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AN EXTENSION OF VENN DIAGRAMS

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In §15 of *Methods of Logic*, Quine states some limitations of Venn Diagrams as a decision-procedure. He considers a problem ("the class of '00") of which the pattern is:

(1) All F who are G are $H \supset$ some F are not G
All F are G v All F are H
∴ All F who are H are $G \supset$ some F who are not H are G,

and comments that Venn Diagrams are suited to handling expressions like 'all F who are G are H', and the techniques of propositional logic to handling connectives like '$\supset$' and 'v'. He asks 'just how may we splice the two techniques in order to handle a combined inference of the above kind?' (p. 82). In the sections that follow he introduces quantifiers, and eventually (§21) reaches a decision-procedure for problems like (1).

What follows here is an extension of Venn Diagrams (EVD) to deal with such problems. The method provides, not only a simple decision-procedure, but a pedagogically useful introduction to types of reasoning found in deductive systems.

A Venn Diagram depicts an assertion concerning the emptiness or non-emptiness of certain classes. Hatching is an assertion or a conjunction of assertions of emptiness; the 'cross' an assertion and the 'bar' a disjunction of assertions of non-emptiness. Consequently, the assertion depicted in any Venn Diagram may be negated by interchanging hatching and a cross, or hatchings and bars.* We can call this 'Diagrammatic Negation'; EVD will cite it as 'Diag. Neg.'.

The superposition of Venn Diagrams will depict the conjunction of the assertions made by each of them. Such superposition must respect the method of representation of the diagrams: the distinctness of classes must not be destroyed by the superposition, and the subclasses engendered by the classes considered must be fully represented. Examples of superposition are:

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*See Addendum to this article for some suggestions here.

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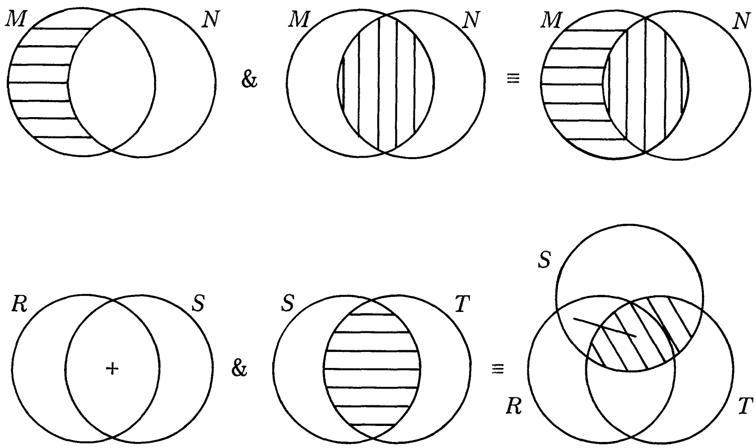
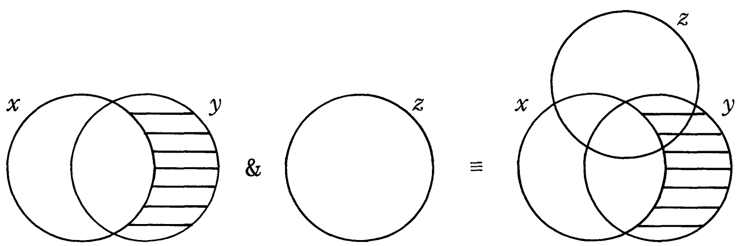
(2)



The superposition may be vacuous, adding only an extra, unmarked circle:

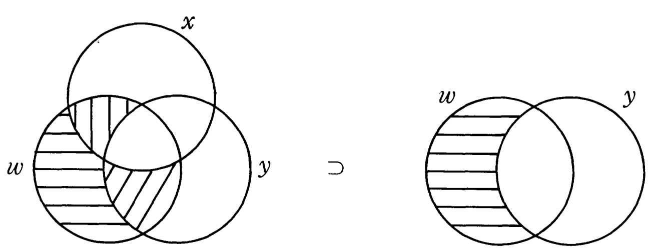
(3)



Notice, in the second example in (2), how superposition demands that the cross between R and S be replaced by the bar of alternation; and how the hatching between S and T excludes one of the alternatives. EVD will cite the procedure of diagrammatic superposition as 'Diag. Sup.'.

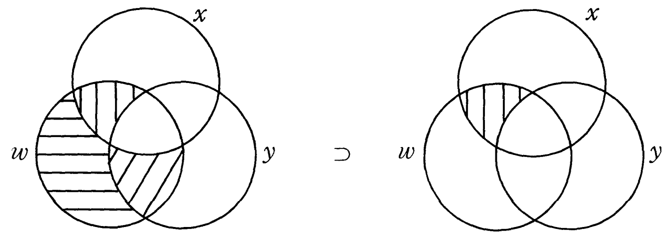
Diagrammatic Extraction is simply the foregoing procedure reversed.

(4)



The antecedent of (4) asserts, *inter alia,* the emptiness of the class 'W and not-Y'; and it is just this that the consequent asserts. We could replace the consequent of (4) by a diagram in which the right-hand circle depicted the class X, with no other alteration made to the existing consequent. But notice that no two-circled diagram making assertions about the classes X and Y can be extracted from the antecedent of (4). All we are entitled to extract is a diagram depicting the assertion that no X *which is a* W is a Y:

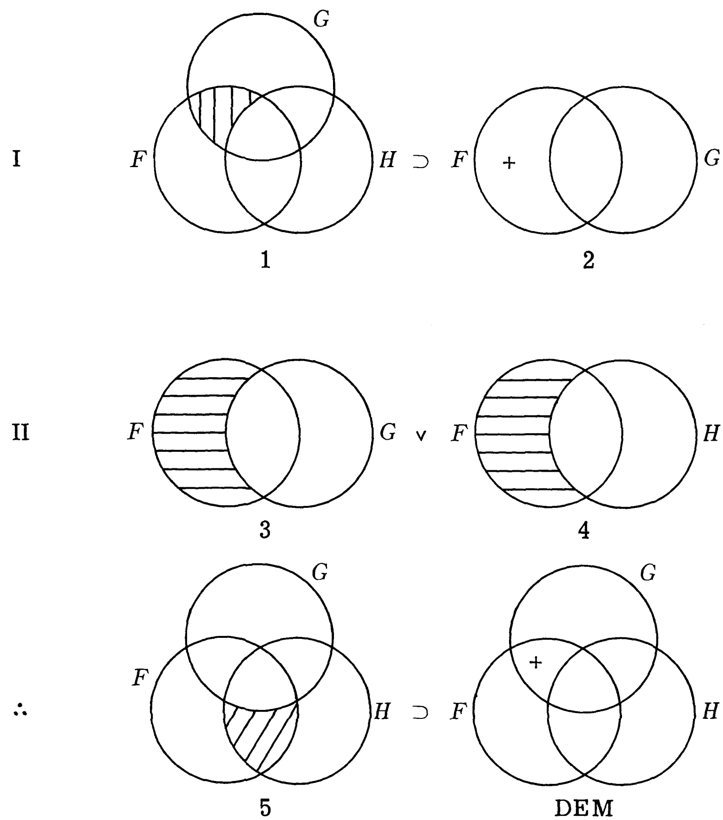
(5)



EVD will cite this procedure as 'Diag. Extr.'

All procedures of EVD concern the structure of the diagrams which depict assertions; so EVD links them, as any assertions might be linked, with the connectives of propositional logic, and employs the basic manipulations of that logic. EVD uses also the Rule of Conditionalisation, by which a proof of B on the hypothesis A is taken as a proof, independent of that hypothesis, that $A \supset B$. Such uses of propositional logic are cited as 'PL'.

(6)



Given I and II; to prove that $5 \supset$ DEM. Suppose 5 to be true; on this added hypothesis, to prove DEM. Either 3 is true or 4 is true (II).

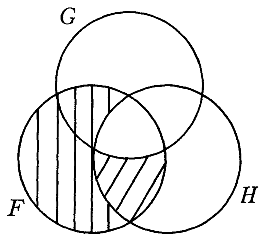
Suppose 3 is true

$3 \equiv \sim 2$ (Diag. Neg.)

$\sim 2 \supset \sim 1$ (I, PL)
$\sim 1 \equiv$ DEM (Diag. Neg.)
So $3 \supset$ DEM (PL)
So DEM (PL)

Suppose 4 is ture

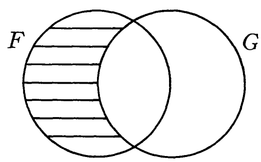
$4 \;\&\; 5 \equiv 7$ (Diag. Sup.)



7

$7 \supset 3$ (Diag. Extr.)

$3 \supset$ DEM (Proved above)



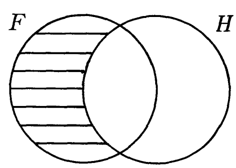
So DEM (PL).

So, on the hypothesis that 5 is true, DEM is true. So $5 \supset$ DEM (PL).

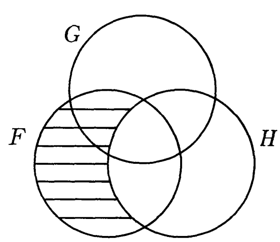
Notice that EVD displays a point not immediately evident in Quine's treatment of the problem: one of the elements in the premises (our 4) is inconsistent with the consequent of the conclusion (our DEM). Indeed, our consideration of the hypothesis that 4 is true might have led us to the desired conclusion by a different route:

Suppose 4 is true

4: (hypothesis)

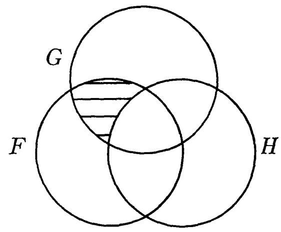


So 8: (4, Diag. Sup.)



8

So 9: (8, Diag. Extr.)



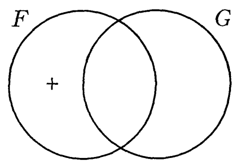
9

$9 \equiv 1$ (Diag. Extr.)

$1 \supset 2$ (I)

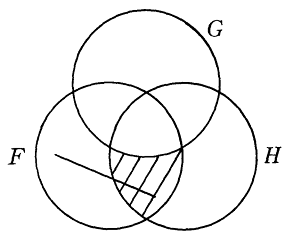
So $4 \supset 2$ (PL)

So 2 (PL)



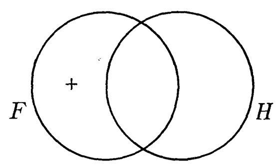
5 is true (hypothesis)

$2 \ \& \ 5 \equiv 10$ (Diag. Sup.)



10

$10 \supset 11$ (Diag. Extr.)



11

$11 \equiv \sim 4$ (Diag. Neg.)

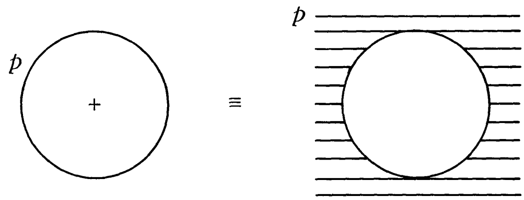
So $2 \ \& \ 5 \supset \sim 4$ (PL)

So ~ 4 (PL)

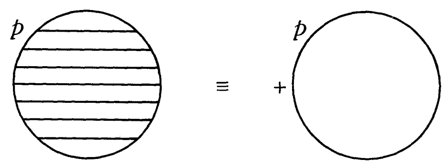
So the hypothesis that 4 is true leads to its contradictory; so the hypothesis must be rejected, and the conclusion of the hypothesis that 3 is true must be accepted: which is DEM.

No change in this method is necessitated by problems where, in EVD, the connectives link sentence-letters as well as diagrams. (Quine gives such a problem towards the end of §21, "the Bissagos Report"). But since quantificational notation also allows sentence-letters to occur within the scope of quantifiers, it may be asked whether EVD can provde expressions analogous to such formulae.

There is no need for EVD to be able to do this, since any such letter may always be exported from the scope of the quantifier within which it falls. But unmodified formulae like '$(x)\ (Fx \supset p)$' can be depicted in EVD, and so subjected to its deductive processes. We simply allot circles to the sentence-letters as if they were classes, but observe the following Propositional Convention ('Prop. Conv.'):

(7) 

Each diagram in (7) depicts the proposition that p. Prop. Conv. embodies the two-valued character of propositional logic; it points to the incompleteness of the analogy between that logic and the calculus of classes. Diag. Neg. applied to (7) yields

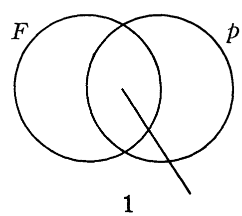
(8) 

Each diagram in (8) depicts the proposition that not-p.

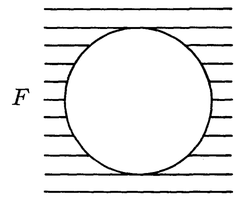
We prove one equivalence involving such formulae: the others present no special difficulty.

(9) $(\exists x)\ (Fx \supset p) \equiv ((x)\ (Fx) \supset p)$

Let us first prove that LHS $\supset$ RHS. The antecedent of this conditional is depicted thus:

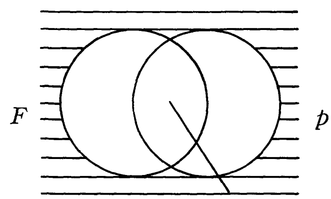


(The justice of the transcription will be more easily seen if we think of '$Fx \supset p$' as '$\sim Fx \vee p$'). Now to prove the consequent of the conditional on this hypothesis. We suppose $(x)\ (Fx)$ to be true; we must now show that p is true. $(x)\ (Fx)$ is depicted as:



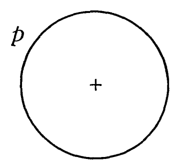
2

1 & 2 ≡ 3: (Diag. Sup.)



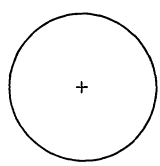
3

3 ⊃ 4: (Diag. Extr.)



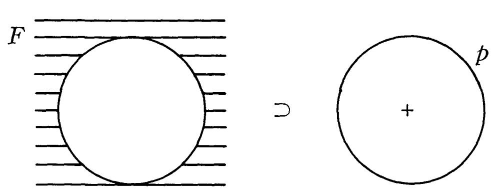
4

So (1 and 2 are hypotheses)



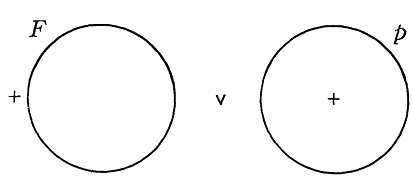
So, granted that 1 is true, 4 follows from the hypothesis that 2. That is, p is true, given that $(x)(Fx)$ is true. So RHS.

Now to prove that RHS ⊃ LHS: that is, given $(x)(Fx) \supset p$, to prove $(\exists x)(Fx \supset p)$. $(x)(Fx) \supset p$ we may transcribe as



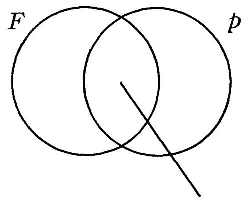
1 2

But 1 ⊃ 2 is equivalent to



(PL; Diag. Neg.)

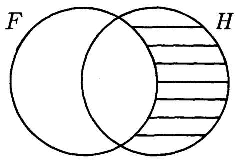
And this disjunction of assertions we may, by superposition, depict thus:

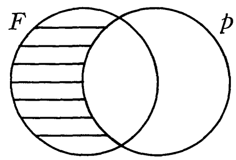


which, as we have seen, depicts $(\exists x)(Fx \supset p)$. So RHS $\supset$ LHS.

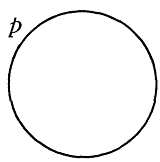
We conclude with a problem involving these procedures. Its quantificational expression would be:

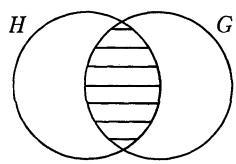
(10) $((x)(Hx \supset Fx)\ \&\ (x)(Fx \supset p)) \supset (\sim p \supset (x)(Hx \supset \sim Gx))$.

I 

& 

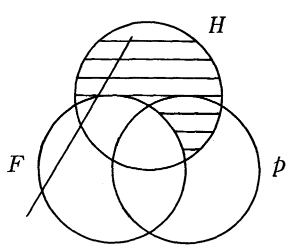
1 2

∴ + 

⊃ 

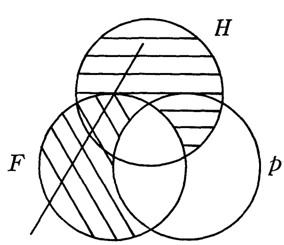
3 DEM

Given that I is true; to show that, if 3 is true, DEM is true.

1 & 3 ≡ 4: 

(Diag. Sup.) (Notice how the cross in 3 has to be changed)

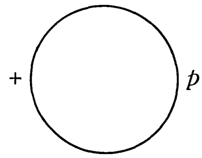
4

4 & 2 ≡ 5: 

(Diag. Sup.)

5

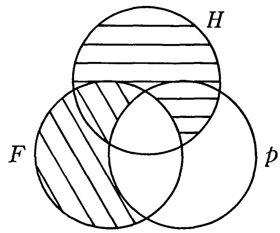
$5 \supset 6$:



6

(Diag. Extr.)

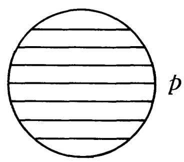
$5 \supset 7$:



7

(Diag. Extr.)

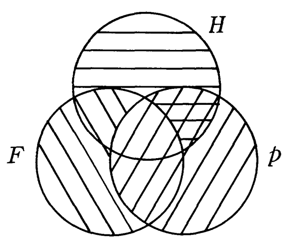
$6 \equiv 8$:



8

(Prop. Conv.)

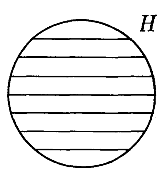
$7 \,\&\, 8 \equiv 9$:



9

(Diag. Sup.)

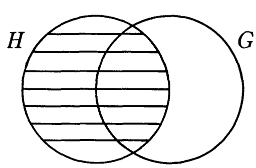
$9 \supset 10$:



10

(Diag. Extr.)

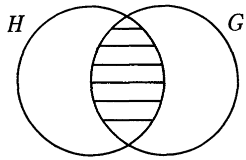
$10 \equiv 11$:



11

(Diag. Sup.)

11 ⊃ DEM



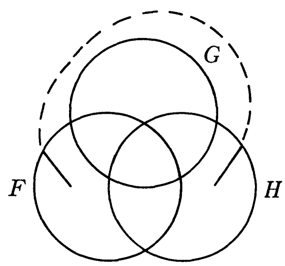
(Diag. Extr.)

So 1 & 2 & 3 ⊃ DEM (PL)
So DEM (PL)

Addendum

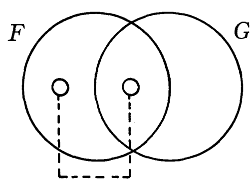
The usual 'bar' notation presupposes that the areas of the diagram with which it is concerned are contiguous. It is not difficult to adapt the notation to cases where contiguity does not obtain.

(11)



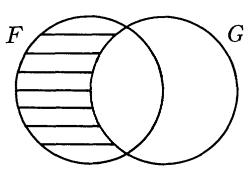
In (11) it is asserted that one or both of two classes is non-empty: *F*'s that are neither *G* nor *H*, and *H*'s that are neither *F* nor *G*. The dotted line does no more than link the two elements of the bar; it says nothing of the class depicted by the area outside the circles. As for the 'hatching' notation, it asserts emptiness of a class, or a conjunction of such assertions. Should cases arise when the *disjunction* of such assertions needs to be depicted, it might be effected thus:

(12)

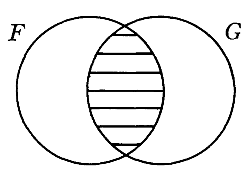


What (12) depicts can, of course, be expressed with the aid of the sign of propositional disjunction:

(13)



v



For EVD, the choice between (12) and (13) is a matter of tactics, not of principle.

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