Notre Dame Journal of Formal Logic
Volume XIV, Number 2, April 1973
NDJFAM

THE CONCEPT OF LOGIC

A. P. RAO

According to Frege, corresponding to every unambiguous concept there is a definite and complete definition with reference to which it can be decided, for any given object, whether it falls under that concept or not. In the absence of such a definition the concept must be treated as ambiguous. In this regard, Frege and his followers are in the tradition of Aristotle, as they would subscribe to the Aristotelian view that ambiguity creeps in when the principle of the excluded middle is violated. And the concept of logic, which Frege himself brought into vogue, and has gathered wide acceptance since then, namely the one (to use the currently fashionable jargon) under which fall all studies of abstract structures, is ambiguous, in the sense that it does not have sharp boundaries, and in the absence of sharp boundaries there is nothing to preclude violations of the principle of excluded middle.

But, even those who hold such a broad conception of logic would categorise 'the denial of a conjunction and the disjunction of the denied conjuncts have the same truth-values' as *logical*, and 'for any three numbers the first multiplied by the sum of the second and the third is identical to the sum of the first multiplied by the second and the first multiplied by the third' as *arithmetical*, as if they are two different categories (an idea which cannot be sustained when once the broad concept of logic has been accepted). And they would also categorise certain statements as "half-logical and half-mathematical" as Church characterised the axiom of infinity.

Such a conception of logic, with its hazy boundaries, obviously resists a definite and complete definition, explaining why Church does not offer a definition, but only gives a "descriptive account" of logic, on the basis of which he tries to indicate its boundaries. "It," he says, "is at any rate much less than the total content of a mathematical library, or even a few good mathematical books".[1] But from this it is not clear whether he

1. "Mathematics and Logic," in *Logic, Methodology, and Philosophy of Science*, E. Nagel, P. Suppes, and A. Tarski, eds., Stanford University Press (1962), p. 185.

Received November 2, 1970

intends to include in logic the whole of the projected second volume of his *Introduction*. Nor is it clear why even the whole of the first volume of it should be included. On his own account any such issue is "terminological." If it is precisely that, he ought not to have raised non-terminological objections to the inclusion of the whole mathematical library in logic, or to taking it as the "logic of Aristotle plus further developments in this immediate context".[2] His objections are not to the effect that 'logic' has not been, or ought to be, used in this or that way. They have sprung from his desire to save the logistic thesis, of course, in its weak formulation, which he, like Frege, thinks to be the correct description of the real nature of logic.

And surprisingly, Kneale, who is a trenchent critic of the terminological interpretation of logical laws, shares with Church the terminological approach to 'logic'. The relegation of identity theory and the theory of sets from the realm of logic, for him, is a "linguistic legislation",[3] of which he is keen on pushing through. The terminological interpretation of 'logic' has become imperative, as Kneale himself is aware, by the absence of an "agreed definition" of it.[4] Then not to proceed in search of a definition and try to find out convincing arguments as to the acceptability of that definition, and instead resort to an appeal to language, is to explain away an issue of serious philosophical importance. The issue involved is not only of utmost significance but also equally difficult for an adequate criterion of adequacy itself will have to be found out.

Further, the delimiting of logic, by defining it, ought to be in consonance with the Aristotle-Bolzano-Frege view of it, which, at least in bare essentials, is accepted even today, in so far as it is conceived to be the study of the substructure of all structures. That logic, though not a substantive science itself, is "a part of general culture which everyone should undergo before he studies any science, and which alone will enable him to know what sorts of propositions he should demand proof and what sorts of proof he should demand of them"[5] is shared even by the contemporary logicians with Aristotle and every logician that followed him. Also, any definition that would be offered ought not to vitiate the intuitive differentiation that we usually make between logic and what falls outside it (say arithmetic). This is not to prejudge that arithmetic is not a part of logic, but only to state a fact. Do we not say that Fermat's last theorem belongs to arithmetic and that the Herbrand-Tarski theorem belongs to logic? The delimiting of logic should not, however, be carried out with the intention of not departing or departing slightly, from the hitherto accepted

---

2. *ibid.*, p. 181.

3. *The Development of Logic*, Oxford University Press (1962), p. 741.

4. *ibid.*, p. 740.

5. Ross, W. D., *Aristotle*, Oxford University Press, p. 20.

conception of logic as Kneale[6] and Martin[7] seem to suggest. But if the delimitation providing a sharp boundary to logic makes it coextensive with the traditional boundary of logic, it would only show that the traditional conception is unambiguous despite the fact that the corresponding definition, by which ambiguity becomes eliminable, has not been formulated explicitly.

Now, if the delimiting of the concept of logic ought not to be usage-oriented, i.e. either aimed at the confirmity to the accepted use or at suggesting an acceptable use, what should be the direction in which we should proceed? The aim should clearly be providing sharp boundaries to the concept. The very ideas of confirmity and departure from tradition are irrelevant here. It might be the case that the two millennia old tradition is wrong, or it might be the case that the contemorary logicians have a glorious illusion and Kant is right.[8] However, if logic is taken, or intended, to be the theory of all theories, whatever is included in it must be applicable to itself. Lest it will fall short of complete generality. To illustrate, and not to prejudge, if the concept of logic is defined in such a way that the law of excluded middle falls under it, then the concept of logic itself must be within the range of the applicability of the law. Secondly, as logic is conceived to be the common core of all theories, it should not share with any specific theory some feature not found in each and all theories. On these two points, I hope, an agreement can be reached. But, as shall be shown in the sequel, when logic is delimited keeping these two points in view, it becomes coextensive with what is called *unextended first order predicate logic*.

Though the inclusion of identity in logic is insufficient but necessary to make set theory a part of it, the exclusion of it from logic is sufficient (and, perhaps, necessary) to relegate from it set theory. So, instead of raising the boundary issue at a point where no convincing answer can be found (say with reference to the logical nature of the axioms needed for formalising set theory), we can retreat a bit and take a safer position at identity itself, for it is strategically important to fight a decisive battle. The reason adduced for the inclusion of identity in logic is this: "its persistent recurrence in all sorts of theories and its relevance to all sorts of universes of discourse is . . . (responsible for its being) . . . customarily

6. *op. cit.*, p. 471.

7. "Ontology and the province of logic," in *Contributions to Logic and Methodology in Honour of J. M. Bocheński*, A. T. Tymieniecka, ed. (1965), p. 273.

8. " . . . unless we choose to consider as improvements the removal of some unnecessary subtleties of the clear exposition of its doctrine both of which are forto the elegance rather than to the solidity of the science . . . it is remarkable . . . that to the present day (logic) has not been able to advance a step and is thus to all appearances complete and perfect." *Critic of Pure Reason*, trans. N. K. Smith, Preface.

considered under the head logic".[9] Thus factual omnipresence has been accepted as a sufficient reason for bestowing logical status to identity. Considering this omnipresence let us see whether it belongs to the theories proper, or to that part of the various theories which is common to all those theories (that is to logic). As an example, let us take ST, set theory, to present which, it is usual to assume the first order predicate logic, FPL for short. Now, augment FPL by say Hao Wang's axiom containing the binary predicate '=', namely

(i) $Fy \equiv (x)\ ((x = y)\ \&\ Fx)$

which guarantees the substitutivity of identicals, and all other properties that are usually assigned to identity. The result of such an augmentation is called, as it usually has been, *extended first order predicate logic*, EFPL for short. To have ST, EFPL is further augmented by suitable axioms in which the only binary predicate occurring will be 'ϵ'. It has become customary to include in the set of axioms added to EFPL one containing the identity predicate. This is the axiom of extensionality. But when the relationship between 'ϵ' and '=' is definitionally fixed as

(ii) $(x = y) \equiv_{df} (z)\ ((z \epsilon x) \equiv (z \epsilon y))$

the axiom of extensionality, namely

(iii) $(z)\ ((z \epsilon x) \equiv (z \epsilon y)) \rightarrow (x = y)$

can be deduced from (i) and (ii), by substituting the definiens for '$x = y$' and '$x =$' for 'F' in (i), and applying only the rules of sentential logic.

Now let ST* be the theory which is the result of augmenting FPL by the usual set theoretic axioms containing the binary predicate 'ϵ', such that an augmentation of ST* by (i) and (ii) will result in ST. A comparison of ST* and ST will show certain important things. First, according to ST the cardinality of sets having the same members is the same, whereas there is no way of either asserting or denying this with respect to ST*. It provides rubrics for logical equivalence, by virtue of containing '$\equiv$', but not for numerical identity as is the case with ST. This is the contribution of (i) and (ii). But the same thing could be accomplished by adding, as an axiom, the following to ST*.

(iv) $(x = y) \equiv ((P^1_1 x \equiv P^1_1 y)\ \&\ \ldots\ \&\ (P^1_2 x \equiv P^1_2 y)\ \&\ \ldots\ \&\ (x_1) \ldots (x_n)$
$(P^n_1 x, x_1, \ldots, x_n \equiv P^n_1 y, x_1, \ldots, x_n)\ \&\ \ldots\ \&\ (x_1) \ldots (x_n)$
$P^n_n x, x_1, \ldots, x_n \equiv P^n_n y, x_1, \ldots, x_n))$

where 'P' with i as its superscript and j as its subscript is the jth predicate of i arguments. The equant of (iv) might be considered to be belonging to FPL in the sense that '=' may be considered as the first two-place predicate of FPL. Taking '=' thus, and by replacing 'P^2_1' by 'ϵ' in (iv)

9. Quine, W. V. O., *Set Theory and its Logic,* Harvard University Press (1963), p. 13.

we can have the necessary apparatus to assert the unity of the equivalents, for after such replacement (iii) will turn out to be a consequence of (iv). But there is a snag here. Neither (iv), nor its equants, is a truth, that is a theorem of FPL. So a justification of its acceptance can be offered only on non-logical grounds. However, (iv), it might be argued, can be given logical status, for example by treating it as a "meaning postulate", in the Carnapian sense of the term. But this can be done only by smuggling into the import of (iv) our intentions and intuitions. Thus when once (iv) is accepted it will have to be admitted that identity can be introduced into a theory only in terms of the non-logical truths of that theory. This may be treated as a sufficient reason to banish the notion of identity from the realm of logic.

Identity is needed, and is introduced into theories only to provide conditions under which we can decide for any two given expressions whether they refer to, or report about, one object or two objects; and as such it has something to do with the language in which they are expressions, and the nature of the objects referred to, or reported about in that language. Thus identity is linked to a host of other problems, which will have to be settled with reference to the nature of objects, and for which there can be no fruitful blanket logical solution. One such problem is: what are the conditions under which numerical distinction could be affirmed between objects? Such issues get suitable answers by ontologico-epistemological inquiries, and not by arbitrarily accepting some conditions and bestowing on them logical status. Identity is thus linked to the notion of counting, and logic need not provide an abacus for counting however badly we might need one.

Martin[10] has shown that the acceptance of the definiens of (iv) presupposes, firstly that the number of predicates is finite, secondly that no predicate is universal. The second and the third presuppositions are taken care of in FPL by the following truth of it:

(v) $(x)\,Ax \rightarrow (\mathrm{E}x)Ax.$

No

(vi) $(x)P_i^k x_i, \ldots, x_n \; (i \leq j \leq n)$

is a truth in FPL. The first presupposition raises some epistemological and ontological problems; in fact it has all the affiliations of the Leibnizian concept of identity and hence all the difficulties associated with it. And the second presupposition has been objected to, and attempts have been made to purge FPL from (v). The restriction on the domains of interpretation to the effect that they are non-empty, which we find in the standard or classical interpretations of FPL is due to the presence of (v) in it. This, in a sense, implies that "identity is already contained" in FPL "whether we

10. Martin, R. M., *The Notion of Analytic Truth,* Philadelphia (1959).

like it or not"[11]; and this is due to the reason that identity requires existence of entities, and existence of entities requires identity. And they being so extricably related together, and as existence of at least one entity is taken for granted by FPL, it can be entertained that FPL provides for identity too. But, if we want to banish identity from FPL, we want it not because we want to strip off FPL from (v), but for a different reason.

It is true that FPL takes only non-empty domains as its models. But a little more careful attention brings to our notice that the restriction on the domains of interpretation of FPL is still more stringent. If we want a truth of FPL to hold for all nonempty domains under all interpretations of the predicates occurring in it, then those domains should not be just sets of objects, but sets of overlaping non-empty sets of objects *and* either unit sets or objects themselves.[12] The variables when tagged to monadic predicates take unit sets (or objects) as values, and when tagged to n-adic predicates, the n variables together take n-tuples of unit sets (or objects) as values. Anyhow, what is worth noting is that in that case the domains of interpretation cannot consist of homogeneous entities. And the ontological assumptions implicit in the axiom of pairing of classical set theory (to which is geared FPL) underwrites the existence of such sets, and thereby safeguards the validity of (v) with respect to non-monadic predicates. Lest the non-monadic predicates of FPL would be empty clinching the validity of (v).

This shows how soaked in ontology is FPL. To assume *being* instead of *non-being* is one thing, and to assume the *plurality of being* is another. The choice between the unity and the plurality of being is an ontological choice; and the acceptance of the latter presupposes a taxonomical analysis of being, which is the business of an ontologist and not of a logician. With respect to monadic predicates (v) assumes being, but with respect to non-monadic predicates plurality of being or multiple types of being.

In order to come out of this ontological mire, one might treat the variables tagged to non-monadic predicates to be just like those tagged to monadic predicates, in the sense that they severally, that is each in itself and not together, take objects (or unit sets) as values. In that case the following will have to be admitted.

(vii) $(x_1) \ldots (x_n) P^n x_1, \ldots, x_n \equiv_{df} (x_1) P^* x_1 \;\&\; \ldots \;\&\; (x_n) P^* x_n$

where 'P^*x_i' is '$Q^{n-1} x_{i-1}, \ldots, x_n, x_1, \ldots, x_{i+1}$' when '$Q^{n-1}$' is a predicate which when tagged to n - 1 variables results in a well-formed expression. Now by parity of reasoning we will have to admit

(viii) $(x_1) \ldots (x_n) P^n x_1, \ldots, x_n \equiv_{df} (\ldots ((P_n' x_n)' x_{n-1})' \ldots) x_1$

11. See footnote 7 above.

12. It depends on the way ordered pairs are defined, that is whether they are defined as $(x, y) = \{\{x\}, \{x, y\}\}$ or as $(x, y) = \{x, \{x, y\}\}$.

It is clear that the monadic predicate occurring in the definiens of (viii) does not occur in the list of the monadic predicates of FPL. But FPL could be reformulated, say as FPL* whose predicates are all monadic, such that there exists a function F, and for each (a, b) with a in FPL and b in FPL*, (a, b) is in F, and a is a theorem of FPL if and only if b is a theorem of FPL*. (viii) itself indicates the existence of such a function, and the reformulation can be carried out by taking the predicates of FPL* to be the monadic predicates of FPL plus the monadic predicates

$$P^*_{i_1}, \ldots, P^*_{i_n}, (i = 1, 2, \ldots)$$

such that for any formula S of FPL in which

$$P^{i+1} x_1, \ldots x_{i-1}$$

occurs is a theorem if and only if there is a formula S^* of FPL* in which '$P^*_{i_1}$' occurs and is a theorem of FPL*. (viii) guarantees the non-emptiness of these new predicates as it does of the old ones. However, as the variables tagged to these two sets of predicates are the same, in the sense that a variable tagged either to the predicates which FPL* shares with FPL or to the predicates introduced into it, refer to the entities of the domain individually. Hence the entities of the domain of interpretation of FPL* can be of just one category, which implies that FPL* takes only *being* (and not types of being) for granted. FPL indulges us in a taxonomy of being, for the notion of predicate with reference to it is not an undifferentiated or univocal one. As predicates are tagged to arguments, they will have to be differentiated with reference to the kind of arguments to which they are tagged. A monadic predicate, for instance, is tagged to one argument, and a diadic one to two arguments *taken together* (thus really to one argument). And this (usually ignored) difference brings in the difference between the types of values of the arguments, say in '*Pa*' and '*Qab*.' In contradistinction to this, the notion of predicate in FPL* is univocal as all of its predicates are tagged to the same type of arguments. Thus the predicates, the arguments, and the entities of the domains of interpretations all are uniformly homogeneous with reference to FPL*. The apparent "difference" between the two sets of predicates of FPL* is only due to the *factual complexity* of the properties exemplified by the entities of the domain and (or) due to the subjective difficulty in picking up those properties to assign to the predicates of FPL* while providing an interpretation to it.[13] This, however, is neither a logical drawback nor a drawback in logic. Logic is not intended either for simplifying the universe or for simplifying our knowledge of it. That is why, we can use the apparently two kinds of predicates indifferently as long as we are confined to logical activity. This means that 'logic' and 'first order monadic predicate logic' are co-extensive expressions.

---

13. The epistemological issues involved here need not bother us in the present context, for they do not have any serious bearing on the point under discussion.

There have been a spate of objections, since the time of Russell's *Introduction to Mathematical Philosophy*, to the consideration of 'first order monadic predicate logic' and 'logic' as co-extensive terms. These objections are in relation to (v), which guarantees the non-emptiness of the predicates, or to use different terminology, the existence of entities. By applying the rules of inference accepted in FPL*, we can deduce, from (v),

(ix) $(\mathrm{E}x)\,(Ax \vee \sim Ax)$.

Considering (ix) to be "empirical" in nature, it was demanded to exclude it, (v), and their ilk from logic. This was thought to be incumbent, if, as Carnap puts it, "logic is to be independent of empirical knowledge".[14] This, no doubt, is an idea which is accepted by every logician as an ideal. But what is important to note is that the possibility of raising such an objection to (v) and (ix) requires certain untenable presuppositions, one of which is the acceptance of '*a* exists' as a factual statement just as '*a* is red'. The former is taken to be containing empirical knowledge as the latter does; and, as is obvious, it would be impossible to assume this if 'exists' is not taken as a natural property in the same way as 'is red' is. Further, such a treatment of the predicate 'exists' presupposes that knowledge of an object constitutes knowing *what* an object is just as much as it constitutes knowing that it *is*. But are these assumptions and presuppositions less arbitrary or less dubious than those that are found to be underlying FMPL by those who objected to the inclusion of (v) and (ix) in logic? Raising objections to FMPL on such doubtful assumptions seems to be an attempt to jump from the frying pan of ontology to the fire of epistemology. If logic should be free from ontological contamination, why should it not be free from epistemological contamination? If the purity of logic would be lost even on the minimum ontic commitment to the effect that the predicates are not empty, how could it be saved if it indulges in the assumptions that existence is a natural property, and that knowledge of existence is empirical?

The attempts to guard logic from ontology were made in the direction of syntax as well as semantics, sharing, however, a common feature, namely a narrow conception of logical truth, namely the one which is defined not in terms of all non-empty domains, but in terms of all domains, be they of cardinality zero or aleph null. Those who opted the syntactical approach suggest a reformulation of FMPL by a change either in the axiom set or by placing restrictions on the rules of inference so as to eliminate (v), and all those truths that follow from it. Let the result of such a reformulation be called free first order monadic predicate logic FFMPL. Those who opted the semantical approach suggest that there is nothing wrong with FMPL as such, but only with the standard interpretations offered to it. Hence they suggest that either of the following should be followed: 1) to make provi-

14. Carnap, R., *The Logical Syntax of Language*, London (1937), p. 140.

sion for an empty individual *in* the domain of interpretation, and accept the usual method of interpreting the formulas, or 2) to find fault with the conventional interpretation of the quantifiers, and provide fresh interpretation in order to eliminate the presumed ontological commitments. In their zeal to reformulate FMPL, which has a perceptable Aristotelian bias in so far as it admits only non-empty predicates, the reformists ended up in the world of Platonic essences (essences, pure and simple, and in need of no entity to exemplify them). Thus FFMPL has its own ontic commitments and consequences, and the freedom it exhibits is the freedom to choose one kind of ontology in preference to another, and not freedom from ontology altogether.

(v) and (ix) can be given logical status, and thereby 'logic' and 'monadic predicate logic' can be equated, on purely non-empirical grounds, and for reasons that have nothing to do with ontological decisions. Nor will these reasons be expediency oriented. We can justify this equation without indulging in the taxonomy of being or the morphology of knowledge. Also, when we are equating these we are not doing so due to any fascination for Aristotelian metaphysic, but only due to the reason that such a conclusion is forced on us by the very concept of logic. This is to say that there is an *a priori* justification for that equation. "Logic", as we said earlier, is a sub-theory of all theories. Now let us pin down our attention to these concepts and their mutual relationship. A "theory", by definition, is an ordered n-tuple where the first term, say T is a non-empty set, and the remaining terms are functions or relations defined over T, or are unique members of T. Thus, by definition, when T is a null set there is no theory, and hence no question of a subtheory of a non-existent theory. We can, no doubt, keep the empty set in T; the definition of a theory as formulated above permits this. But a theory of an empty domain of entities is impossible.[15]

At the outset itself we remarked that logic, in order to be the common core of all theories should not share with any particular theory something that is not shared by all theories. Now, the impossibility of a theory when T is an empty set shows that (v) and (ix) do not belong exclusively to any specific theory, but constitute the common assumption of all theories—in

15. What, in fact, is needed is neither a free logic nor a theory of the empty domain, but a simple and elegant semantic apparatus for a class of theories containing individual variables, say $x, x', x'', \ldots$, individual constants, say $a, a', a'', \ldots$, and function letters, say $f, f', f'', \ldots$ taking individual variables, individual constants and functions themselves as arguments, such that the set of terms T has as its members individual variables, individual constants and well-formed functions, and also such that there is a non-empty proper subset of T, for whose members we can find no correlatable objects in the domains of interpretations over which range all of their variables. Historically speaking the suspecting of FPL and the attempts to build FFPL are products of investigations into this field.

fact, an assumption that makes theories possible, an assumption but for which theories would be impossible. Hence the imperativeness of their inclusion in logic. The second requirement that we laid down earlier is that the concept of logic itself should be within the range of the applicability of whatever is asserted in it. This is to say that the laws of logic should apply to the concept of logic itself. Now, when "logic" is equated with "first order monadic predicate logic" this requirement will be satisfied, for, as FMPL is a decidable theory it would be possible for us to decide in a finite number of steps whether a given formula is one that belongs to logic or not. Thus we have a precise and definite concept of logic.

The delimiting of logic in this way need not dishearten those who believe that there has been progress in logic, for it does not deprive logicians of their achievement, but only puts it in proper perspective, and indicates the points where the claim for the achieved progress can be made. An analogy might be of some help to drive the point home. Consider what belongs to the Euclidian Geometry. Most of it was known to the predecessors of Euclid. But what has happened to this quanta of knowledge with Euclid, and twenty two hundred years later, with Hilbert! What these two together have achieved in two millennia in Euclidian Geometry, Frege, Hilbert, Gödel and Skolem achieved in less than half a century in logic. But this achievement had to wait until man started theorising in the century in which logic acquired the simplicity of a theory. With the deductive unification of logic, logical truths got a "logical" vindication. To recognize this, one has only to compare Aristotle's *Analytics* with Church's *Introduction.*

Indian Institute of Advanced Study
Rashtrapati Nivas, Simla, India