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A NEW REPRESENTATION OF S5

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We consider first a modal language with propositional constants (and no variables) and show that there is a unique set H of formulas of this language meeting certain attractive syntactical conditions; moreover H is the set of theses of a very simple calculus. We then show that the theses of S5 are characterized by the fact that all their instances are in H.*

Let $\mathcal{L}_c$ be the language having an infinite set of "propositional constants" and connectives $\neg$, $\vee$, and $\Box$ used in the usual way. As usual, other connectives are used as abbreviations. If S is a string of symbols, $s_1, \ldots, s_n$ are distinct symbols, and $S_1, \ldots, S_n$ are strings of symbols, then $S(S_1, \ldots, S_n/s_1, \ldots, s_n)$ is the result of replacing each symbol $s_i (i = 1, \ldots, n)$ in S by the string S_i. A *tautology* is a string of the form $X(S_1, \ldots, S_n/x_1, \ldots, x_n)$ where X is a tautology of the classical propositional calculus and $x_1, \ldots, x_n$ are propositional variables. A set H of formulas of $\mathcal{L}_c$ is *correct* if for all formulas A and B of $\mathcal{L}_c$

(1) If A is a tautology then $A \in H$.
(2) If A has no occurrences of $\Box$ and $A \in H$, then A is a tautology.
(3) If $A \in H$ and $A \Rightarrow B \in H$, then $B \in H$.
(4) $A \in H$ if and only if $\Box A \in H$.
(5) Either $A \in H$ or $\neg\Box A \in H$.

Let $\mathcal{L}_v$ be the language which is like $\mathcal{L}_c$ except that $\mathcal{L}_v$ has a countably infinite set of "propositional variables" rather than propositional constants. A set J of formulas of $\mathcal{L}_v$ is said to be correct if it consists of all formulas X of $\mathcal{L}_v$ such that every formula of $\mathcal{L}_c$ of the form $X(A_1, \ldots, A_n/x_1, \ldots, x_n)$ is a member of H, where H is a correct set of formulas of $\mathcal{L}_c$.

Let $\mathfrak{C}$ be the formal system whose language is $\mathcal{L}_c$, whose axioms are an appropriate set of tautologies and all formulas of the form

$$\Diamond \,\&\, \{a_i^* \mid i = 1, \ldots, n\}$$

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where $a_1, \ldots, a_n$ are distinct propositional constants and each a_i^* is either a_i or $\neg a_i$, and whose rules are detachment and the following

(6) From $A \Rightarrow B$ infer $\Box A \Rightarrow \Box B$.
(7) From A infer $\Box A$.

If $\mathfrak{S}$ is any formal system then $\mathrm{Thm}(\mathfrak{S})$ is the set of thesis of $\mathfrak{S}$, and $\mathfrak{S} \vdash X$ if and only if $X \in \mathrm{Thm}(\mathfrak{S})$.

Theorem 1. *There is exactly one correct set of formulas of* $\mathcal{L}_c$, *and it is* $\mathrm{Thm}(\mathfrak{C})$.

Proof. We first establish a semantics for $\mathfrak{C}$. Let Con be the set of propositional constants and Fla be the set of formulas of $\mathcal{L}_c$. A *truth value assignment* is a function V:Con $\rightarrow \{\mathbf{T}, \mathbf{F}\}$. Such a V can be uniquely extended to a function V^*:Fla $\rightarrow \{\mathbf{T}, \mathbf{F}\}$ in the obvious way—in particular $V^*(\Box A) = \mathbf{T}$ if and only if $W^*(A) = \mathbf{T}$ for all W:Con $\rightarrow \{\mathbf{T}, \mathbf{F}\}$. We say A is *valid* if $V^*(A) = \mathbf{T}$ for all truth value assignments V. In terms of the partial truth tables originally used by Kripke [1] in defining validity in modal propositional logic, A is valid if and only if A is assigned the value $\mathbf{T}$ in every row of every partial truth table for A which is full, i.e., has all 2^n rows if A has n propositional constants.

A few brief computations suffice to show that the axioms of $\mathfrak{C}$ are valid and that the rules of $\mathfrak{C}$ preserve validity, and hence that every thesis of $\mathfrak{C}$ is valid. The converse is proved by a slight modification of Kalmár's proof of the analogous result for classical propositional calculus. For any formula B and truth value assignment V, let $B^V = B$ or $B^V = \neg B$ according as $V^*(B) = \mathbf{T}$ or $V^*(B) = \mathbf{F}$. It suffices to prove that if $a_1, \ldots, a_n$ are distinct propositional constants including all those occurring in A then $\mathfrak{C} \vdash \& \{a_i^V \mid i = 1, \ldots, n\} \Rightarrow A^V$. (For then if A is valid and $V_1, \ldots, V_{2^n}$ are appropriate truth value assignments then

$$\mathfrak{C} \vdash \vee \{\& \{a_i^{V_j} \mid i = 1, \ldots, n\} \mid j = 1, \ldots, 2^n\} \Rightarrow A$$

and

$$\mathfrak{C} \vdash \vee \{\& \{a_i^{V_j} \mid i = 1, \ldots, n\} \mid j = 1, \ldots, 2^n\}.)$$

This proof proceeds by induction on the length of A. Leaving the easy cases to the reader, we suppose $A = \Box B$. If $V^*(A) = \mathbf{T}$ then $W^*(B) = \mathbf{T}$ for all truth value assignments W, so by the induction hypothesis $\mathfrak{C} \vdash \& \{a_i^W \mid i = 1, \ldots, n\} \Rightarrow B$ for all W. As noted above, it follows that $\mathfrak{C} \vdash B$; but then also $\mathfrak{C} \vdash \Box B$ (by (7)) and $\mathfrak{C} \vdash \& \{a_i^V \mid i = 1, \ldots, n\} \Rightarrow \Box B$, as required. If $V^*(A) = \mathbf{F}$ then $W^*(B) = \mathbf{F}$ for some W, so by the induction hypothesis $\mathfrak{C} \vdash \& \{a_i^W \mid i = 1, \ldots, n\} \Rightarrow \neg B$. Using (6), $\mathfrak{C} \vdash \Diamond \& \{a_i^W \mid i = 1, \ldots, n\} \Rightarrow \neg \Box B$. But $\Diamond \& \{a_i^W \mid i = 1, \ldots, n\}$ is an axiom of $\mathfrak{C}$, so that $\mathfrak{C} \vdash \neg \Box B$ and $\mathfrak{C} \vdash \& \{a_i^V \mid i = 1, \ldots, n\} \Rightarrow \neg \Box B$, as required.

From this semantics for $\mathfrak{C}$ it follows immediately that for every formula A exactly one of $\mathfrak{C} \vdash A$ and $\mathfrak{C} \vdash \neg \Box A$ holds, and also that $\mathrm{Thm}(\mathfrak{C})$ is a correct set of formulas. If H is any correct set of formulas, then by (1), (5), and (2) all the axioms of $\mathfrak{C}$ are members of H; moreover H is closed under detachment and the rule (7) because of (3) and (4). It is not difficult to prove by induction on the length of A that if A is completely modalized (i.e., every occurrence of a constant in A is within the scope of an occurrence of $\Box$) then either $A \in H$ or $\neg A \in H$. (Consider the cases $A = \Box B$, $A = \neg B$ where B is completely modalized, and $A = B \Rightarrow C$ where B and C are completely modalized.) Also, by (1) and (3), for no formula A do both $A \in H$ and $\neg A \in H$ hold. It follows that if $\Box A \Rightarrow \Box B \notin H$ then $A \in H$ and $B \notin H$ so that $A \Rightarrow B \notin H$. Thus H is also closed under the rule (6). Hence $\mathrm{Thm}(\mathfrak{C}) \subseteq H$. Suppose $A \in H$. Then $\Box A \in H$ so $\neg\Box A \notin H$, so $\mathfrak{C} \nvdash \neg\Box A$ so $\mathfrak{C} \vdash A$. Hence $\mathrm{Thm}(\mathfrak{C}) = H$.

Theorem 2. *There is exactly one correct set of formulas of $\mathcal{L}_v$, and it is* $\mathrm{Thm}(\mathrm{S5})$.

Proof. Since $\mathrm{Thm}(\mathfrak{C})$ is the only correct set of formulas of $\mathcal{L}_c$, it suffices to prove that $X \in \mathrm{Thm}(\mathrm{S5})$ if and only if every formula of $\mathcal{L}_c$ of the form $X(A_1, \ldots, A_n/x_1, \ldots, x_n)$ is a member of $\mathrm{Thm}(\mathfrak{C})$. We shall have no need for an axiomatization of S5, but we shall review the original truth-table semantics for S5 due to Kripke [1, pp. 11ff]. A *truth value assignment* is a map V from the set of propositional variables to $\{\mathsf{T}, \mathsf{F}\}$. A *complete assignment* is a pair (V, K) where K is a set of truth value assignments and $V \in K$. One may visualize a complete assignment as a "partial truth table with designated row." Then $(V, K)^*(X)$ is defined by

$$(V, K)^*(x) = V(x)$$
$$(V, K)^*(\neg X) = \mathsf{T} \text{ iff } (V, K)^*(X) = \mathsf{F}$$
$$(V, K)^*(X \vee Y) = \mathsf{T} \text{ iff } (V, K)^*(X) = \mathsf{T} \text{ or } (V, K)^*(Y) = \mathsf{T}$$
$$(V, K)^*(\Box X) = \mathsf{T} \text{ iff } (W, K)^*(X) = \mathsf{T} \text{ for all } W \in K.$$

X is *valid* in S5 if $(V, K)^*(X) = \mathsf{T}$ for all complete assignments (V, K). Then $\mathrm{S5} \vdash X$ if and only if X is valid in S5.

Now if X is valid in S5 then $X(a_1, \ldots, a_n/x_1, \ldots, x_n)$ is plainly valid in $\mathfrak{C}$. Moreover, if X is valid in S5 then so is every formula $X(X_1, \ldots, X_n/x_1, \ldots, x_n)$. Hence if X is valid in S5 then every formula of $\mathcal{L}_c$ of the form $X(A_1, \ldots, A_n/x_1, \ldots, x_n)$ is valid in $\mathfrak{C}$. The converse is rather more difficult.

Let $x_1, \ldots, x_n$ be distinct propositional variables, and $a_1, \ldots, a_n$ distinct propositional constants. If V is a truth value assignment to $x_1, \ldots, x_n$ (i.e., $V \in \{\mathsf{T}, \mathsf{F}\}^{\{x_1, \ldots, x_n\}}$) then there corresponds naturally a truth value assignment to $a_1, \ldots, a_n$, which for the sake of notational convenience we shall also call V. We claim first that if K is a non-empty set of truth value assignments to $x_1, \ldots, x_n$, then there are formulas $A_1, \ldots, A_n$ of $\mathcal{L}_c$ such that

(8) There are no symbols in A_i ($i = 1, \ldots, n$) other than $a_1, \ldots, a_n$, $\neg$, $\vee$.
(9) For all $V \in K$ and $i = 1, \ldots, n$, $V^*(A_i) = V(x_i)$.
(10) For all $V \notin K$, $\mathfrak{C} \vdash \neg\& \{A_i^{V(x_i)} \mid i = 1, \ldots, n\}$, where the meaning of $A_i^{V(x_i)}$ is given by $A^{\mathsf{T}} = A$ and $A^{\mathsf{F}} = \neg A$.

For by the functional completeness of classical propositional logic we know that for every $\alpha:\{\mathsf{T}, \mathsf{F}\}^{\{x_1, \ldots, x_n\}} \to \{\mathsf{T}, \mathsf{F}\}$ there is a formula A (having no symbols other than $a_1, \ldots, a_n, \neg, \vee$) such that $\alpha(V) = V^*(A)$ for all $V \in \{\mathbf{T}, \mathbf{F}\}^{\{x_1, \ldots, x_n\}}$. Choose $V_0 \in K$, and for $i = 1, \ldots, n$ define α_i by

$$\alpha_i(V) = \begin{cases} V(x_i) \text{ if } V \in K \\ V_0(x_i) \text{ if } V \notin K. \end{cases}$$

Then there are formulas $A_1, \ldots, A_n$ satisfying (8), such that $V^*(A_i) = \alpha_i(V)$ for all i and V. So if $V \in K$ then $V^*(A_i) = \alpha_i(V) = V(x_i)$, and (9) is satisfied. Moreover if $V \notin K$ and W is any truth value assignment, then $W^*(A_i) \neq V(x_i)$ for some i. Now $W^*(A_i^{V(x_i)}) = \mathbf{T}$ if and only if $V(x_i) = W^*(A_i)$; thus if $V \notin K$ and W is any truth value assignment we have $W^*(\& \{A_i^{V(x_i)} \mid i = 1, \ldots, n\}) = \mathbf{F}$, so (10) is satisfied and our first claim is established.

Now let X be a formula of $\mathcal{L}_v$ having no variables other than $x_1, \ldots, x_n$, and let $\emptyset \neq X \subseteq \{\mathbf{T}, \mathbf{F}\}^{\{x_1, \ldots, x_n\}}$. Let $A_1, \ldots, A_n$ satisfy (8)-(10). Then we claim that for every $V \in K$

$$V^*(X(A_1, \ldots, A_n/x_1, \ldots, x_n)) = (V, K)^*(X).$$

Establishing this claim will complete the proof of the theorem. We proceed by induction on the length of X.

Case 1: $X = x_i$. Then $V^*(X(A_1, \ldots, A_n/x_1, \ldots, x_n)) = V^*(A_i) = V(x_i) = (V, K)^*(X)$. *Case 2:* $X = \neg Y$ or $X = Y \vee Z$. This case is trivial. *Case 3:* $X = \Box Y$. If $V^*(X(A_1, \ldots, A_n/x_1, \ldots, x_n)) = \mathbf{T}$ then for every truth value assignment W, $W^*(Y(A_1, \ldots, A_n/x_1, \ldots, x_n)) = \mathbf{T}$. By the induction hypothesis, $(W, K)^*(Y) = \mathbf{T}$ for all $W \in K$, i.e., $(V, K)^*(X) = \mathbf{T}$. On the other hand, if $V^*(X(A_1, \ldots, A_n/x_1, \ldots, x_n)) = \mathbf{F}$ then there is a truth value assignment W such that $W^*(Y(A_1, \ldots, A_n/x_1, \ldots, x_n)) = \mathbf{F}$. Define V_1 by $V_1(x_i) = W^*(A_i)$. Now $V_1 \in K$, for otherwise $\mathfrak{C} \vDash \neg\& \{A_i^{V_1(x_i)} \mid i = 1, \ldots, n\}$ by (10), but $W^*(\& \{A_i^{V_1(x_i)} \mid i = 1, \ldots, n\} = \mathbf{T}$. Since $V_1 \in K$, $V_1^*(A_i) = V_1(x_i) = W^*(A_i)$ $(i = 1, \ldots, n)$ so $V_1^*(Y(A_1, \ldots, A_n/x_1, \ldots, x_n)) = W^*(Y(A_1, \ldots, A_n/x_1, \ldots, x_n)) = \mathbf{F}$. By the induction hypothesis $(V_1, K)^*(Y) = V_1^*(Y(A_1, \ldots, A_n/x_1, \ldots, x_n)) = \mathbf{F}$. Hence $(V, K)^*(X) = \mathbf{F}$. Q.E.D.

We wonder whether it is possible to represent modal logics weaker than S5 in a similar fashion.

REFERENCE

[1] Kripke, Saul A., "A completeness theorem in modal logic," *The Journal of Symbolic Logic*, vol. 24 (1959), pp. 1-14.

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