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ON SOME MODELS OF MODAL LOGICS

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The purpose of this note is to show that the models of the modal logics M, S4, *Brouwersche* and S5 defined by Drake in [1], following McKinsey [3], can be presented as models of the type defined by Kripke in [2].

Models based on a Boolean algebra $\mathfrak{A}$ were defined in [1] as follows. A model is a triple $\langle \mathfrak{A}, D, S\rangle$ where $\mathfrak{A}$ is a Boolean algebra, D is a maximal additive ideal of $\mathfrak{A}$, and S is a set of operators defined on elements of $\mathfrak{A}$ satisfying

a1) $\mathrm{s}(a \cup b) = \mathrm{s}(a) \cup \mathrm{s}(b)$,
a2) $\mathrm{s}(-a) = -\mathrm{s}(a)$ ($-a$ is the complement of a)

and

a3) *there is an* $\mathrm{s}_0 \in S$ *such that* $\mathrm{s}_0(a) = a$ *for all* $a \in A$.

In addition S may be assumed to satisfy one or both of

a4) *for each* $\mathrm{s}, \mathrm{s}' \in S$, *there is an* $\mathrm{s}'' \in S$ *such that* $\mathrm{s}\{\mathrm{s}'(a)\} = \mathrm{s}''(a)$ *for all* $a \in A$,
a5) *for any* $a_1, \ldots, a_n \in A$ *and* $\mathrm{s} \in S$, *there is an* $\mathrm{s}' \in S$ *such that* $\mathrm{s}\{\mathrm{s}'(a_1)\} = a_1, \ldots, \mathrm{s}\{\mathrm{s}'(a_n)\} = a_n$.

Defining an operation

$$*a = \bigcup_{\mathrm{s} \in S} \mathrm{s}(a),$$

corresponding to the modal operation of possibility, Drake showed that, if S is assumed to satisfy a1)-a3) {a1)-a4), a1)-a5), resp.}, then the triples $\langle \mathfrak{A}, D, S\rangle$ are characteristic for M(S4, S5). It is easy to show by the methods of [1] that, if S satisfies a1)-a3) and a5) then $\langle \mathfrak{A}, D, S\rangle$ is characteristic for the *Brouwersche* system.

In [2], triples $\langle G, K, R\rangle$ are defined with $G \in K$ and K is to be interpreted as a set of possible worlds. R is a relation on K and these triples are called model structures. A model structure is an M- (S4-, *Brouwersche*-, S5-, resp.) model structure if R is reflexive (reflexive and transitive, reflexive and symmetric, an equivalence relation). A model is a function $\Phi(A, H)$ where A ranges over subformulae of the given formula and H ranges

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over possible worlds with $\Phi(A, H) \epsilon \{T, F\}$. The interpretation of possibility is that $\Phi(A,H) = T$ if there is an $H' \epsilon K$ with $H R H'$ such that $\Phi(A, H') = T$. In [2] it is shown that the model structures are characteristic for the modal logics given in the name.

Now let $\langle A, D, S\rangle$ be a model of the first type and define, for $s_1, s_2, s_3 \epsilon S$, $s_1 s_2 = s_3$ if $s_1\{s_2(a)\} = s_3(a)$ for all $a \epsilon A$. Define a relation $\rho \subseteq S \times S$ as follows: $\langle s_1, s_2\rangle \epsilon \rho$ if and only if there is an $s \epsilon S$ such that $s_1 s = s_2$.

Lemma. *If S satisfies* a3) {a4), a5), resp.}, *then ρ is reflexive (transitive, symmetric).*

Proof: If a3) holds, then $ss_0 = s$ for all $s \epsilon S$, i.e., $\langle s, s\rangle \epsilon \rho$ and ρ is reflexive. If $\langle s, s'\rangle$ and $\langle s', s''\rangle \epsilon \rho$, then $s'' = s' s_1$ and $s' = ss_2$ for some $s_1, s_2 \epsilon S$. If a4) holds, then $s_2 s_1$ is defined so that $s'' = s\langle s_2 s_1\rangle$ and $\langle s, s''\rangle \epsilon \rho$ as required. Finally, suppose that $\langle s, s'\rangle \epsilon \rho$ so that $ss_1 = s'$ for some $s_1 \epsilon S$. A consequence of a5) [3] is that, for each $s_1 \epsilon S$, there is an $s_2 \epsilon S$ such that $s_1\{s_2(a)\} = a$ for all $a \epsilon A$ so that $s = s' s_2$ and $\langle s', s\rangle \epsilon \rho$.

For $a \epsilon A$ and $s \epsilon S$ define $\Phi(a, s)$ as follows:

$$\begin{cases} \Phi(a, s) = T \text{ if } s(a) \epsilon D, \\ \Phi(a, s) = F \text{ if } s(a) \notin D. \end{cases}$$

It follows from the properties of D that $\Phi(a \cup b, s) = T$ if and only if at least one of $\Phi(a, s) = T$, $\Phi(b, s) = T$ holds and that $\Phi(-a, s) = T$ if and only if $\Phi(a, s) = F$. Consider now $\Phi(*a, s) = \Phi\left\{\bigcup_{s\epsilon S} s'(a), s\right\}$. Then, by definition, $\Phi(*a, s) = T$ if and only if $\bigcup_{s\epsilon S} s\{s'(a)\} \epsilon D$, i.e., if and only if $ss'(a) \epsilon D$ for some $s' \epsilon S$. That is, $\Phi(*a, s) = T$ if and only if there is an $s_1 \epsilon S$ such that $\Phi(a, s_1) = T$ and $\langle s, s_1\rangle \epsilon \rho$.

Taking the lemma above with these considerations, we have

Theorem. *If $\langle \mathfrak{A}, D, S\rangle$ is a Boolean algebra model of the appropriate type, then $\langle s_0, S, \rho\rangle$ is a model structure of the same type with $\Phi(a, s)$ an associated model.*

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