *Adequacy.* In **2** it was shown that no finite matrix is adequate for Cn. In this section we show that there are at least two ways of constructing infinite adequate matrices. For certain natural extensions, however, no adequate matrix exists.

We first define the *direct-sum* of a collection of models. Suppose that for each member i of an index set $I, M_i = \langle \mathfrak{A}_i, B_i\rangle$ is a model, and let $\mathfrak{A}_i = \langle \mathfrak{A}_i, \neg_i, \rightarrow_i, \equiv_i\rangle$. We further suppose that A_i and A_j are disjoint when $i \neq j$, $(i, j \in I)$. (When this is the case, we say M_i and M_j are disjoint). The model $M = \langle \mathfrak{A}, B\rangle$, $\mathfrak{A} = \langle A, \dot\neg, \dot\rightarrow, \dot\equiv\rangle$ is defined as follows. Let

$$A = \bigcup_{i\in I} A_i, \text{ and } B = \bigcup_{i\in I} B_i.$$

The operations will be defined so that B is a prime normal filter in $\mathfrak{A}$. First select some member of B, say 1, and some member of $A - B$, say O.

We denote members of the set A_i by the letters a_i and $b_i (i \epsilon I)$. Now define $\dot{\neg}$, $\dot{\rightarrow}$, $\dot{\equiv}$ by:

$$\dot{\neg}(a_i) = \neg_i a_i$$

$$a_i \dot{\rightarrow} b_j = \begin{cases} a_i \rightarrow_i b_i & \text{if } i = j; \\ O & \text{if } a_i \epsilon B_i \text{ but } b_j \notin B_j; \\ 1 & \text{otherwise.} \end{cases}$$

$$a_i \dot{\equiv} b_j = \begin{cases} a_i \equiv_i b_i & \text{if } i = j; \\ O & \text{otherwise.} \end{cases}$$

It is quite easy to check the following facts.

$$a_i \dot{\rightarrow} b_j \notin B \Longleftrightarrow a_i \epsilon B \text{ and } b_j \notin B$$

$$a_i \notin B \Longleftrightarrow \dot{\neg} a_i \epsilon B$$

$$a_i \equiv b_j \notin B \Longleftrightarrow a_i \neq b_j.$$

Hence B is indeed a prime, normal filter in A, so that M is a model. We call M the *direct sum* of the models M_i. We use the direct-sum construction in the proofs of **3.4** and **3.6**.

3.3 *Definition.* Let Φ be a set of formulas. *The support of* Φ, in symbols, $s\Phi$, is the set consisting of those variables which occur in at least one formula in Φ.

3.4 Theorem. *Let* $\mathfrak{L}$ *be any* SCI*-language, and let* Φ_1 *and* Φ_2 *be sets of formulas of* $\mathfrak{L}$. *If both* Φ_1 *and* Φ_2 *are consistent and* $s\Phi_1 \cap s\Phi_2 = \emptyset$, *then* $\Phi_1 \cup \Phi_2$ *is also consistent.*

Proof: By **1.9**, there are models M_1 and M_2, and valuations v_1 and v_2 such that v_i satisfies Φ_i in M_i, $i = 1, 2$. We may clearly suppose M_1 and M_2 disjoint. Let M be the direct sum of M_1 and M_2. Let v be any valuation of M satisfying the condition that for any variable p in $s\Phi_i$, $v(p) = v_i(p)$. At least one such valuation v exists since $s\Phi_1 \cap s\Phi_2 = \emptyset$. v satisfies $\Phi_1 \cup \Phi_2$ in M since the operations in M agree with the operations in M_i (for arguments belonging to M_i) and v_i satisfies Φ_i in M_i. Thus, again by **1.9** $\Phi_1 \cup \Phi_2$ is consistent.

3.5 *Remark.* We note that the following conditions are equivalent for sets of formulas Φ_1, Φ_2 with disjoint supports.

(a) if Φ_1 and Φ_2 are separately consistent, so is $\Phi_1 \cup \Phi_2$.
(b) if $\varphi \epsilon \mathrm{Cn}(\Phi_1 \cup \Phi_2)$ and $s\{\varphi\} \cap s\Phi_2 = \emptyset$ then either $\varphi \epsilon \mathrm{Cn}(\Phi_1)$ or Φ_2 is inconsistent.

Any consequence operation satisfying condition (b) was called *uniform* in [3]. ((a) and (b) are equivalent for any consequence operation C on a sentential language having a unary connective # such that $\alpha \epsilon \mathrm{C}(\Gamma)$ iff $\mathrm{C}(\Gamma, \#\alpha) =$ all formulas.)

3.6 Theorem. Cn *has an adequate model.*[7]

Proof: Let $\{\Phi_i\}_{i\in I}$ be the collection of all maximal consistent theories in $\mathfrak{L}$. By **1.9**, for each i in I there is a model M_i, and a valuation v_i such that v_i satisfies Φ_i in M_i. Without loss of generality, we suppose the models M_i are mutually disjoint. Let $M = \langle \mathfrak{A}, B\rangle$ be the direct sum of the models M_i. We will show $C_M = \mathrm{Cn}$. By **1.9** $\mathrm{Cn} \subseteq C_M$ for *any* model M. Hence we need only prove $C_M \subseteq \mathrm{Cn}$. Suppose $\varphi \notin \mathrm{Cn}(\Gamma)$ for some formula φ, and some set of formulas Γ. By **2.1** there is a maximal consistent set, say, Φ_{i_0} containing Γ but not containing φ. Let h be the valuation of M defined by $h(p) = v_{i_0}(p)$ for each variable p. That is, h is v_{i_0} considered as a valuation of M. Since the image of any formula under v_{i_0} is in A_{i_0} it follows by the same argument as that in **3.4** that

$$\Phi_{i_0} \subset h^{-1}(B), \text{ i.e. } \Phi_{i_0} \subset \mathrm{Sat}_h(M).$$

But since B is a prime filter, $h^{-1}(B)$ is a maximal consistent set. Since Φ_{i_0} itself is maximal, it follows that $\Phi_{i_0} = h^{-1}(B)$; i.e. $\Phi_{i_0} = \mathrm{Sat}_h(M)$. Now since $\varphi \in \Phi_{i_0}$, $\varphi \notin \mathrm{Sat}_h(M)$. But this shows $\varphi \notin C_M(\Phi_{i_0})$ and hence $\varphi \notin C_M(\Gamma)$. Thus $C_M \subseteq \mathrm{Cn}$, and the proof is complete.

3.7 Corollary. *The model M* (constructed in the proof of **3.6**) *has the following property*: *Φ is a maximal consistent theory if and only if there is a valuation h of M such that $\Phi = \mathrm{Sat}_h(M)$.*

Call any model (or matrix) having the property given in **3.7** *special*. The idea of the proof of **3.6** was to construct a special model and then point out that any *special model is adequate*. It may be asked whether any matrix M such that $C_M = \mathrm{Cn}$ will be special. From the next theorem it follows that the answer is negative, since the inverse image of a filter which is not prime will not be a *maximal* consistent theory.

3.8 Theorem. *There is a matrix $M = \langle \mathfrak{A}, B\rangle$* such that $C_M = \mathrm{Cn}$ *and B is not a prime filter in $\mathfrak{A}$.*

Proof: (II of **3.6**) Let $\{\Phi_i : i \in I\}$ be the collection of all Cn-consistent theories (*not just the maximal theories*). If $\mathfrak{L}$ is generated from the set of variables $\{p_k : k \in K\}$ let $\mathfrak{L}^*$ be the **SCI** language generated from the set of variables $\{p^i_k : i \in I, k \in K\}$. Cn^* is the consequence operation on $\mathfrak{L}^*$ (defined by modus ponens and the same axiom schemata **1.1** that defines Cn on $\mathfrak{L}$ only extended to the language $\mathfrak{L}^*$). Let $e_i : \mathfrak{L} \mapsto \mathfrak{L}^*$ be the unique monomorphism taking p_k onto p^i_k.

3.9 *Lemma.*

(a) *For any formula φ of $\mathfrak{L}$, any set of formulas Φ of $\mathfrak{L}$ and any i,* $\varphi \in \mathrm{Cn}(\Phi) \Longleftrightarrow e_i(\varphi) \in \mathrm{Cn}^*(e_i(\Phi))$

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7. This theorem is a corollary of the main theorem of [3]. However our proof I is totally different from the one given there. Proof II (**3.8**) is a minor modification of the Łoś-Suszko proof. We give it in order to get Corollary **3.10**.

(b) Φ *is* Cn*-consistent iff* $e_i(\Phi)$ *is* Cn^**-consistent.*
(c) *Let* $B_0 = \bigcup_{i\in I} e_i(\Phi_i)$. *Then* B_0 *is* Cn^**-consistent.*

Proof of **3.9**: (a) is easy, and is omitted. (b) follows immediately from (a). In order to prove (c), assume B_0 is not Cn^*-consistent. Since proofs are finite, for some $i_1, \ldots, i_n$ in I,

$$e_{i_1}(\Phi_{i_1}) \cup \ldots \cup e_{i_n}(\Phi_{i_n}),$$

is Cn^*-inconsistent. But each set $e_i(\Phi_i)$ is separately consistent by (b). Hence, by **3.4** (which applies to any **SCI** language) so is any finite union of these sets, contradicting the assumption. Thus B_0 is Cn^*-consistent.

We now define the matrix M as the pair $\langle \mathfrak{L}^*, \mathrm{Cn}^*(B_0)\rangle$. Before proving $\mathrm{C}_M = \mathrm{Cn}$ we show that $B = \mathrm{Cn}^*(B_0)$ is *not* a prime filter in $\mathfrak{L}^*$. Indeed, let Φ_{i_0} be any non-maximal Cn-theory, and let ϕ be a formula such that neither φ nor $\neg\varphi$ belong to $\mathrm{Cn}(\Phi_{i_0})$. From **3.9** (a) it follows that neither $e_{i_0}(\varphi)$ nor $e_{i_0}(\neg\varphi) = \neg e_{i_0}(\phi)$ are in $\mathrm{Cn}^*(e_{i_0}(\Phi_{i_0}))$. If $e_{i_0}(\varphi) \in \mathrm{Cn}^*(B_0) = \mathrm{Cn}^*(e_{i_0}(\Phi_{i_0}) \cup \bigcup_{j\neq i_0} e_j(\Phi_j))$, then, by **3.4** and **3.5**, either $e_{i_0}(\phi) \in \mathrm{Cn}^*(e_{i_0}(\Phi_{i_0}))$ or $\bigcup_{j\neq i_0} e_j(\Phi_j)$ is Cn^*-inconsistent. But both alternatives have been shown false. Hence $e_{i_0}(p) \notin \mathrm{Cn}^*(B_0)$. Similarly $\neg e_{i_0}(\varphi) \notin \mathrm{Cn}^*(B_0)$. Thus B is not prime.

$\mathrm{Cn} \subseteq \mathrm{C}_M$ by **1.9**. In order to show $\mathrm{C}_M \subseteq \mathrm{Cn}$, suppose that $\varphi \notin \mathrm{Cn}(\Phi)$. Let $\Phi_{i_0} = \mathrm{Cn}(\Phi \cup \{\neg\varphi\})$. Φ_{i_0} is a Cn-consistent theory. Thus $e_{i_0}(\Phi_{i_0}) \subseteq B$. We show $e_{i_0}(\varphi) \notin B$. For if $e_{i_0}(\varphi) \in B$, then $e_{i_0}(\varphi) \in \mathrm{Cn}^*(e_{i_0}(\Phi) \cup \bigcup_{j\neq i_0} e_j(\Phi_j))$. By the argument given two paragraphs above it follows that $e_{i_0}(\varphi) \in \mathrm{Cn}^*(e_{i_0}(\Phi_{i_0}))$ and hence $\varphi \in \mathrm{Cn}(\Phi_{i_0})$. But then Φ_{i_0} is Cn-inconsistent, a contradiction. Hence e_{i_0} is a valuation of $\mathfrak{L}^*$ such that $e_{i_0}(\Phi) \subseteq B$, but $e_{i_0}(\varphi) \notin B$. This shows $\varphi \notin \mathrm{C}_M(\Phi)$ and completes the proof.

3.10 *Corollary. There are two matrices*

$$M = \langle \mathfrak{A}, B\rangle \text{ and } M' = \langle \mathfrak{A}', B'\rangle$$

such that B *is a prime filter in* $\mathfrak{A}$, B' *is not a prime filter in* $\mathfrak{A}'$, *but* $\mathrm{C}_M = \mathrm{C}_{M'}$

Proof: Let M be the model constructed in the proof of **3.7** and let M' be the matrix of **3.8**. $\mathrm{C}_M = \mathrm{C}_{M'} = \mathrm{Cn}$.

We give two examples of consequence operations having no adequate matrices. In the first example, the language is extended but the consequence operation remains "the same." Let $\mathfrak{L}^+$ be the language obtained from $\mathfrak{L}$ by the addition of some sentential constants, say $\{c_i\}_{i\in I}$, and perhaps some connectives. The notion of an "$\mathfrak{L}^+$-matrix", M^+, must be modified so that an "$\mathfrak{L}^+$-algebra" contains constants $\{\bar{c}_i\}_{i\in I}$ and functions corresponding to the new connectives. Every valuation of M^+ must take c_i onto $\bar{c}_i$. Cn^+ is defined from the same axiom schemata that define Cn. We show Cn^+ can have no adequate matrix. Let c be any sentential constant of $\mathfrak{L}^+$, and let φ_c be the formula $(c \to c) \equiv (c \equiv c)$. Then, by constructing appropriate $\mathfrak{L}^+$-models, it may be easily seen that both φ_c and $\neg\varphi_c$ are Cn^+-consistent. Now let $M^+ = \langle \mathfrak{A}, B\rangle$ be any $\mathfrak{L}^+$-matrix. If $\overline{c}$ is the constant of M^+ corresponding to c, let d be $(\overline{c} \dot{\rightarrow} \overline{c}) \doteq (\overline{c} \doteq \overline{c})$. Either $d \in B$ or $d \notin B$. If $d \in B$, then φ_c is true in M^+; if h is any valuation of $A^+, h(\varphi_c) = d \in B$. Thus $\neg\varphi_c$ is C_{M+}-contradictory: i.e. $C_{M+} \neq Cn^+$. Similarly, if $d \notin B$, then φ_c is C_{M+} contradictory, and *hence* $C_{M+} \neq Cn$ *in this case too. Thus* M^+ *is not adequate for* Cn, *concluding* the proof.

In the second example, we keep $\mathfrak{L}$ fixed and extend the consequence operation Cn. If Γ is any set of formulas of $\mathfrak{L}$, Cn_Γ is the consequence operation defined by $Cn_\Gamma(\Phi) = Cn(\Gamma \cup \Phi)$. It seemed natural to suppose that if Γ is an *invariant* (consistent) set of formulas (see **1.4**), then Cn_Γ would have an adequate matrix. However, for some consistent invariant Γ, Cn_Γ has no adequate matrix. In order to prove this we first prove:[8]

3.11 *Lemma. If* $M = \langle \mathfrak{A}, B\rangle$ *is any matrix, and* Φ_1, Φ_2 are each C_M-*consistent sets with disjoint supports, then* $\Phi_1 \cup \Phi_2$ *is* C_M-*consistent.*

Proof: We note first that any set Φ of formulas is C_M-consistent iff there is some valuation h of M such that $h(\Phi) \subseteq B$. Since Φ_1 and Φ_2 are separately C_M-consistent, there are valuations h_1, h_2 such that $h_i(\Phi_i) \subseteq B$, $i = 1, 2$. Since $s\Phi_1 \cap s\Phi_2 = \varphi$, we may define a valuation h by

$$h(p) = \begin{cases} h_1(p), & p \in s\Phi_1 \\ h_2(p), & \text{otherwise} \end{cases}$$

Then h agrees with h_1 on Φ_1 and with h_2 on Φ_2. Hence $h(\Phi_1 \cup \Phi_2) \subset B$, and by the first remark, $\Phi_1 \cup \Phi_2$ is C_M-consistent.

Now let Γ be the set of all formulas in $\mathfrak{L}$ of the form (a) or (b):

$$\text{(a)}\ (\varphi \rightarrow \varphi) \equiv (\psi \rightarrow \psi)$$
$$\text{(b)}\ (\varphi \equiv \varphi) \equiv (\psi \equiv \psi)$$

Γ is clearly invariant and Cn-consistent. We claim Cn_Γ has no adequate matrix. Indeed, if φ_0 and ψ_0 are the formulas

$$\varphi_0 = (p_1 \rightarrow p_1) \equiv (p_1 \equiv p_1)$$
$$\psi_0 = \neg((p_2 \rightarrow p_2) \equiv (p_2 \equiv p_2))$$

then φ_0 and ψ_0 clearly have disjoint supports and each is separately Cn_Γ-consistent. However $\varphi_0 \wedge \psi_0$ is Cn_Γ-inconsistent, as may be easily verified. By **3.11** then, for no M is $Cn_\Gamma = C_M$, which completes the proof.

4. *A Particular* **SCI** *theory.* We will not study here the intuitionist version of the **SCI** which is obtained by changing our axiom (c_2) to the following weaker form: $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi)$. Instead we will consider a particular theory Λ based on our (classical) version of **SCI** and study the relation of Λ to contemporary modal logic.

4.1 *Definition.* Let $\Lambda = Cn(\Lambda_0)$, where Λ_0 is the set of all equations of the forms

8. Using remark **3.5** it may be seen that a theorem equivalent to **3.11** was stated without proof in [3].

(a_0) $\varphi \equiv \psi$, all $\varphi, \psi \in \mathrm{Cn}(\emptyset)$, the set of logical theorems.
(a_1) $(\varphi \equiv \neg\psi) \equiv (\neg\varphi \equiv \psi)$
(a_2) $((\varphi \to \psi) \equiv (\psi \to \varphi)) \equiv (\varphi \equiv \psi)$.

Λ is clearly an invariant theory[9], and in view of (a_0), we let 1 be an abbreviation for some member of $\mathrm{Cn}(\emptyset)$, say $p \vee \neg p$, and let 0 be an abbreviation of $p \wedge \neg p$. Hence

(a_3) $\varphi \equiv 1 \in \Lambda$ if $\varphi \in \mathrm{Cn}(\emptyset)$, and
$\varphi \equiv 0 \in \Lambda$ if $\neg\varphi \in \mathrm{Cn}(\emptyset)$.

Note that 1 and $(\varphi \equiv \varphi) \equiv 1$ are in Λ. Also (a_4) - (a_{10}) are in Λ.

(a_4) $\neg\neg\varphi \equiv \varphi$
(a_5) $(\varphi \equiv 1) \to \varphi$
(a_6) $(\varphi \equiv \psi) \leftrightarrow ((\varphi \equiv \psi) \equiv 1)$
(a_7) $((\varphi \to \psi) \equiv 1 \to ((\psi \to \varphi) \equiv 1) \to \varphi \equiv \psi$
(a_8) All "Boolean" equations: e.g.
$(\varphi \to \psi) \equiv (\neg\varphi \vee \psi)$
$\varphi \wedge (\psi \vee \theta) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \theta)$
$\neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi$
(a_9) $((\varphi \wedge \psi) \equiv 1) \leftrightarrow ((\varphi \equiv 1) \wedge (\psi \equiv 1))$
(a_{10}) $((\varphi \to \psi) \equiv 1) \to ((\varphi \equiv 1) \to (\psi \equiv 1))$

Proof (*sketch*): (a_4) follows from (a_1). By (a_3) and (RL) (in **1**) one obtains (a_5) and (a_6). (a_7) follows from (a_2). All formulas in (a_8) follow from (a_3) and (a_7). (a_9) may be derived from (a_0) and (RL); (a_{10}) from (a_0), (RL) and (a_7).

4.3 *Proposition. The theory* Λ *has the following property* (*which we call property N*): $\varphi \in \Lambda$ iff $\varphi \equiv 1 \in \Lambda$.

Proof: If φ is a logical axiom (*TFA* or *IDA*) or an axiom of Λ, then $\varphi \equiv 1 \in \Lambda$ by (a_0) and (a_6). By (a_{10}) this property is preserved under modus ponens. The converse is trivial.

We may now verify that the following formulas are also in Λ:

(a_{11}) $(\varphi \equiv 1) \equiv 1$ if $\varphi \in \Lambda$
(a_{12}) $((\varphi \wedge \psi) \equiv 1) \equiv ((\varphi \equiv 1) \wedge (\psi \equiv 1))$
(a_{13}) $((\varphi \equiv 1) \equiv 1) \equiv (\varphi \equiv 1)$
(a_{14}) $((\varphi \equiv 1) \to \varphi) \equiv 1$

Our aim is to show that Λ is the set of all formulas of $\mathfrak{L}$ which are valid in every topological Boolean algebra.

A topological Boolean algebra (**T.B.A.**) $\mathfrak{B}$ is a Boolean algebra (with respect to the usual operations $-, \cap, \cup$, having maximal and minimal elements 1 and 0 respectively) together with an *interior operator* I on $\mathfrak{B}$ which satisfies the following conditions:

9. (a_1) is redundant. All Boolean equations, e.g. $\neg\neg\varphi \equiv \varphi$ and $(\varphi \equiv \neg\Psi) \leftrightarrow (\neg\varphi \equiv \Psi)$ follow from (a_0) and (a_2). (a_1) follows from (a_7) and Proposition **4.3**.

$$\mathrm{I}(a \cap b) = \mathrm{I}(a) \cap \mathrm{I}(b); \ \mathrm{I}\mathrm{I}a = \mathrm{I}a; \ \mathrm{I}a \dot{\rightarrow} a = \mathrm{I}1 = 1$$

where $a \dot{\rightarrow} b = -a \cup b$. Now define the operations $\dot{\equiv}$ and C by: $\mathrm{C}a = -\mathrm{I}(-a)$ and $a \dot{\equiv} b = \mathrm{I}((a \dot{\rightarrow} b) \cap (b \dot{\rightarrow} a))$. Then $\mathrm{I}a = a \dot{\equiv} 1$ and $\mathrm{C}a = -(a \dot{\equiv} 0)$; also $a \dot{\equiv} b = 1$ iff $a = b$. It is now easy to show that the **SCI**-algebra $\mathfrak{B} = \langle B, -, \dot{\rightarrow}, \dot{\equiv} \rangle$ has the property that the **SCI**-filters (definition **1.6**) are precisely the proper *Boolean* filters of B. (Our notation is not consistent: we should write $\dot{\neg}$ for the operation - in the Boolean algebra.) Consequently, one may consider the formulas which are *true* in the (topological) matrix $\langle \mathfrak{B}, F \rangle$ where F is any **SCI** (i.e. proper-Boolean) filter in $\mathfrak{B}$, and the formulas which are *valid* in $\mathfrak{B}$. Since the intersection of any collection of filters is a filter, a formula is valid in $\mathfrak{B}$ iff it is true in $\langle \mathfrak{B}, \{1\} \rangle$.

A filter F in $\mathfrak{B}$ is *topological* if $\mathrm{I}a \in F$ whenever $a \in F$. It is easily checked that the prime (**SCI**) filters in $\mathfrak{B}$ are precisely the prime Boolean filters and the normal (**SCI**) filters are those filters F such that the *only* topological filter contained in F is $\{1\}$, the unit filter.

4.4 *Lemma. Every formula in* Λ *is valid in any topological Boolean algebra.* (*Hence* Λ *is consistent*).

Proof: One need only check that all formulas (a_0), (a_1) and (a_2) are valid in any **T.B.A.** and note that validity is preserved by modus ponens.

Consider the canonical matrix $\langle \mathfrak{L}, \Lambda \rangle$ and the corresponding quotient matrix $M_\Lambda = \langle \mathfrak{L}/\sim, \Lambda/\sim \rangle$, where $\sim$ is the congruence defined by the filter Λ: (see remarks following **1.12**).

4.5 *Lemma.* $\mathfrak{L}/\sim$ *is a topological Boolean algebra, and* $\Lambda/\sim$ *is the unit filter in* $\mathfrak{L}/\sim$.

Proof: The operations $-$, $\dot{\rightarrow}$, $\dot{\equiv}$ are defined on $\mathfrak{L}/\sim$ as indicated after **1.12**. The additional operations I, $\cap$, and the elements 0 and 1 are defined by:

$$\mathrm{I}|\varphi| = |\varphi \equiv \mathit{1}|; \ |\varphi| \cap |\psi| = |\varphi \wedge \psi|;$$
$$1 = |\mathit{1}| (= |p \rightarrow p|), \ 0 = |\mathit{0}| \ (= |p \wedge (\neg p)|)$$

where $|\varphi|$ is the congruence class of the formula φ. By (a_8) $\mathfrak{L}/\sim$ is a Boolean algebra. Since Λ has the property N, we see that $\Lambda/\sim$ is the unit filter. Finally, from (a_{11}) - (a_{14}), it follows that I is an interior operator on $\mathfrak{L}/\sim$.

Remark. Since Λ is invariant and consistent it follows that $|p_i| \neq |p_j|$ if $i \neq j$. On the other hand, the family $VAR/\sim$ of all cosets $|p_i|$ clearly generates the whole algebra $\mathfrak{L}/\sim$. Moreover, using the techniques of universal algebra we are able to show that $\mathfrak{L}/\sim$ is freely generated by $VAR/\sim$ in both following versions: (1) every mapping of $VAR/\sim$ into $\mathfrak{L}/\sim$ can be extended to an endomorphism of $\mathfrak{L}/\sim$ and (2) every mapping of $VAR/\sim$ into any topological algebra $\mathfrak{A}$ can be extended to an homomorphism of $\mathfrak{L}/\sim$ into $\mathfrak{A}$.

4.6 Theorem. *The theory* Λ *is the set of all formulas of* $\mathfrak{L}$ *which are valid in every topological Boolean algebra.*

Proof: It suffices, in view of lemma **4.4** to show that for every formula φ which is not in Λ there exists a topological matrix $M \equiv \langle \mathfrak{B}, F\rangle$ and a valuation of which does not satisfy φ in M. The matrix M_Λ and the natural homomorphism of $\mathfrak{L}$ onto $\mathfrak{L}/\sim$ will do. If $\varphi \epsilon \Lambda$ then there exists a complete superset Φ of Λ so that $\varphi \notin \Phi$. It follows that $|\varphi| \notin \Phi/\sim$. But $\Phi/\sim$ is a prime filter in $\mathfrak{L}/\sim$. Hence, $|\varphi| \neq 1$. Thus φ is not valid in M_Λ.

The last theorem shows that Λ is the set of all theorems of the system S_4 of the modal logic introduced by C. I. Lewis. See [2] and [5]. To see this more distinctly the reader may enlarge the language by adding new connectives $\Box$, $\Diamond$, and supplement the set Λ_0 by two definitional formulas of the form: $\Box\varphi \equiv (\varphi \equiv 1)$ and $\Diamond\varphi \equiv \neg(\varphi \equiv 0)$. The theory Λ so enlarged will then contain formulas of the form $(\varphi \equiv \psi) \equiv \Box(\varphi \leftrightarrow \psi)$.

One might conclude that S_4 modal logic simply is the **SCI** supplemented by additional logical axioms like (a_0), (a_1), (a_2) or, in other words, that the modal theories based on S_4 simply are extensions of Λ, i.e. those theories in our sense which include Λ. However, we think this is not so. The crucial point is the Gödel rule:

From φ infer $\Box\varphi$, i.e. $\varphi \equiv 1$.

There exist axiomatizations of the system S_4 which do not use the Gödel rule, [5]. However, many papers on modal logic do make essential use of this rule. (Compare [6], Ch. 11, and the collection of papers in [1].) We conjecture that simplicity is not the only reason for the use of the Gödel rule in modal theories. We think that the very meaning of the necessity connective $\Box$ which underlies modal logic and which we dare to consider intensional, forces the modal consequence and modal theories to be closed under the Gödel rule. To clarify the relation of **SCI** to the modal logic let us introduce some precise definitions.

4.7 *Definition*. $C^*_M(\Phi)$ is the smallest set of formulas of $\mathfrak{L}$, closed under the rule modus ponens and the Gödel rule which contains *TFA*, *IDA*, (a_0), (a_1), (a_2) and the set Φ. A set of formulas Φ is called an S_4 modal theory if $C^*_M(\Phi) = \Phi$.

4.8 *Proposition. The operation C^*_M, called S_4 modal consequence, has the properties* C1 - C3. *Moreover, Φ is a S_4 modal theory if and only if Φ is an extension of Λ which has the property N, i.e. $\varphi \epsilon \Phi$ if and only if $(\varphi \equiv 1) \epsilon \Phi$. In particular, Λ is the smallest S_4 modal theory, $\Lambda = C^*_M(\emptyset)$.*

The difference between S_4 modal theories and extensions of Λ is reflected in the Lindenbaum-Tarski algebra $\mathfrak{L}/\sim$.

4.9 *Proposition. The mapping $\Phi \mapsto \Phi/\sim$ is a one to one correspondence* (1) *between consistent extensions of Λ and proper Boolean filters in $\mathfrak{L}/\sim$ and* (2) *between consistent S_4 modal theories and proper topological filters in $\mathfrak{L}/\sim$.*

4.10 *Corollary. There exist consistent extensions of* Λ *which are not* S_4 *modal theories. In particular, there exist consistent extensions of* Λ *which, for some* φ, *contain* φ *and* $\neg(\varphi \equiv 1)$, *while any* S_4 *modal theory which, for some* φ, *contains* φ *and* $\neg(\varphi \equiv 1)$, *obviously is inconsistent.*

The Gödel rule is essential for the modal consequence and theories in the sense that it cannot be replaced by any set of axioms.

4.11 *Proposition. There exists no set of formulas* Φ_0 *such that for every* Φ, $C^*_M(\Phi) = Cn(\Lambda \cup \Phi_0 \cup \Phi)$

Proof: Suppose to the contrary, that Φ_0 is such a set. Since $C^*_M(\varphi) = \Lambda = Cn(\Lambda \cup \Phi_0)$ we infer that $\Phi_0 \subseteq \Lambda$. Hence, $C^*_M(\Phi) = Cn(\Lambda \cup \Phi)$ for all $\Phi \subseteq Fm$. This is, however, not possible because $Cn(\Lambda \cup \{p_1, \neg(p_1 \equiv 1)\}) \neq Fm$ but $C^*_M(\{p_1, \neg(p_1 \equiv 1)\}) = Fm$.

Remark. The relation noted above, of the **SCI** to the modal theory S_4, may be extended to the case of Lewis' system S_5. To this end, one has to replace the theory Λ by a stronger one, say Λ^*, which is defined as $Cn(\Lambda^*_0)$ where Λ^*_0 arises from Λ_0 by the addition of all formulas of the following form:

$$((\varphi \equiv \psi) \equiv 0) \equiv \neg(\varphi \equiv \psi)$$

Here, instead of arbitrary topological algebras one has to consider the class of "self-dual" topological Boolean algebras (i.e. those in which every open element is closed) whose elements satisfy

$$(a \dot{\equiv} b) \dot{\equiv} 0 = \dot{\neg}(a \dot{\equiv} b).$$

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Stephen L. Bloom

Stevens Institute of Technology
Hoboken, New Jersey

Roman Suszko

Stevens Institute of Technology
Hoboken, New Jersey

and

The Polish Academy of Sciences
Warszawa, Poland