Notre Dame Journal of Formal Logic
Volume XIV, Number 1, January 1973
NDJFAM

A NOTE ON NEWMAN'S ALGEBRAIC SYSTEMS

BOLESŁAW SOBOCIŃSKI

This note possesses a purely supplementary and informative character with respect to the papers [2], [3], [4], [5] and [6].[1] Namely, in order to describe the systems investigated in those papers more completely the definitions of the dual associative Newman algebras which are mentioned only casually in [6], p. 536, and of the dual mixed associative Newman algebras will be established. Additionally, a rather bad misprint and erroneous statement which both appear in [3] will be corrected.

1 It has been established in [4] that the associative Newman algebras can be defined, as follows:

Any algebraic structure

$$\mathfrak{D} = \langle A, +, \times, -\rangle$$

where $+$ *and* $\times$ *are two binary operations, and* $-$ *is a unary operation defined on the carrier set* A, *is an associative Newman algebra, if it satisfies the following postulates:*

P1 $[ab]:a,b \in A\,.\supset.\, a = a + (b \times -b)$ [Axiom $F1$ in [4]]
P2 $[ab]:a,b \in A\,.\supset.\, a = a \times (b + -b)$ [$F2$ in [4]]
P3 $[abc]:a,b,c \in A\,.\supset.\, a \times (b + c) = (c \times a) + (b \times a)$ [$H1$ in [4]]
P4 $[abc]:a,b,c \in A\,.\supset.\, a \times (b \times c) = (a \times b) \times c$ [$L1$ in [4]]

Therefore, it is self-evident that the dual associative Newman algebras can be defined as follows:

Any algebraic structure

$$\mathfrak{N} = \langle A, +, \times, -\rangle$$

1. An acquaintance with the papers [2]-[6] is presupposed. Concerning the symbols used in this note it should be remarked that instead of "$\overline{a}$" which is used in [2], [3] and [4] I am using here "$-a$". An enumeration of the algebraic tables, *cf.* section **3** below, is a continuation of the enumeration of such tables given in [2], [4], [5] and [6].

Received November 3, 1971

where + and × are two binary operations, and - is a unary operation defined on the carrier set A, is a dual associative Newman algebra, if it satisfies the following postulates:

$R1$ $[ab]:a,b \in A .\supset. a = a \times (b + -b)$
$R2$ $[ab]:a,b \in A .\supset. a = a + (b \times -b)$
$R3$ $[abc]:a,b,c \in A .\supset. a + (b \times c) = (c + a) \times (b + a)$
$R4$ $[abc]:a,b,c \in A .\supset. a + (b + c) = (a + b) + c$

Since $\mathfrak{M}19$, *cf.* [6], p. 542, verifies the axioms $R1$-$R4$, but falsifies the law of idempotency with respect to the operation ×, we know that system $\mathfrak{R}$ is not necessarily a Boolean algebra. In section **3**, point (1), below the mutual independency of the postulates $R1$-$R4$ will be proved.

Using the deductions entirely analogous to those which are given in [4] we can prove easily that in the field of the fixed carrier set A the axioms $R1$-$R4$ are inferentially equivalent to the following formulas: $R1$, $R2$, $R4$ and

$R5$ $[ab]: a,b \in A .\supset. a + b = b + a$
$R6$ $[abc]:a,b,c \in A .\supset. a + (b \times c) = (a + b) \times (a + c)$

and, moreover, that $R1$-$R4$ imply

$R7$ $[a]:a \in A .\supset. a = a + a$

Hence, *cf.* an analogous case in [4], we can conclude:

A dual associative Newman algebra can be considered as a semi-lattice with respect to the binary operation + to which the additional postulates are added concerning the properties of the operations × and -.

2 In [5], p. 418, an equational axiomatization of the mixed associative Newman algebras has been established. Analogously, we can define the dual mixed associative Newman algebras as follows:

Any algebraic structure

$$\mathfrak{S} = \langle A, +, \times, \rightarrow \rangle$$

where +, × and → are three binary operations defined on the carrier set A, is a dual mixed associative Newman algebra, if it satisfies the following postulates:

$S1$ $[abc]:a,b,c \in A .\supset. a + (b \times c) = (a + b) \times (a + c)$
$S2$ $[ab]:a,b \in A .\supset. a + b = b + a$
$S3$ $[ab]:a,b \in A .\supset. (a \rightarrow b) \times (a + b) = b$
$S4$ $[abc]:a,b,c \in A .\supset. (a \rightarrow b) + (a + b) = c \rightarrow c$

Concerning the primitive binary operation → of the system $\mathfrak{S}$ it should be remarked that this operation is not a pseudo-complement operation $\Rightarrow$ which is a familiar primitive operation in the relatively pseudo-complemented lattices. It will be shown in section **3**, point (2), below that $\mathfrak{M}23$ verifies $S1$-$S4$, but falsifies a formula:

(β) $[ab]:a,b \in A .\supset. (a \rightarrow (b \times c)) + (a \rightarrow b) = a \rightarrow b$

which corresponds to the well-known formula of relatively pseudo-complemented lattices, *cf.* [1], p. 62:

$$a \Rightarrow (b \times c) \leq a \Rightarrow b$$

In section **3**, point (3) below the mutual independency of the postulates *S1-S4* will be proved. Again, using deductions entirely analogous to those given in [5], see p. 418, Theorem 2, we can establish that:

A dual mixed associative Newman algebra can be considered as a semi-lattice with respect to the primitive operation + to which the additional postulates are added concerning the properties of the primitive operations × and →.

3 In order to establish the independencies which are announced in sections **1** and **2** above we use the following algebraic tables: $\mathfrak{M}14$, $\mathfrak{M}15$, $\mathfrak{M}21$, *cf.* [6], p. 541 and p. 545, and:

$\mathfrak{M}22$

+	α	1	0
α	α	1	0
1	1	1	0
0	0	0	0

×	α	1	0
α	α	1	α
1	1	1	1
0	α	1	0

x	$-x$
α	0
1	0
0	1

$\mathfrak{M}23$

+	0	η
0	0	0
η	0	η

×	0	η
0	0	η
η	η	0

→	0	η
0	0	η
η	0	0

$\mathfrak{M}24$

+	0	α	β	γ	δ
0	0	0	0	0	0
α	0	α	0	0	α
β	0	0	β	0	β
γ	0	0	0	γ	γ
δ	0	α	β	γ	δ

×	0	α	β	γ	δ
0	0	α	β	γ	δ
α	α	α	δ	δ	δ
β	β	δ	β	δ	δ
γ	γ	δ	δ	γ	δ
δ	δ	δ	δ	δ	δ

→	0	α	β	γ	δ
0	0	α	β	γ	δ
α	0	0	β	γ	β
β	0	α	0	γ	γ
γ	0	α	β	0	α
δ	0	0	0	0	0

$\mathfrak{M}25$

+	α	1	0
α	α	1	0
1	α	1	0
0	0	0	0

×	α	1	0
α	α	1	α
1	1	1	1
0	α	1	0

→	α	1	0
α	0	0	0
1	0	0	0
0	α	1	0

$\mathfrak{M}26$

$+$	0	α
0	0	0
α	0	0

$\times$	0	α
0	0	α
α	α	0

$\rightarrow$	0	α
0	0	0
α	0	0

$\mathfrak{M}27$

$+$	0	α	β	γ
0	0	0	0	0
α	0	α	γ	β
β	0	γ	β	α
γ	0	β	α	γ

$\times$	0	α	β	γ
0	0	α	β	γ
α	α	0	γ	β
β	β	γ	0	α
γ	γ	β	α	0

$\rightarrow$	0	α	β	γ
0	0	α	β	γ
α	0	0	α	α
β	0	β	0	β
γ	0	γ	γ	0

(1) Since: (a) $\mathfrak{M}15$ verifies *R2, R3* and *R4,* but falsifies *R1, cf.* [6], p. 542; (b) $\mathfrak{M}22$ verifies *R1, R3* and *R4,* but flasifies *R2* for a/α and $b/1$: (i) $\alpha = \alpha$, (ii) $\alpha + (1 \times -1) = \alpha + (1 \times 0) = \alpha + 0 = 0$; (c) $\mathfrak{M}14$ verifies *R1, R2* and *R4,* but falsifies *R3* for a/γ, b/α and c/β: (i) $\gamma + (\alpha \times \beta) = \gamma + 1 = \gamma$, (ii) $(\beta + \gamma) \times (\alpha + \gamma) = \beta \times 0 = 0$; and (d) $\mathfrak{M}21$ verifies *R1, R2* and *R3*, but falsifies *R4, cf.* [6], p. 545, the proof that the axioms *R1-R4* are mutually independent is complete.

(2) Since $\mathfrak{M}23$ verifies *S1, S2, S3* and *S4,* but falsifies the formula (β) for $a/0$, b/η and c/η: (i) $(0 \rightarrow (\eta \times \eta) + (0 \rightarrow \eta) = (0 \rightarrow 0) + \eta = 0 + \eta = 0$, (ii) $0 \rightarrow \eta = \eta$, we know that ($\beta$) is not a consequence of *S1-S4.*

(3) Since: (a) $\mathfrak{M}24$ verifies *S2, S3* and *S4*, but falsifies *S1* for a/α, b/β and c/γ: (i) $\alpha + (\beta \times \gamma) = \alpha + \delta = \alpha$, (ii) $(\alpha + \beta) \times (\alpha + \gamma) = 0 \times 0 = 0$; (b) $\mathfrak{M}25$ verifies *S1, S3* and *S4,* but falsifies *S2* for a/α and $b/1$: (i) $\alpha + 1 = 1$, (ii) $1 + \alpha = \alpha$; (c) $\mathfrak{M}26$ verifies *S1, S2* and *S4,* but falsifies *S3* for $a/0$ and b/α: (i) $(0 \rightarrow \alpha) \times (0 + \alpha) = 0 \times 0 = 0$, (ii) $\alpha = \alpha$; and (d) $\mathfrak{M}27$ verifies *S1, S2* and *S3,* but falsifies *S4* for a/α, b/β and $c/0$: (i) $(\alpha \rightarrow \beta) + (\alpha + \beta) = \alpha + \gamma = \beta$, (ii) $0 \rightarrow 0 = 0$, the proof that the axioms *S1-S4* are mutually independent is complete.

4 Corrections:

(A) The proof of *F3* in [3], p. 268, lines 8-13, contains rather bad misprints. It should be given, as follows:

F3 $[ab]:a,b \epsilon A . \supset . a = (b + \bar{b}) \times a$

PR $[ab]:\mathrm{Hp}(1) . \supset .$

$$a = a \times (b + \bar{b}) = (\bar{b} \times a) + (b \times a) = ((\bar{b} \times (b + \bar{b})) \times a) + ((b \times (b + \bar{b})) \times a) \quad [1; F2; H1; F2]$$

$$= (\bar{b} \times ((b + \bar{b}) \times a)) + (b \times ((b + \bar{b}) \times a)) \quad [A10; L1]$$

$$= ((b + \bar{b}) \times a) \times (b + \bar{b}) = (b + \bar{b}) \times a \quad [H1; F2]$$

(B) Since $\mathfrak{M}5$, *cf.* [2], p. 263, falsifies *H1* for $a/0$, b/α and $c/1$: (i) $0 + (\alpha \times 1) = 0 + 1 = 1$, (ii) $(1 + 0) \times (\alpha + 0) = 1 \times \alpha = \alpha$, the statement "$\mathfrak{M}5$ verifies *F1* and *H1,* but falsifies *F2,*" which due to a manuscript mix-up appeared in [3], p. 268, lines 30-31, is obviously false. It should be substituted by the correct one:

"an algebraic table

$\mathfrak{M}28$

$+$	α	1	0
α	α	1	α
1	1	1	1
0	α	1	0

$\times$	α	1	0
α	α	1	0
1	1	1	0
0	0	0	0

x	$-x$
α	0
1	0
0	1

verifies $F1$, $H1$ and $L1$, but falsifies $F2$ for a/α, and $b/1$: (i) $\alpha = \alpha$, (ii) $\alpha \times ((1 + \bar{1}) = \alpha \times (1 + 0) = \alpha \times 1 = 1$''.

REFERENCES

[1] Rasiowa, H., and R. Sikorski, "*The mathematics of metamathematics,*" Monografie Matematyczne, tom 41. Państwowe Wydawnictwo Naukowe, Warszawa (1963).

[2] Sobociński, B., "A new formalization of Newman algebras," *Notre Dame Journal of Formal Logic,* vol. XIII (1972), pp. 255-264.

[3] Sobociński, B., "An equational axiomatization of associative Newman algebras," *Notre Dame Journal of Formal Logic,* vol. XIII (1972), pp. 265-269.

[4] Sobociński, B., "A semi-lattice theoretical characterization of associative Newman algebras," *Notre Dame Journal of Formal Logic,* vol. XIII (1972), pp. 283-285.

[5] Sobociński, B., "An equational axiomatization and a semi-lattice theoretical characterization of mixed associative Newman algebras," *Notre Dame Journal of Formal Logic,* vol. XIII (1972), pp. 407-423.

[6] Sobociński, B., "Solution to the problem concerning the Boolean bases for cylindric algebras," *Notre Dame Journal of Formal Logic,* vol. XIII (1972), pp. 529-545.

University of Notre Dame
Notre Dame, Indiana