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ON A PROPERTY OF CERTAIN PROPOSITIONAL FORMULAE

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In [1] section 4 Łukasiewicz gives a theorem concerning the law of syllogism. The present paper presents a much more general theorem from which the Łukasiewicz theorem can be derived. Sections 1 and 2 present our theorem; a brief discussion of its application and its relationship to the Łukasiewicz theorem is given in section 3.

1. *Preliminaries and Statement of Theorem.* We use 'P', 'Q', 'R', with and without subscripts to denote well-formed propositional formulae. '$\{P_1, \ldots P_n\}$', '$\{Q_1, \ldots Q_m\}$' and so on denote ordered sets of such formulae. 'Φ' is used for a constant operation under the substitution rule; 'Φ^n' denotes n repetitions of the operation; '$\Phi\{P_1, \ldots P_n\}$' is an abbreviation for '$\{\Phi P_1, \ldots \Phi P_n\}$'. '$\cup$' and '$\subset$' have their usual meanings. We use '$\sim$' to denote a relationship between an ordered set of propositional formulae and a single formula which is defined as follows.

Definition $\{P_1, \ldots P_n\} \sim Q$ is defined inductively in two steps:
a. Let Q be a member of $\{P_1, \ldots P_n\}$: then $\{P_1, \ldots P_n\} \sim Q$.
b. For some R, let $\{P_1, \ldots P_n\} \sim R$ and let $\{P_1, \ldots P_n\} \sim CRQ$: then $\{P_1, \ldots P_n\} \sim Q$.

Less formally, our relationship holds between a formula and any ordered set of formulae of which it is a member, or from which it can be obtained by one or more applications of Modus Ponens. Our theorem can now be stated.

Theorem *For any well-formed formula of the form* $CP_1 \ldots CP_nCQ_1 \ldots CQ_{m-1}Q_m$ $(m, n \geq 1)$ *if the following three conditions are satisfied:*

a. $\Phi CQ_1 \ldots CQ_{m-1}Q_m = CP_1 \ldots CP_nCQ_1 \ldots CQ_{m-1}Q_m$
b. Q_m *is elementary*
c. $\{P_1, \ldots P_n, Q_1, \ldots Q_{m-1}\} \sim Q_m$

then

$$\Phi\{P_1, \ldots P_n\} \cup \Phi^2\{P_1, \ldots P_n\} \cup \ldots \Phi^m\{P_1, \ldots P_n\} \sim CP_1 \ldots CP_nCQ_1 \ldots CQ_{m-1}Q_m$$

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To illustrate the theorem we may take $CP_1 \ldots CP_nCQ_1 \ldots CQ_{m-1}Q_m = CCpCqrCqCpr$, and $\Phi = p|q,\ q|CpCqr,\ r|Cpr$. The three conditions of the hypothesis are satisfied:

a. $\Phi CqCpr = CCpCqrCqCpr$
b. $Q_3 = r$ is elementary
c. $\{CpCqr,\ q,\ p\} \sim r$

Hence by our theorem $CCpCqrCqCpr$ is closed by the union of the following sets:

$\Phi\{CpCqr\} = CqCCpCqrCpr$
$\Phi^2\{CpCqr\} = CCpCqrCCqCCpCqrCprCqCpr$
$\Phi^3\{CpCqr\} = CCqCCpCqrCprCCCpCqrCCqCCpCqrCprCqCprCCpCqrCqCpr$

The reader can easily verify that this is so by noting that:

$$\Phi^3CpCqr = C\Phi CpCqrC\Phi^2CpCqrCCpCqrCqCpr$$

2. *Proof of Theorem.** Before proceeding to the proof of our theorem we give five lemmas.

Lemma 1 *For* $m < n$, $\Phi CP_1 \ldots CP_{n-1}P_n = C\Phi P_1 \ldots C\Phi P_m\Phi CP_{m+1} \ldots CP_{n-1}P_n$.

Lemma 2 *If* $\{P_1, \ldots P_n\} \sim Q$, *then* $\Phi^m\{P_1, \ldots P_n\} \sim \Phi^mQ$.

Lemma 3 *If* $\Phi\{P_1, \ldots P_n\} = \{Q_1, \ldots Q_m\}$, *then* $\Phi^l\{P_1, \ldots P_n\} = \Phi^{l-1}\{Q_1, \ldots Q_m\}$.

Proof. This follows from the lemma's hypothesis in virtue of the fact that $\Phi^l\{P_1, \ldots P_n\} = \Phi^{l-1}\,\Phi\{P_1, \ldots P_n\}$.

Lemma 4 *If* $\Phi CQ_1 \ldots CQ_{m-1}Q_m = CP_1 \ldots CP_nCQ_1 \ldots CQ_{m-1}Q_m (m, n \geq 1)$, *then for* $kn - n + l \leq m - 1$ *where* $k \geq 1$ *and* $l \leq n$

$$\Phi^k\{Q_{kn-n+1}, \ldots Q_{kn-n+l}\} = \{P_1, \ldots P_l\}.$$

Proof. We assume that the lemma's hypothesis is satisfied. Then by Lemma 1 we have

(1) $\Phi\{Q_1, \ldots Q_l\} = \{P_1, \ldots P_l\}$
(2) $\Phi\{Q_{n+1}, \ldots Q_{n+l}\} = \{Q_1, \ldots Q_l\}$.

From (2) by Lemma 3 and the hypothesis, we have

(3) $\Phi^k\{Q_{kn-n+1}, \ldots Q_{kn-n+l}\} = \Phi^{k-1}\{Q_{(k-1)n-n+1}, \ldots Q_{(k-1)n-n+l}\}$

Our lemma follows from (3) and (1).

Lemma 5 *If* $\Phi CQ_1 \ldots CQ_{m-1}Q_m = CP_1 \ldots CP_nCQ_1 \ldots CQ_{m-1}Q_m$ $(m, n \geq 1)$ *then*

*The author is indebted to Lars Svenonius for help given him in 1957 with this proof.

$$\Phi^l CQ_1 \ldots CQ_{m-1}Q_m = CR_1 \ldots CR_k CP_1 \ldots CP_n CQ_1 \ldots CQ_{m-1}Q_m$$

$(l > 1,\ k \geq 1)$ *where* $\{R_1, \ldots R_k\} \subset \Phi\{P_1, \ldots P_n\} \cup \ldots \Phi^{l-1}\{P_1, \ldots P_n\}$.

Proof. We assume the lemma's hypothesis. Then by Lemma 1 we have

(1) $\Phi^l CQ_1 \ldots CQ_{m-1}Q_m = C\Phi^{l-1}P_1 \ldots C\Phi^{l-1}P_n\Phi^{l-1}CQ_1 \ldots CQ_{m-1}Q_m$.

Our lemma follows from (1) and the hypothesis.

To prove our theorem we assume that conditions (a) through (c) of the hypothesis are satisfied. From Lemma 4 we have:

(1) $\Phi^{k+1}\{Q_{kn-n+1}, \ldots Q_{kn-n+l}\} = \Phi\{P_1, \ldots P_l\}$

and hence by the meaning of $\Phi\{P_1, \ldots P_n\}$,

(2) $\Phi^{k+1}Q_{kn-n+l} = \Phi P_l$.

By the hypothesis of Lemma 4, $kn - n + l \leq m - 1$. Hence for $n = 1$, $k \leq m - 1$ and $k + 1 \leq m$. Therefore from (2) we have

(3) $\Phi^m Q_{m-1} = \Phi^{m-k}P_l$.

Purely from the meaning of the symbols involved, we can assert

(4) $\Phi^m\{P_1, \ldots P_n, Q_1, \ldots Q_{m-1}\} = \Phi^m\{P_1, \ldots P_n\} \cup \Phi^m\{Q_1, \ldots Q_n\} \cup \ldots$
$\Phi^m\{Q_{kn-n+1}, \ldots Q_{kn-n+l}\}$

for $kn - n + l \leq m - 1$ where $k \geq 1$ and $l \leq n$. From (4) by Lemma 4 and (3) it follows that

(5) $\Phi^m\{P_1, \ldots P_n, Q_1, \ldots Q_{m-1}\} = \Phi^m\{P_1, \ldots P_n\} \cup \Phi^{m-1}\{P_1, \ldots P_n\} \cup \ldots$
$\Phi^{m-k}\{P_1, \ldots P_l\}$

and hence we derive

(6) $\Phi^m\{P_1, \ldots P_n, Q_1, \ldots Q_{m-1}\} \subset \Phi\{P_1, \ldots P_n\} \cup \ldots \Phi^m\{P_1, \ldots P_n\}$.

From condition (c) of the hypothesis, we have, by Lemma 2

(7) $\Phi^m\{P_1, \ldots P_n, Q_1, \ldots Q_{m-1}\} \sim \Phi^m Q_m$.

From (6) and (7) it follows that

(8) $\Phi\{P_1, \ldots P_n\} \cup \ldots \Phi^m\{P_1, \ldots P_n\} \sim \Phi^m Q_m$.

We now turn attention to $\Phi^m Q_m$. By condition (a) of the hypothesis we have

(9) a. $\Phi Q_m = CQ_{m-n} \ldots CQ_{m-1}Q_m$ when $n < m - 1$
b. $\Phi Q_m = CQ_1 \ldots CQ_{m-1}Q_m$ when $n = m - 1$
c. $\Phi Q_m = CP_m \ldots CP_n CQ_1 \ldots CQ_{m-1}Q_m$ when $n > m - 1$.

From (9) by condition (a) again and Lemma 1, we derive

(10) where l is the least integer such that $ln \geq m - 1$
a. $\Phi^{l+1}Q_m = CP_1 \ldots CP_n CQ_1 \ldots CQ_{m-1}Q_m$ when $ln = m - 1$
b. $\Phi^{l+1}Q_m = C\Phi P_{n-ln+m} \ldots C\Phi P_n CP_1 \ldots CP_n CQ_1 \ldots CQ_{m-1}Q_m$ when $ln > m - 1$.

By the hypothesis in (10) $m \geq l + 1$. Further $\Phi\{P_{n-ln+m} \ldots P_n\} \subset \Phi\{P_1, \ldots P_n\}$. Therefore from (10) by Lemma 5 it follows that

(11) a. $\Phi^m Q_m = CP_1 \ldots CP_n CQ_1 \ldots CQ_{m-1}Q_m$ or
b. $\Phi^m Q_m = CR_1 \ldots CR_k CP_1 \ldots CP_n CQ_1 \ldots CQ_{m-1}Q_m$

where $\{R_1, \ldots R_k\} \subset \Phi\{P_1, \ldots P_n\} \cup \ldots \Phi^{m-l-1}\{P_1, \ldots P_n\}$ $(k \geq 1)$. From (11) by the definition of $\sim$, it follows that

(12) $\Phi\{P_1, \ldots P_n\} \cup \ldots \Phi^{m-l-1}\{P_1, \ldots P_n\} \cup \{\Phi^m Q_m\} \sim CP_1 \ldots CP_n CQ_1 \ldots CQ_{m-1}Q_m$.

In virtue of the fact that $\Phi\{P_1, \ldots P_n\} \cup \ldots \Phi^{m-l-1}\{P_1, \ldots P_n\} \subset \Phi\{P_1, \ldots P_n\} \cup \ldots \Phi^m\{P_1, \ldots P_n\}$ our theorem follows from (8) and (12).

3. *Application of Theorem.* Our theorem is useful for discovering derivations for formulae within Propositional Calculus. One example is the by no means obvious derivation of *CCpCqrCqCpr* from *CqCCpCqrCpr* given in section 1. Another is given by taking $CP_1 \ldots CP_n CQ_1 \ldots CQ_{m-1}Q_m = CCpqCCqrCpr$, and $\Phi = p|Cpq,\ q|\alpha,\ r|CCqrCpr$. By our theorem $\Phi\{Cpq, Cqr\} \cup \Phi^2\{Cpq, Cqr\} \sim CCpqCCqrCpr$. The union yields the four formulae

(1) $CCpq\alpha$
(2) $C\alpha CCqrCpr$
(3) $CCpq\alpha\Phi\alpha$
(4) $C\Phi\alpha C C\alpha CCqrCprCCpqCCqrCpr$

From (1) and (3) we get $\Phi\alpha$ by Modus Ponens. With $\Phi\alpha$ and (2), two applications of Modus Ponens to (4) yield the original formula. This instance of our theorem is a proof of the Łukasiewicz theorem referred to above, which states that from any two formulae of the form (1) and (2) the law of syllogism can be derived using only Substitution and Modus Ponens. Useful derivations result when α is so chosen that $CCpq\alpha$ and $C\alpha CCqrCpr$ are theses.

REFERENCES

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