Notre Dame Journal of Formal Logic
Volume XIV, Number 2, April 1973
NDJFAM

ON THE NUMBER OF OVERLAPPING SUBSETS OF A SET

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In the course of some other research, *cf.* [2], the question arose whether or not, if we have two sets A and B, which are subsets of a given set M, and which overlap, i.e., $A \cap B \neq \phi$, $A - B \neq \phi$ and $B - A \neq \phi$, are there any more subsets of M which possess the same overlap property among themselves as well as with A and B. The theorem herein answers this affirmatively.

The following notation indicates the cardinality of the sets under investigation.

$\mathfrak{a} = \overline{\overline{A - B}}$
$\mathfrak{b} = \overline{\overline{B - A}}$
$\mathfrak{c} = \overline{\overline{\mathsf{C}(A \cup B)}}$, where $\mathsf{C}(A)$ is the complement of A, relative to M,
$\mathfrak{d} = \overline{\overline{A \cap B}}$

Let $D_n \subset A \cap B$ denote a subset of $A \cap B$ such that there exists an element $d_n \in A \cap B$ and $D_n = A \cap B - \{d_n\}$. Let D denote the set of all such sets. Let a_i denote elements of $A - B$, b_j elements of $B - A$, c_m elements of $\mathsf{C}(A \cup B)$ and d_n elements of $A \cap B$. The following sets are used at various places in the proof:

$E^1 = \{D_k \cup \{a_i\} \cup \{b_j\} \mid D_k \in D,\ a_i \in A - B,\ b_j \in B - A\}$
$E^2 = \{D_k \cup \{c_m\} \mid D_k \in D,\ c_m \in \mathsf{C}(A \cup B)\}$
$E^3 = \{D_k \cup \{a_i\} \cup \{c_m\} \mid D_k \in D,\ a_i \in A - B,\ c_m \in \mathsf{C}(A \cup B)\}$
$E^4 = \{D_k \cup \{b_j\} \cup \{c_m\} \mid D_k \in D,\ b_j \in B - A,\ c_m \in \mathsf{C}(A \cup B)\}$
$E^5 = \{\{a_i\} \cup \{b_j\} \cup \{c_m\} \mid a_i \in A - B,\ b_j \in B - A,\ c_m \in \mathsf{C}(A \cup B)\}$
$E^6 = \{\{d_k\} \cup \{a_i\} \cup \{b_j\} \mid d_k \in A \cap B,\ a_i \in A - B,\ b_j \in B - A\}$
$E^7 = \{\{d_1\} \cup \{c_m\} \mid d_1 \in A \cap B,\ c_m \in \mathsf{C}(A \cup B)\}$
$E^8 = \{(A \cap B) \cup \{c_m\} \mid c_m \in \mathsf{C}(A \cup B)\}$
$E^9 = \{\{a_i\} \cup \{b_j\} \cup \mathsf{C}(A \cup B) \mid a_i \in A - B,\ b_j \in B - A\}$
$E^{10} = \{\{d_k\} \cup \mathsf{C}(A \cup B) \mid d_k \in A \cap B\}$

Define the predicate $\mathsf{P}(A, B)$ as follows:

$$[AB]: \mathsf{P}(A, B) .\equiv. A \cap B \neq \phi .\ A - B \neq \phi .\ B - A \neq \phi .$$

Received February 11, 1971

Notice that if A, $B \subset M$, then $M = (A \cap B) \cup (A - B) \cup (B - A) \cup C(A \cup B)$, where $C(A \cup B)$ is the complement relative to M. Also notice that the pairwise intersection of these components of M is empty; hence, we have

$$\overline{\overline{M}} = \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d}$$

Lemma 1. *If M is a finite set with $\overline{\overline{M}} = \mathfrak{m}$, and if $A \subset M$, $B \subset M$, and $P(A, B)$, then there are at least m subsets of M, $A_1, \ldots, A_m$ such that $A, B \in \{A_i | i \leqq \mathfrak{m}\}$ and $P(A_i, A_j)$ for all $i, j, \leqq \mathfrak{m}$, $i \neq j$.*

Proof: We shall work by cases.

a) $\mathfrak{d} > 2$, $\mathfrak{a}, \mathfrak{b} > 1$. There are d distinct sets in D and if $D_k \in D$ then $\overline{\overline{D}}_k = \mathfrak{d} - 1$. Clearly for D_1, $D_2 \in D$, $D_1 \neq D_2$ we have $P(D_1, D_2)$. Let E_1, $E_2 \in E^1$, F_1, $F_2 \in E^2$, then by the disjointness of the components of M we have $P(E_1, E_2)$ for $E_1 \neq E_2$, $P(F_1, F_2)$ for $F_1 \neq F_2$, $P(E_1, A)$, $P(E_1, B)$, for $E_1 \in E^1$, $P(F_1, A)$, $P(F_1, B)$, for $F_1 \in E^2$, $P(E_1, F_1)$, for $E_1 \in E^1$, $F_1 \in E^2$, and finally $P(A, B)$ by assumption. $\overline{\overline{E^1}} = \mathfrak{d}\mathfrak{a}\mathfrak{b}$, $\overline{\overline{E^2}} = \mathfrak{d}\mathfrak{c}$ hence with A and B there are $\mathfrak{d}\mathfrak{a}\mathfrak{b} + \mathfrak{d}\mathfrak{c}$ sets satisfying P, but since $\mathfrak{d} > 2$ and $\mathfrak{a}, \mathfrak{b} > 1$ we have $\mathfrak{d}\mathfrak{a}\mathfrak{b} + \mathfrak{d}\mathfrak{c} = \mathfrak{d}(\mathfrak{a}\mathfrak{b} + \mathfrak{c}) \geqq \mathfrak{d}(\mathfrak{a} + \mathfrak{b} + \mathfrak{c}) \geqq \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = \mathfrak{m}$.

b) $\mathfrak{d} > 2$, $\mathfrak{a} = 1$, $\mathfrak{b} > 1$, $\mathfrak{c} \neq 0$. Let E_1, $E_2 \in E^4$ then if $E_1 \neq E_2$, $P(E_1, E_2)$. Also we have $P(E_1, A)$, $P(E_1, B)$, for $E_1 \in E^4$, and $P(A, B)$. Therefore we have, including A and B, $\mathfrak{d}\mathfrak{b}\mathfrak{c} + 2$ sets satisfying P, also $\mathfrak{d}\mathfrak{b}\mathfrak{c} + 2 \geqq (\mathfrak{d} + \mathfrak{b})\mathfrak{c} + 2 \geqq \mathfrak{d} + \mathfrak{b} + \mathfrak{c} + 1 = \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = \mathfrak{m}$.

c) $\mathfrak{d} > 2$, $\mathfrak{a} = 1$, $\mathfrak{b} > 1$, $\mathfrak{c} = 0$. Let E_1, $E_2 \in E^1$ then from above in a), we have $\mathfrak{d}\mathfrak{b} + 2$ sets satisfying P. But $\mathfrak{d}\mathfrak{b} + 2 \geqq \mathfrak{d} + \mathfrak{b} + 2 \geqq \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = \mathfrak{m}$. Similarly using E^3, we can show b) and c) are true when $\mathfrak{d} > 2$, $\mathfrak{a} > 1$, $\mathfrak{b} = 1$.

d) $\mathfrak{d} > 2$, $\mathfrak{a} = \mathfrak{b} = 1$, $\mathfrak{c} \neq 0$. Let E_1, $E_2 \in E^4$, F_1, $F_2 \in E^5$ then we have for $E_1 \neq E_2$ $P(E_1, E_2)$, $P(E_1, A)$, $P(E_1, B)$, and $P(A, B)$ as before in b). Since $F_1 \neq F_2$, there are c_{m_1} and c_{m_2} such that $c_{m_1} \in F_1$, and $c_{m_2} \in F_2$ and $c_{m_1} \neq c_{m_2}$. Hence, we also have $P(F_1, F_2)$ for $F_1 \neq F_2$. Clearly, $P(F_1, A)$ and $P(F_1, B)$ for $F_1 \in E^5$ and finally we have $P(E_1, F_1)$ since they have at least $\{b_j\}$ in common and $a_i \neq D_k$. Hence there are at least $2 + \mathfrak{d}\mathfrak{c} + \mathfrak{c}$ sets satisfying P and $2 + \mathfrak{d}\mathfrak{c} + \mathfrak{c} \geqq \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = \mathfrak{m}$.

e) $\mathfrak{d} > 2$, $\mathfrak{a} = \mathfrak{b} = 1$, $\mathfrak{c} = 0$. Using E^1 again we have $2 + \mathfrak{d} = \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = \mathfrak{m}$ sets satisfying P.

f) $\mathfrak{d} = 2$, $\mathfrak{a}, \mathfrak{b} > 1$. Let $A \cap B = \{d_1, d_2\}$. Let E_1, $E_2 \in E_1^6$ where $E_1^6 = \{E \in E^6 | d_k = d_1\}$ and F_1, $F_2 \in E^7$. We have $P(E_1, E_2)$ for $E_1 \neq E_2$ and $P(E_1, A)$, $P(E_1, B)$, $P(F_1, A)$ and $P(F_1, B)$ as usual. For $F_1 \neq F_2$, $P(F_1, F_2)$ and finally $P(E_1, F_1)$. Hence the set satisfying P has at least $2 + \mathfrak{a}\mathfrak{b} + \mathfrak{c}$ elements but $2 + \mathfrak{a}\mathfrak{b} + \mathfrak{c} \geqq \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = \mathfrak{m}$.

g) $\mathfrak{d} = 2$, $\mathfrak{a} = 1$, $\mathfrak{b} > 1$. If we consider sets in E^6 and E^8 we see that they, along with A and B, satisfy P. So we have $2\mathfrak{b} + \mathfrak{c} + 2$ such sets and $2\mathfrak{b} + \mathfrak{c} + 2 > \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = \mathfrak{m}$.

h) $\mathfrak{d} = 2$, $\mathfrak{a} > 1$, $\mathfrak{b} = 1$. By symmetry in E^6 we proceed as in g).

i) $\mathfrak{d} = 2, \mathfrak{a} = \mathfrak{b} = 1$. With this we have $\overline{\overline{E^6}} = 2$ and $\overline{\overline{E^8}} = \mathfrak{c}$ and the sets in E^6 and E^8, along with A and B, satisfy P as above. Hence, we have at least $2 + \mathfrak{c} + 2$ sets satisfying P and $2 + \mathfrak{c} + 2 = \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = \mathfrak{m}$.

j) $\mathfrak{d} = 1, \mathfrak{a}, \mathfrak{b} > 1$. Consider again E_1^6 and E^8 and we have $\mathfrak{a}\mathfrak{b} + \mathfrak{c} + 2$ sets satisfying P and $\mathfrak{a}\mathfrak{b} + \mathfrak{c} + 2 \geqq \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + 2 > \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = \mathfrak{m}$.

k) $\mathfrak{d} = 1, \mathfrak{a} = 1, \mathfrak{b} > 1$. If we consider sets in E^8 and E^9, we find they satisfy P, along with A and B, and there are $\mathfrak{c} + \mathfrak{a}\mathfrak{b} + 2$ such sets, while $\mathfrak{a}\mathfrak{b} + \mathfrak{c} + 2 = \mathfrak{b} + \mathfrak{c} + 2 = \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = \mathfrak{m}$.

l) $\mathfrak{d} = 1, \mathfrak{a} > 1, \mathfrak{b} = 1$. Similar to k) since E^9 is symmetric in A and B.

m) $\mathfrak{d} = \mathfrak{a} = \mathfrak{b} = 1, \mathfrak{c} \neq 0$. Then the sets in E^5 and E^{10}, with A and B, satisfy P so there are $\mathfrak{c} + 1 + 2$ such sets and $\mathfrak{c} + 1 + 2 = \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = \mathfrak{m}$.

n) $\mathfrak{d} = \mathfrak{a} = \mathfrak{b} = 1, \mathfrak{c} = 0$. In this case, $\overline{\overline{M}} = 3$ and the result is obvious.

Thus by a)-n) we have that for any two overlapping subsets A and B of a finite set M, whose cardinality is $\mathfrak{m}$, there are at least $\mathfrak{m}$ subsets, including A and B, which overlap.

Lemma 2. *If M is a set which is not finite, such that $\overline{\overline{M}} = \mathfrak{m}$, and if $A \subset M$, $B \subset M$, and* P(A, B), *then there are at least $\mathfrak{m}$ subsets of M, including A and B, which satisfy* P *pairwise.*

The proof of this theorem requires the use of the axiom of choice. As before, $\overline{\overline{M}} = \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = \mathfrak{m}$, but since $\mathfrak{m}$ is not finite, by theorems of Iseki and Leśniewski, *cf.* [1], p. 414, we have that $\mathfrak{a} = \mathfrak{m}$ or $\mathfrak{b} = \mathfrak{m}$ or $\mathfrak{c} = \mathfrak{m}$ or $\mathfrak{d} = \mathfrak{m}$.

Proof:

a) $\mathfrak{d} = \mathfrak{m}$. There are $\mathfrak{m}$ sets in E^1 which, along with A and B, satisfy P, so there are at least $\mathfrak{m}$ subsets of M satisfying P.

b) $\mathfrak{a} = \mathfrak{m}$ or $\mathfrak{b} = \mathfrak{m}$. Here we have $\mathfrak{m}$ subsets of M in E^6 which, including A and B, overlap pairwise.

c) $\mathfrak{c} = \mathfrak{m}$. Using E^8, A, and B, we have at least $\mathfrak{m}$ overlapping subsets of M satisfying P.

Hence if M has non-finite cardinality $\mathfrak{m}$, and A and B are overlapping subsets of M, there are at least $\mathfrak{m}$ subsets of M, including A and B which satisfy P. Thus from Lemmas 1 and 2 we have:

Theorem. *If $A \subset M$, $B \subset M$, $\overline{\overline{M}} = \mathfrak{m}$, and* P$(A, B)$, *then there are at least $\mathfrak{m}$ subsets of M, including A and B, which satisfy* P.

Note that the proof for M a non-finite cardinal depends upon the axiom of choice, while the proof for the finite case does not.

REFERENCES

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