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FORMATION SEQUENCES FOR PROPOSITIONAL FORMULAS

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Formation sequences play a central role in Smullyan's elegant development of the propositional calculus, given in [1] and [2]. In the following, we modify the treatment of [1] in that we take only $\sim$ and $\vee$ as our undefined logical connectives; the definitions given in [1] are altered accordingly.*

Let $\mathcal{P}_0$ be a denumerable collection of symbols, called *propositional variables*. Let the four symbols

$$\sim, \vee, (,)$$

be distinct from each other and from the propositional variables. A *formation sequence* is defined, recursively, to be a finite sequence each of whose terms is either

(i) a propositional variable,
(ii) of the form $\sim P$, where P is an earlier term of the sequence, or
(iii) of the form $(P \vee Q)$, where P and Q are earlier terms of the sequence.

P is called a *formula* if there is a formation sequence, $\langle P_0, P_1, \ldots, P_N\rangle$ in which $P_N = P$; $\langle P_0, P_1, \ldots, P_N\rangle$ is then called a *formation sequence for P*. It follows directly from this definition that if $\langle P_0, P_1, \ldots, P_N\rangle$ is a formation sequence, then for each $K \leq N$, $\langle P_0, P_1, \ldots, P_K\rangle$ is a formation sequence for P_K. As the name suggests, formation sequences yield information concerning the manner in which formulas are constructed from propositional variables by means of connectives. Clearly, a formation sequence for a formula, P, is not unique.

Formulas of type (ii) are called *negations*; those of type (iii) are called *disjunctions*. It is well-known that for every formula P, exactly one of the following holds:

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(i) $P \in \mathcal{P}_0$;
(ii) $P = \sim Q$ for some unique formula Q;
(iii) $P = (Q \vee R)$ for unique formulas Q and R.

Consequently, if $\sim P$ is a term of a formation sequence, then P must be an earlier term of this sequence; if $(P \vee Q)$ is a term of this sequence, both P and Q must be earlier terms.

It will be convenient to consider the following recursive definition of the degree of a formula. Let $P \in \mathcal{P}$. We define the *degree of P,* written "$d(P)$" as follows:

(i) If $P \in \mathcal{P}_0$, let $d(P) = 0$.
(ii) If $P = \sim Q$ for some formula Q, let $d(P) = d(Q) + 1$.
(iii) If $P = (Q \vee R)$ for some formulas Q and R, let $d(P) = d(Q) + d(R) + 1$.

An elementary inductive argument on the degrees of formulas establishes the following *Principle of Induction for Formulas.*

Let $\mathcal{L}$ be a subset of $\mathcal{P}$ which is such that

(i) $\mathcal{P}_0 \subseteq \mathcal{L}$;
(ii) *if* $P \in \mathcal{L}$, *then* $\sim P \in \mathcal{L}$;
(iii) *if* $P, Q \in \mathcal{L}$, *then* $(P \vee Q) \in \mathcal{L}$.

Then $\mathcal{L} = \mathcal{P}$.

In [1], the notion of a subformula is introduced by first defining an *immediate subformula* as follows:

(i) Propositional variables have no immediate subformulas;
(ii) $\sim P$ has P as its unique immediate subformula;
(iii) The immediate subformulas of $(P \vee Q)$ are P and Q, and only these.

P is defined to be a *subformula of* Q if there is a finite sequence, $\langle P_0, P_1, \ldots, P_N \rangle$, in which $P_0 = Q$, $P_N = P$, and P_{I+1} is an immediate subformula of P_I for all $I = 0, 1, \ldots, N-1$. We call such a sequence a (P, Q)-*subformula sequence.*

Clearly every formula P is a subformula of itself as well as of $\sim P$; if Q is also a formula, then P and Q are each subformulas of $(P \vee Q)$. If P is a subformula of Q and Q is a subformula of R, then P is a subformula of R; in particular, if P is a subformula of Q, then P is also a subformula of $\sim Q$, and for every formula R, P is a subformula of $(Q \vee R)$.

A subformula of a formula P, other than P itself, is called a *proper subformula of P.*

Let $s = \langle P_0, P_1, \ldots, P_N \rangle$ be a formation sequence. We say that P *appears in s* if P is a term of s—i.e., if for some I, $1 \leq I \leq N$, $P = P_I$; we say that P *appears K times in s* if there are exactly K indices I, $1 \leq I \leq N$, for which $P = P_I$.

Lemma. (a) *If* $P \in \mathcal{P}_0$, *then P is its own unique subformula.*
(b) *If P is a proper subformula of* $\sim Q$, *then P is a subformula of Q.*
(c) *If P is a proper subformula of* $(Q \vee R)$, *then P is a subformula of Q or P is a subformula of R.*

Proof. (We prove only part (b).) If P is a proper subformula of $\sim Q$ and if $\langle P_0, P_1, \ldots, P_N\rangle$ is a $(P, \sim Q)$—subformula sequence, then $P_0 = \sim Q$ and $P_1 = Q$. Thus if $Q_I = P_{I+1}, I = 0, 1, \ldots, N-1$, $\langle Q_0, Q_1, \ldots, Q_{N-1}\rangle$ is a (P, Q)—subformula sequence.

Theorem 1. *Let P be a formula and let $\langle P_0, P_1, \ldots, P_N\rangle$ be a formation sequence for P. Then every subformula of P appears in $\langle P_0, P_1, \ldots, P_N\rangle$.*

Proof. Let $\mathcal{L} = \{P \epsilon \mathcal{P}$: whenever s is a formation sequence for P, then every subformula of P appears in $s\}$. We apply the Principle of Induction for Formulas to show that $\mathcal{L} = \mathcal{P}$.

We first note that every formula P appears in each formation sequence for itself; thus we need only consider proper subformulas of P.

Part (a) of the lemma implies $\mathcal{P}_0 \subseteq \mathcal{L}$.

Suppose $P \epsilon \mathcal{L}$ and let $s = \langle P_0, P_1, \ldots, P_N\rangle$ be any formation sequence for $\sim P$. Since $P_N = \sim P$, P must appear in s; thus $P = P_K$ for some $K < N$, and $\langle P_0, P_1, \ldots, P_K\rangle$ is a formation sequence for P. Each subformula of P appears in $\langle P_0, P_1, \ldots, P_K\rangle$, and hence in s. Part (b) of the lemma guarantees that every subformula of $\sim P$ appears in s.

Suppose $P, Q \epsilon \mathcal{L}$ and let $s^* = \langle P_0, P_1, \ldots, P_N\rangle$ be any formation sequence for $(P \vee Q)$. Both P and Q must appear in s^*; say $P = P_K$ and $Q = P_L$, $K, L < N$. $s_1 = \langle P_0, P_1, \ldots, P_K\rangle$ and $s_2 = \langle P_0, P_1, \ldots, P_L\rangle$ are formation sequences for P and for Q, respectively. Each subformula of P appears in s_1—hence in s^*; each subformula of Q appears in s_2—hence in s^*. Part (c) of the lemma indicates that every subformula of $(P \vee Q)$ appears in s^*.

In [2], a *proper formation sequence for P* is defined to be a formation sequence for P, $\langle P_0, P_1, \ldots, P_N\rangle$, which is such that

(i) $P_J \neq P_K$, $0 \leq J < K \leq N$ and
(ii) for each $K \leq N$, P_K is a subformula of P.

From this definition and theorem 1 it immediately follows that a formation sequence, s, for P is proper if and only if every subformula for P appears once in s and every formula which appears in s is a subformula of P. Moreover, if $s = \langle P_0, P_1, \ldots, P_N\rangle$ is a formation sequence for P, then s is proper if and only if we have besides (i) above,

(iii) $\{P_0, P_1, \ldots, P_N\} \subseteq \{Q_0, Q_1, \ldots, Q_M\}$ for every formation sequence, $\langle Q_0, Q_1, \ldots, Q_M\rangle$, for P.

By definition, each formula has a formation sequence; we apply Induction for Formulas to show that each formula has a proper formation sequence.

Theorem 2. *For each formula P, there exists a proper formation sequence for P.*

Proof. Let $\mathcal{L} = \{P \epsilon \mathcal{P}$: there exists a proper formation sequence for $P\}$. For $P \epsilon \mathcal{P}_0$, $\langle P\rangle$ is a proper formation sequence for P; thus $\mathcal{P}_0 \subseteq \mathcal{L}$. Let $P \epsilon \mathcal{L}$ and let $s = \langle P_0, P_1, \ldots, P_N\rangle$ be a proper formation sequence for P.

Since $P_N = P$, $\sim P$ cannot appear in s because otherwise, P would appear at least twice in s. It follows that $\langle P_0, P_1, \ldots, P_N, \sim P\rangle$ is a proper formation sequence for $\sim P$.

Let $P, Q \in \mathcal{L}$ and let $s_1 = \langle P_0, P_1, \ldots, P_M\rangle$ and $s_2 = \langle Q_0, Q_1, \ldots, Q_N\rangle$ be proper formation sequences for P and for Q, respectively. Let $\langle Q_{I_0}, Q_{I_1}, \ldots, Q_{I_S}\rangle$, $0 \leq I_0 < I_1 < \ldots < I_S \leq N$, be the subsequence of s_2 obtained by deleting from s_2 those formulas which appear in s_1. Now $P_M = P$ and $Q_N = Q$; it follows that $(P \vee Q)$ cannot appear either in s_1 or in s_2, for if it did, then either P would appear at least twice in s_1 or Q would appear at least twice in s_2. Consequently, $\langle P_0, P_1, \ldots, P_M, Q_{I_0}, Q_{I_1}, \ldots, Q_{I_S}, (P \vee Q)\rangle$ is a proper formation sequence for $(P \vee Q)$.

The proof of theorem 2 also yields the following:

Corollary. *For all formulas P and Q,*
(a) *$\sim P$ is not a subformula of P;*
(b) *$(P \vee Q)$ is neither a subformula of P nor of Q.*

Theorem 3. *Let P and Q be formulas. Then P is a subformula of Q if and only if every formation sequence for Q has an initial which is a formation sequence for P.*

Proof. Let P be a subformula of Q and let $\langle Q_0, Q_1, \ldots, Q_N\rangle$ be any formation sequence for Q. Then $P = Q_K$ for some $K \leq N$. Consequently, $\langle Q_0, Q_1, \ldots, Q_K\rangle$ is a formation sequence for P.

Suppose P is not a subformula of Q; let $s = \langle R_0, R_1, \ldots, R_M\rangle$ be a proper formation sequence for Q. Then P does not appear in s; hence no initial of s can be a formation sequence for P.

Corollary. *Let P and Q be formulas. Then P is a subformula of Q if and only if every formation sequence for Q has a subsequence which is a formation sequence for P.*

Proof. If P is a subformula of Q, then every formation sequence for Q has an initial—hence has a subsequence—which is a formation sequence for P.

Conversely, suppose every formation sequence for Q has a subsequence which is a formation sequence for P. Then there is a proper formation sequence for Q in which P appears. Therefore P is a subformula of Q.

Note that theorem 3 is false if "proper formation sequence" replaces each instance of "formation sequence" in the statement of the theorem; for example, if $P, Q \in \mathcal{P}_0$, then $\langle P, Q, (P \vee Q)\rangle$ is a proper formation sequence for $(P \vee Q)$ none of whose initials is a proper formation sequence for Q. Note, further, that if P and Q are arbitrary formulas and if $s = \langle P_0, P_1, \ldots, P_M\rangle$ is any formation sequence for P, then s is an initial of some formation sequence for Q. In fact, if $\langle Q_0, Q_1, \ldots, Q_N\rangle$ is any formation sequence for Q, then $\langle P_0, P_1, \ldots, P_M, Q_0, Q_1, \ldots, Q_N\rangle$ is also a formation sequence for Q. Thus the following theorem will be false if we replace each instance of "proper formation sequence" by "formation sequence" in the statement of the theorem.

Theorem 4. *Let P and Q be formulas. Then P is a subformula of Q if and* *only if every proper formation sequence for P is an initial of some proper formation sequence for Q.*

Proof. P appears in every (proper) formation sequence for P; hence if P is not a subformula of Q, then no proper formation sequence for P can be an initial of a proper formation sequence for Q. Let $\mathcal{Z} = \{Q \in \mathcal{P} :$ whenever P is a subformula of Q, then every proper formation sequence for P is an initial of some proper formation sequence for $Q\}$. Clearly, $\mathcal{P}_0 \subseteq \mathcal{Z}$.

Suppose $Q \in \mathcal{Z}$ and suppose that P is any subformula of $\sim Q$. We may suppose that P is a proper subformula of $\sim Q$. Then P is a subformula of Q. Every proper formation sequence, s, for P is an initial of some proper formation sequence, $\langle Q_0, Q_1, \ldots, Q_N\rangle$, for Q. Therefore s is an initial of the proper formation sequence $\langle Q_0, Q_1, \ldots, Q_N, \sim Q\rangle$ for $\sim Q$.

Suppose Q, $R \in \mathcal{Z}$ and suppose that P is any subformula of $(Q \vee R)$; again it suffices to assume P is a proper subformula of $(Q \vee R)$. Then either P is a subformula of Q or P is a subformula of R. We consider the case where P is a subformula of Q; the other case is proved similarly. Every proper formation sequence, s^*, for P is an initial of a proper formation sequence, $s_1 = \langle Q_0, Q_1, \ldots, Q_M\rangle$, for Q. Let $s_2 = \langle R_0, R_1, \ldots, R_N\rangle$ be any proper formation sequence for R; delete from s_2 those formulas which appear in s_1, and let $\langle R_{I_0}, R_{I_1}, \ldots, R_{I_S}\rangle$, $0 \leq I_0 < I_1 < \ldots < I_S \leq N$, be the resulting subsequence of s_2. Then s^* is an initial of the proper formation sequence $\langle Q_0, Q_1, \ldots, Q_M, R_{I_0}, R_{I_1}, \ldots, R_{I_S}, (Q \vee R)\rangle$ for $(Q \vee R)$.

Corollary. *Let P, $Q \in \mathcal{P}$. Then P is a subformula of Q if and only if every proper formation sequence for P is a subsequence of a proper formation sequence for Q.*

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