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A NON-STANDARD PROOF IN THE THEORY OF INTEGRATION

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In the process of studying the relationships between the standard and non-standard definitions of the Lebesgue and Riemann integrals, the following proof occurred and seemed to have a certain amount of intrinsic interest of its own. Although patterned after the usual classical proofs (*cf.* [3], pp. 248-249), it is simpler in that one can do the whole proof using only one partition of the domain, and more natural in that one just decides what ought to be true and then computes that it is true.

For the terminology of non-standard analysis and the elementary properties of the Riemann and Lebesgue integrals when viewed non-standardly, we refer the reader to [2], especially Chapters III and V.

Theorem: *Let* f *be a bounded real-valued function defined on an interval* $A = [a,b]$ *of the real line. Suppose that the Riemann integral* $\int_a^b f dx$ *exists; then* f *is continuous except on a set of measure zero, so that* f *is measurable. Further, the Lebesgue integral* $\int_a^b f dm$ *is equal to the Riemann integral* $\int_a^b f dx$.

Proof: Let $a = x_0 < x_1 < \ldots < x_\omega = b$ be an internal fine partition of the interval A, so that in particular, ω is a *natural number, i.e., a natural number in an enlargement of the reals, and for all i, $x_i \simeq x_{i+1}$. Define

(1) $y_k = \text{lub}\,\{f(x) \mid x_{k-1} \leqq x \leqq x_k\}$

and

(2) $z_k = \text{glb}\,\{f(x) \mid x_{k-1} \leqq x \leqq x_k\}$,

(lub is least upper bound and glb is greatest lower bound).

Using the non-standard definition of the Riemann integral (*cf.* [3], pp. 72 ff.), we have that

(3) $\sum_{k=1}^{\omega} z_k(x_k - x_{k-1}) \simeq \int_a^b f dx \simeq \sum_{k=1}^{\omega} y_k(x_k - x_{k-1})$,

(4) $\sum_{k=1}^{\omega} (z_k - y_k)(x_k - x_{k-1}) \simeq 0$.

Now let n be any standard integer and set

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(5) $A_n = \left\{x \epsilon [a, b] \mid \forall\, \delta > 0\ \exists y \left(|x - y| < \delta \text{ and } |f(x) - f(y)| > \frac{1}{n}\right)\right\}$

and

(6) $T = \left\{k \mid y_k - z_k > \frac{1}{n}\right\}.$

We then have the following lemmas.

Lemma 1: $\sum_{k \epsilon T} (x_k - x_{k-1}) \simeq 0$.

Proof: Since $(z_k - y_k)(x_k - x_{k-1}) \geqq 0$, we have

(7) $$0 \leqq \sum_{k\epsilon T} \frac{1}{n} (x_k - x_{k-1}) < \sum_{k\epsilon T} (y_k - z_k)(x_k - x_{k-1}) \leqq \sum_{k=1}^{\omega} (y_k - z_k)(x_k - x_{k-1}) \simeq 0 .$$

Hence

(8) $\frac{1}{n} \sum_{k\epsilon T} (x_k - x_{k-1}) \simeq 0$ and so $\sum_{k\epsilon T} (x_k - x_{k-1}) \simeq 0$. Q.E.D.

Lemma 2: *Let* m *be Lebesgue measure*; *then* $m(A_n) = 0$.

Proof: Let $x \epsilon {}^*A_n$ with $x_i < x < x_{i-1}$ (where ${}^*A_n$ is the usual non-standard set corresponding to A_n). Then

(9) $\forall \delta > 0,\ \exists y \left(|x - y| < \delta \text{ and } |f(x) - f(y)| > \frac{1}{n}\right).$

In particular,

(10) $\exists y \left(y \epsilon [x_i, x_{i+1}] \text{ and } |f(x) - f(y)| > \frac{1}{n}\right).$

Hence $i \epsilon T$. Similarly, if $x \epsilon {}^*A_n$ and $x = x_i$ for some i, then (10) holds either exactly as it is or else with $[x_i, x_{i+1}]$ replaced by $[x_{i-1}, x_i]$. In either case, we can conclude that $i - 1 \epsilon T$ or $i \epsilon T$. In all cases we have

(11) ${}^*A_n \subset \bigcup_{i\epsilon T} [x_i, x_{i+1}]$.

Let m^* be Lebesgue outer measure. We conclude

(12) $$0 \leqq m^*(A_n) = m^*({}^*A_n) \leqq m\left(\bigcup_{i\epsilon T} [x_i, x_{i+1}]\right) = \sum_{i\epsilon T} (x_{i+1} - x_i) \simeq 0,$$

by Lemma 1. Thus, $m^*(A_n) \simeq 0$. But $m^*(A_n)$ is standard; hence $m^*(A_n) = 0$ so that A_n is measurable and of measure zero. Q.E.D.

Lemma 3: *Let* $E = \{x \epsilon [a, b] \mid f$ *is discontinuous at* $x\}$. *Then* $m(E) = 0$.

Proof: $E = \bigcup_{n=1}^{\infty} A_n$. Therefore

(13) $0 \leqq m(E) = m\left(\bigcup_{n=1}^{\infty} A_n\right) \leqq \sum_{n=1}^{\infty} m(A_n) = 0$

be Lemma 2. Q.E.D.

Lemma 4: *Under the hypotheses of the theorem*, f *is measurable.*

Proof: Let $B = [\alpha, \beta]$ be an interval properly containing the range of f, and let (x, y) be any open subinterval of B. Then

(14) $f^{-1}[(x, y)] \cap (A - E) = (f|(A - E))^{-1}[(x, y)] \cap (A - E) = 0 \cap (A - E)$

where 0 is open (since $f|(A - E)$ is continuous). By Lemma 3, E is measurable, so that A - E, and hence $0 \cap (A - E)$, is measurable. But since $f^{-1}[(x, y)] \cap E \subset E$ and $m(E) = 0$, $f^{-1}[(x, y)] \cap E$ is measurable. Thus,

(15) $f^{-1}[(x, y)] = (f^{-1}[(x, y)] \cap (A - E)) \cup (f^{-1}[(x, y)] \cap E)$

is measurable. Hence f is measurable. Q.E.D.

We now proceed to show that the two integrals of f are equal. Let $w_1 < w_2 < \ldots < w_\eta$ be the distinct z_i listed in strictly increasing order; similarly, let $v_0 < v_1 < \ldots < v_{\nu-1}$ be the distinct y_i listed in strictly increasing order. Further, let $w_0 = \alpha$ and $v_\nu = \beta$. Define

(16) $g_1(x) = z_k$, for $x \in [x_{k-1}, x_k)$
(16′) $g_2(x) = y_k$, for $x \in (x_{k-1}, x_k]$,

and let $u_0 < u_1 < \ldots < u_\mu$ be an internal fine refinement of $\{w_i\}$ and $\{v_j\}$, listed in strictly increasing order. Then (*cf.* [3], pp. 126 ff.)

(17) $\int_a^b g_1\, dm \simeq \sum_{k=1}^{\mu} u_{k-1}\, m\, \{x \in A | u_{k-1} < g_1(x) \leqq u_k\}$

(18) $\int_a^b g_2\, dm \simeq \sum_{k=1}^{\mu} u_{k-1}\, m\, \{x \in A | u_{k-1} \leqq g_2(x) < u_k\}$.

But we easily compute

(19) $\sum_{k=1}^{\mu} u_k\, m\, \{x \in A | u_{k-1} < g_1(x) \leqq u_k\}$
$= \sum_{k=0}^{\eta-1} w_k\, m\left(\bigcup\{[x_{i-1}, x_i) | z_i = w_k\}\right) = \sum_{k=0}^{\eta-1} w_k \sum^* (x_i - x_{i-1})$
$= \sum_{k=0}^{\eta-1} \sum^* z_1(x_i - x_{i-1}) = \sum_{k=1}^{\omega} z_i(x_i - x_{i-1})$

where $\sum^*$ is the sum over all i such that $z_i = w_k$. Similarly, we compute

(20) $\sum_{k=1}^{\mu} u_{k-1}\, m\, \{x \in A | u_{k-1} \leqq g_2(x) < u_k\} = \sum_{i=1}^{\omega} y_i (x_i - x_{i-1})$.

Comparing (3), (17), (18), (19), and (20), we see that

(21) $\int_a^b g_2\, dm \simeq \int_a^b f dx \simeq \int_a^b g_2\, dm$.

But on A, $g_2(x) \leqq f(x) \leqq g_1(x)$, so that

(22) $\int_a^b f dx \simeq \int_a^b g_2\, dm \leqq \int_a^b f dm \leqq \int_a^b g_1\, dm \simeq \int_a^b f dx$.

Hence, since both $\int f dm$ and $\int_a^b f dx$ are standard, we have

(23) $\int_a^b f dm = \int_a^b f dx$. Q.E.D.

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