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A NOTE ON A THEOREM OF C. YATES

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1. *Introduction.* Let E denote the collection of all non-negative integers. We recall from [2] that a one-to-one function t_n (from E into E) is *regressive* if the mapping $t_{n+1} \to t_n$ has a partial recursive extension; and is *retraceable* if it is both strictly increasing and regressive. An infinite set is said to be *regressive* if it is the range of a regressive function; and is *retraceable* if it is the range of a retraceable function. A one-to-one function a_n is *indexed* if the mapping $a_n \to n$ has a partial recursive extension; and a set is *indexed* if it is the range of an indexed function. In [4] C. Yates proved the following result:

Theorem A. (Yates). *Let α be an infinite set. Then α is strongly hyperhyperimmune $\Longleftrightarrow \alpha$ contains no infinite retraceable subset.*

In this paper we arrive at a new proof of this result. It is somewhat easier than the proof in [4] (see also: [3, pp. 250-251]), and also, it makes use of a basic property of indexed sets.

2. *Indexed sets.* Let $\{w_n\}$ denote the usual effective enumeration of the collection of all recursively enumerable sets. We call a sequence $\{w_{f(x)}\}$ an *array* if

(a) f is a one-to-one recursive function,
(b) for each x, $w_{f(x)} \neq \phi$, and
(c) for each x and y, if $x \neq y$ then $w_{f(x)} \cap w_{f(y)} = \phi$.

We recall from [3, p. 250] that an infinite set α is said to be *strongly hyperhyperimmune* if for every array $\{w_{f(x)}\}$, there is a number x such that $w_{f(x)} \cap \alpha = \phi$.

Theorem 1. *Let α be an infinite set. Then α is a strongly hyperhyperimmune $\Longleftrightarrow$ α contains no infinite indexed subset.*

Proof. ($\Longrightarrow$) Assume that α is strongly hyperhyperimmune and suppose that α contains an infinite indexed subset. Let a_n be an indexed function that ranges over a subset of α and let p denote a partial recursive function such that, for each number n,

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$$a_n \in \delta p \text{ and } p(a_n) = n \quad .$$

It can be readily seen that there is a one-to-one recursive function f such that, for each number n,

$$w_{f(n)} = \{x \mid x \in \delta p \text{ and } p(x) = n\} \quad .$$

It follows that $\{w_{f(n)}\}$ is an array and, for each number n,

$$a_n \in \alpha \cap w_{f(n)} \quad ;$$

and therefore, for each number n, $\alpha \cap w_{f(n)} \neq \phi$. We could conclude then that α would not be strongly hyperhyperimmune, and this we know is not the case. It follows therefore, that α does not contain an infinite indexed subset.

($\Longleftarrow$) Assume that α is not strongly hyperhyperimmune. Let $\{w_{f(n)}\}$ be an array such that, for each number n,

(1) $$\alpha \cap w_{f(n)} \neq \phi \quad .$$

We wish to show that α contains an infinite indexed subset. Let

$$w = \bigcup_{n=0}^{\infty} w_{f(n)} \quad .$$

Because f is a recursive function, it follows that w is a recursively enumerable set. Also, because $\{w_{f(n)}\}$ is an array, we see that the function q defined by

(2) $$\delta q = w \text{ and } q(x) = n \text{ for } x \in w_{f(n)}$$

will be partial recursive. For each number n, let

(3) $$a_n = (\mu y)\left[y \in \alpha \cap w_{f(n)}\right] \quad .$$

In view of (1), we see that a_n is an everywhere defined one-to-one function. In addition, it follows from (2) that $q(a_n) = n$, and therefore a_n is an indexed function. Combining this fact with (3), we can conclude that α contains an infinite indexed subset. This is the desired result and completes the proof.

Remark. It is easy to verify that every regressive function is indexed, and hence, that every regressive set is indexed. We now state two results, the first is due to J. Barback [1] and the second is due to J. Dekker [2]; the second result we will state without proof.

Lemma 1. (Barback). *Let α be an infinite set. Then α contains an infinite indexed subset $\Longleftrightarrow$ α contains an infinite regressive subset.*

Proof. The direction ($\Longleftarrow$) in the lemma is clear. For the direction ($\Longrightarrow$), let a_n be an indexed function that ranges over a subset of α. We may assume that $a_0 \neq 0$. Let the function t_n be defined by

$$t_0 = a_0 \text{ and } t_{n+1} = a_{t_n} \quad .$$

It is readily seen that t_n is a one-to-one function and ranges over a subset of α. In addition, the mapping

$$t_{n+1} = a_{t_n} \to t_n$$

will have a partial recursive extension, since a_n is an indexed function. It follows that t_n is a regressive function and ranges over an (infinite) regressive subset of α.

Lemma 2. (Dekker) [2, p. 90]. *Let α be any set. Then α contains an infinite regressive subset $\Longleftrightarrow$ α contains an infinite retraceable subset.*

Remark. Combining Theorem 1 and Lemmas 1 and 2, we see that one obtains a proof of Yates' Theorem A. In addition, it also follows that for α any infinite set, then the following four conditions are equivalent:

(a) *α is strongly hyperhyperimmune,*
(b) *α contains no infinite indexed subset,*
(c) *α contains no infinite regressive subset,*
(d) *α contains no infinite retraceable subset.*

REFERENCES

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