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A NOTE ON IMPLICATIVE MODELS

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1. *Introduction.* Implicative models were first considered by Leon Henkin who explored the relation between certain formal (logical) systems and certain algebraic structures. More precisely, implicative models correspond to a logical system whose only logical connective is implication and whose laws are satisfied by classical, intuitionistic and modal logics.

Several examples of implicative models are Boolean lattices, Brouwerian semi-lattices, and closures algebras. Henkin's definition of an implicative model has been dualized to conform with common notation for Brouwerian semi-lattices. In this note it is shown that several significant results for Brouwerian semi-lattices also obtain in the setting of implicative models.

2. *Implicative Models*. An implicative model [2] is an algebraic system $\langle X, *, 1\rangle$ where X is a set, 1 is an element of X, and $*$ is a binary operation satisfying the axioms listed below. It is convenient to use the relation $\leq$ defined by $x \leq y$ if $x * y = 1$. The following axioms hold for all x, y, z in X:

A_1 $\quad y \leq x * y$
A_2 $\quad x * (y * z) \leq (x * y) * (x * z)$
A_3 $\quad x \leq 1$
A_4 $\quad x = y$ *when* $x \leq y$ *and* $y \leq x$.

Proposition 1.

(i) *If* $1 \leq x$, *then* $1 = x$.
(ii) $x * 1 = 1$.
(iii) $x * x = 1$.
(iv) $1 * x = x$.
(v) *If* $x \leq y$ *and* $y \leq z$, *then* $x \leq z$.
(vi) $\langle X, \leq\rangle$ *is a partially ordered set.*

Proposition 2.

(i) *If* $x \leq y$, *then* $y * z \leq x * z$.
(ii) $x * (y * z) = y * (x * z)$.

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(iii) $x * (y * z) = (x * y) * (x * z)$.
(iv) $x * (y * z) = x * [(x * y) * z]$.
(v) *If* $x \leq y * z$, *then* $y \leq x * z$.
(vi) *If* $x \leq y$, *then* $z * x \leq z * y$.

The above propositions and the following two propositions are easily proved. Some of the arguments are contained in [2]; also some similar arguments are in [5]. The key computational facts used in this note are contained in Propositions 2 and 3.

Proposition 3.

(i) *If* $x \leq z$ *and* $x \leq z * y$, *then* $x \leq y$.
(ii) *If* $x \leq y$, *then* $x * (y * z) = x * z$.
(iii) *If either of* $z \wedge (z * y)$ *or* $z \wedge y$ *exists, then both exist and they are equal.*
(iv) *If* $x \wedge y$ *exists, then* $x * (y * z) \leq (x \wedge y) * z$.

Let $0 \in X$ be a zero of the partially ordered set $\langle X, \leq \rangle$. An implicative model with such an element will be called *bounded*. Define $*$: $X \to X$ by $x^* = x * 0$. Let X^* be the image of the mapping $*$. For the remainder of this section, let X denote a bounded implicative model.

Proposition 4.

(i) $1^* = 0$ *and* $0^* = 1$.
(ii) *If* $x \leq y$, *then* $y^* \leq x^*$.
(iii) $x \leq x^{**}$.
(iv) $x^* = x^{***}$.

Lemma 5. *For* $x \in X$ *and* $y \in X^*$, $(y * 0) * x \leq (x * 0) * y$. *Hence, if* $x, y \in X^*$, *then* $y * x = x^* * y^*$.

Proof. Since $y \in X^*$, then $y = y^* * 0$. Thus $[(y * 0) * x] * [(x * 0) * y] = [y^* * x] * \{x^* * [y^* * 0]\} = x^* * \{[y^* * x] * [y^* * 0]\} = x^* * \{y^* * [x * 0]\} = x^* * \{y^* * x^*\} = 1$.

Theorem 6. *For* $x, y \in X^*$, $x * y \in X^*$.

Proof. We show that $[(x * y) * 0] * 0 \leq x * y$. Now $y * 0 \leq x * (y * 0) = x * [(x * y) * 0] = \{[(x * y) * 0] * 0\} * (x * 0)$. Thus $[(x * y) * 0] * 0 \leq (y * 0) * (x * 0) = x * y$.

Proposition 7.

(i) *For* $x, z \in X^*$, $x^* \leq z$ *if and only if* $x * z = z$.
(ii) *For* $x, y, z \in X^*$ *with* $x^* \leq z$, $(x * y) * z = x * (y * z) = y * z$.

Proof. (i) $(\Leftarrow)$ $x^* \leq z^* * x^* = x * z = z$.

$(\Rightarrow)$ We first show that $x^* * x = x$. Here $(x^* * x) * x = (x^* * x) * (x^* * 0) = x^* * (x * 0) = 1$. In general, if $x^* \leq z$, then $x \leq z * x \leq x^* * x = x$.

(ii) $(x * y) * z = (x * y) * (x * z) = x * (y * z) = y * (x * z) = y * z$.

Theorem 8.

(i) *For $x, y \in X^*$, the infimum in X^* or in X of $\{x, y\}$, exists and $x \wedge y = (x * y^*)^*$.*

(ii) *For $x, y \in X^*$, the supremum in X^* of $\{x, y\}$ is $x \vee y = x^* * y$.*

(iii) *For $y \in X^*$, y^* is a complement in X^* of y.*

Proof. (i) $(x * y^*)^* \leq y$ since $y^* \leq (x * y^*)$. Since $x * y^* = y * x^*$, then $(x * y^*)^* \leq x$. Now suppose that $z \in X^*$ and $z \leq \{x, y\}$. The statement $z \leq (x * y^*)^*$ is equivalent to $z * (x * y^*)^* = 1$. But $z * (x * y^*)^* = (x * y^*) * z^* = y^* * z^* = 1$ by Proposition 7. Notice that if $z \in X$ with $z \leq \{x, y\}$ then $z \leq z^{**} \leq (x * y^*)^*$.

(ii) follows from (i) by the de Morgan laws.

We have now shown that $\langle X^*, \leq, *, 0, 1\rangle$ is an orthocomplemented lattice. In proving the following theorem we use the fact that an orthocomplemented lattice in which complements are unique is a Boolean lattice (see [3], Theorem 4).

Theorem 9. *$\langle X^*, \leq, *, 0, 1\rangle$ is a Boolean lattice.*

Proof. The computation $x \vee y = (y * 0) * x$ of Theorem 8 makes it clear that $x \vee y = 1$ if and only if $y^* \leq x$. Suppose $x, y \in X^*$ and x is a complement of y. Since $x \vee y = 1$, then $y^* \leq x$. Also since $x \wedge y = 0$, then $x^* \vee y^* = 1$, so $x \leq y^*$. This shows that each member of X^* has a unique complement.

3. *Brouwerian Semi-Lattices*. A *Brouwerian semi-lattice* is a semi-lattice with unit $\langle X, *, 1, \wedge\rangle$ on which $*$ is a binary operation such that $w \leq x * y$ if and only if $x \wedge w \leq y$. Proposition 10 below indicates that any Brouwerian semi-lattice is an implicative model. Conversely, Carol Karp has proved that any implicative model can be embedded, via the filters of Section **4** below, in a complete Brouwerian lattice. This embedding preserves certain logical infima and any suprema which exist in the implicative model. It is convenient to include in this section an example of an implicative model which is markedly non-Brouwerian.

Proposition 10. *A Brouwerian semi-lattice is an implicative model.*

Proof. A useful computational fact in a Brouwerian semi-lattice is that $x \wedge (x * y) = x \wedge y$. To check that condition A_2 is satisfied, note that the following statements are equivalent: $x * (y * z) \leq (x * y) * (x * z)$, $(x * y) \wedge [x * (y * x)] \leq x * z$, $x \wedge (x * y) \wedge [x * (y * z)] \leq z$, $x \wedge y \wedge (y * z) \leq z$, $x \wedge y \wedge z \leq z$.

Example. Let $\langle P, \leq, 1\rangle$ be any partially ordered set with unit. Define $*$: $P \times P \to P$ by $x * y = y$ if $x \not\leq y$, and $x * y = 1$ if $x \leq y$. Then $\langle P, *, 1\rangle$ is an implicative model.

Theorem 11. *Let X be an implicative model in which any two elements have an infimum. The following conditions are equivalent.*

(i) *X is a Brouwerian semi-lattice.*

(ii) *For $x, y, z \in X$, $(x \wedge y) * z = x * (y * z)$.*

(iii) *For* $x, y, z \in X$ *with* $z \leq y$, *then* $x \wedge y \leq z$ *if and only if* $x \leq (y * x)$.
(iv) *For* $x, y \in X$, $x \leq y * (x \wedge y)$.
(v) *For* $x, y, z \in X$, $x * (y \wedge z) = (x * y) \wedge (x * z)$.

Proof. (i) $\Longrightarrow$ (ii). By the definition of a Brouwerian semi-lattice, these are equivalent: $w \leq x * (y * z)$, $x \wedge w \leq (y * z)$, $x \wedge y \wedge w \leq z$, $w \leq (x \wedge y) * z$.

(ii) $\Longrightarrow$ (iii). Condition (iii) may be read as $(x \wedge y) * z = 1$ if and only if $x * (y * z) = 1$.

(iii) $\Longrightarrow$ (iv). Apply (iii) with $z = x \wedge y$.

(iv) $\Longrightarrow$ (v). It is sufficient to show that $(x * y) \wedge (x * z) \leq x * (y \wedge z)$. By Proposition 3 (iv), $[(x * y) \wedge (x * z)] * [x * (y \wedge z)] \geq (x * y) * \{(x * z) * [x * (y \wedge z)]\} = (x * y) * \{x * [z * (y \wedge z)]\}$. Further $z * (y \wedge z) \geq y$, so $(x * y) * \{x * [z * (y \wedge z)]\} \geq (x * y) * (x * y) = 1$.

(v) $\Longrightarrow$ (i) Observe that $x \wedge w \leq y$ implies $w \leq x * y$ since $w \leq x * w = (x * w) \wedge (x * x) = x * (w \wedge x) \leq x * y$.

4. *Filters in Implicative Models*. A *filter* in an implicative model $\langle X, *, 1\rangle$ is a non-empty subset F of X for which (i) $y \in F$ when $x \in F$ and $x * y \in F$, and (ii) $y \in F$ when $x \in F$ and $x \leq y$. Notice that (ii) may be replaced by (ii)$'$ $1 \in F$.

Theorem 12.

(i) *Let* F *be a filter in* X *and let* $x \sim y$ *when* $x * y \in F$ *and* $y * x \in F$. *Then* $\sim$ *is a congruence relation on* X, *and the kernel of* $\sim$ *is* F.
(ii) *For any congruence relation* $\sim$ *on* X, *ker* $\sim$ *is a filter in* X.

Proof. (i) It is immediate that $\sim$ is reflexive and symmetric. To confirm that $\sim$ is transitive we suppose $x \sim y$ and $y \sim z$. By symmetry it is sufficient to show that $x * z \in F$. Since $x * y$, $y * z \in F$, and $y * z \leq x * (y * z) = (x * y) * (x * z)$, then $x * z \in F$.

The justification that $\sim$ is preserved by $*$ is separated into three cases.

Case 1. If $x \sim y$, then $u * x \sim u * y$. The computation $(u * x) * (u * y) = u * (x * y) \geq (x * y)$ indicates that $(u * x) * (u * y) \in F$.

Case 2. If $x \sim y$, then $x * z \sim y * z$. We shall show that $(x * z) * (y * z) \in F$. Observe that $y * x \in F$ and $(y * x) * [(x * z) * (y * z)] = (x * z) * [(y * x) * (y * z)] = (x * z) * [y * (x * z)] = 1$. The conclusion follows.

Case 3. If $x \sim y$ and $u \sim v$, then $x * u \sim y * v$. This follows from the two previous cases and the transitivity of $\sim$.

(ii) First $1 \in \mathrm{ker}(\sim) = \{x \in X | x \sim 1\}$. Suppose that $x, x * y \in \mathrm{ker}(\sim)$. It follows that $y = 1 * y \sim x * y \sim 1$, and thus $y \in \mathrm{ker}(\sim)$.

Let $\langle X, *, 1\rangle$ and $\langle Z, +, 1\rangle$ be implicative models. A function $f : X \to Z$ is called a *homomorphism* providing $f(x * y) = f(x) + f(y)$ for all $x, y \in X$. It is convenient to incorporate several basic properties of homomorphisms into the following:

Proposition 13. *Let* f *be a homomorphism from* $(X, *, 1, 0)$ *to* $(Z, +, 1, 0)$.

(i) $f(1) = 1$.
(ii) *If* $x \leq y$ *in* X, *then* $f(x) \leq f(y)$ *in* Z.
(iii) $\mathrm{ker}(f) = \{x \in X | f(x) = 1\}$ *is a filter in* X.

(iv) *If f is an epimorphism, then $X/\ker(f)$ and Z are isomorphic implicative models.*

(v) *Let $f(0) = 0$. If $x \in X^*$, then $f(x) \in Z^*$. Also, if $z \in Z^*$ and $z = f(x)$, then $z = f(x^{**})$.*

(vi) *If f is an epimorphism and f_1 is the restriction of f to X^*, then $f_1 : X^* \to Z^*$ is an epimorphism of Boolean lattices.*

Theorem 14. *Let $\langle X, *, 1, 0\rangle$ be a bounded implicative model, let X^* be the Boolean lattice of closed elements of X, and let $D = \{x \in X \mid x^{**} = 1\}$ be the set of dense elements of X. Then D is a filter in X, and X/D and X^* are isomorphic as Boolean lattices.*

Proof. Let $f : X \to X^*$ be the closure mapping $f(x) = x^{**}$. We claim that f is a homomorphism. Since $x^{**} * y^{**} = x^* \vee y^{**}$ in X^*, we wish to show that for $x, y \in X$, $(x * y)^{**} = x^* \vee y^{**}$. First $y^{**} \leq (x * y)^{**}$ since $y \leq x * y$, and $x^* \leq (x * y)^{**}$ since $x^* = x * 0 \leq x * y$. Suppose that $u \in X^*$ is an upper bound of $\{x^*, y^{**}\}$. To prove that $(x * y)^{**} \leq u$ it is sufficient to prove that $(x * y) * u = 1$. By Proposition 2 (ii) $x * (y^* * 0) = y^* * (x * 0)$, and hence $[x * y]^* \geq [x * (y^* * 0)]^* = [y^* * (x * 0)]^* = [y^{**} \vee x^*]^*$. Again by Proposition 2 (ii), $(x * y) * u = (x * y) * [u^* * 0] = u^* * [(x * y) * 0] \geq u^* * [y^{**} \vee x^*]^* = [y^{**} \vee x^*] * u = 1$. Since $\ker(f) = D$, then D is a filter in X and furthermore X/D and X^* are isomorphic implicative models.

By Proposition 13 (vi), for the natural isomorphism $g : X/D \to X^*$, the restriction of g to $(X/D)^*$ is an isomorphism onto $X^{**} = X^*$. Since g is an injection, it must be that all elements of X/D are closed.

The previous theorem generalizes a theorem of Glivenko which is contained in [1], Chapter V.11. The following corollary completes the details of the generalization.

Proposition 15. *If the implicative model X is a Brouwerian semi-lattice, then the filters in X are semi-lattice filters.*

Proof. Let F be a filter in X, and let $x, y \in F$. We compute $x * (x \wedge y) = (x * x) \wedge (x * y) = x * y \geq y$. Thus $x, x * (x \wedge y) \in F$ imply $x \wedge y \in F$. Conversely, suppose that F is a semi-lattice filter and that $x, x * y \in F$. Since $x \wedge y = x \wedge (x * y) \in F$ and $x \wedge y \leq y$, then $y \in F$.

Corollary 16. *Let $\langle X, *, 1, 0, \wedge\rangle$ be a bounded Brouwerian semi-lattice. For $x, y \in X$, $x^{**} = y^{**}$ if and only if $x \wedge d = y \wedge d$ for some (dense) $d \in X$ with $d^{**} = 1$.*

Proof. ($\Longrightarrow$) Since $x^{**} = y^{**} = 1$, then $x * y$, $y * x \in D = \{w \mid w^{**} = 1\}$. By Theorem 14 and Proposition 15, $d = (x * y) \wedge (y * x) \in D$. Moreover $x \wedge d = x \wedge (x * y) \wedge (y * x) = x \wedge (x * y) = x \wedge y = y \wedge d$.

($\Longleftarrow$) If $x \wedge d = y \wedge d$ for $d \in D$, then $(x \wedge d)^* = (y \wedge d)^*$. However, by Theorem 11 (ii), $x^* = x * 0 = x * (d * 0) = (x \wedge d) * 0 = (x \wedge d)^* = (y \wedge d)^* = y^*$.

REFERENCES

[1] Birkhoff, G., *Lattice Theory*, 3rd ed., American Mathematical Society (1967).

[2] Henkin, L., "An algebraic characterization of quantifiers," *Fundamenta Mathematicae*, vol. 37 (1950), pp. 63-74.

[3] Holland, S. S., "A Radon-Nikodym theorem in dimension lattices," *Transactions of the American Mathematical Society*, vol. 108 (1963), pp. 66-87.

[4] Karp, C., "Set representation theorems in implicative models," abstract, *American Mathematical Monthly*, vol. 61 (1954), p. 523.

[5] Nemitz, W. C., "Implicative semi-lattices," *Transactions of the American Mathematical Society*, vol. 117 (1965), pp. 128-142.

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