Notre Dame Journal of Formal Logic
Volume XIV, Number 3, July 1973
NDJFAM

ON GENERATING THE FINITELY SATISFIABLE FORMULAS

ARTHUR M. BULLOCK and HUBERT H. SCHNEIDER

Results by B. Trahtenbrot [4] and Th. Hailperin [3] show that the class of finitely valid formulas of first-order predicate logic is not recursively enumerable. In this paper we shall present a system[1] which generates all formulas of first-order predicate logic which are satisfiable in some non-empty finite individual domain. The system forms a kind of calculus for finite satisfiability in the sense that the finitely satisfiable formulas are all obtainable from certain "axioms" by means of several "rules of inference." As in [3] and [4], we shall consider only formulas of the first-order predicate logic without identity. A forthcoming paper will deal with formulas of first-order predicate logic with identity.

The primitive symbols of our first-order predicate logic are the (individual) variables $v_1, v_2, \ldots$, the (individual) constants $c_1, c_2, \ldots$, for each $n \geq 0$ the n-ary predicates $p_1^n, p_2^n, \ldots$, the logical symbols $\neg$, $\wedge$ and $\forall$, and the parentheses (,). Atomic formulas and formulas as well as the concepts of free and bound occurrences of variables in formulas are defined in the customary way, and all occurrences of constants in formulas will be considered free occurrences. If each of $t_1, \ldots, t_n$ is a term (that is, a variable or constant) and each of $t_1', \ldots, t_n'$ is a variable not occurring in the formula φ or is a constant, we denote by $\varphi[t_1/t_1', \ldots, t_n/t_n']$ the formula obtained by replacing each free occurrence of t_i in φ by t_i'.

The model-theoretic notions used here are as in J. L. Bell and A. B. Slomson ([1] pp. 55-56). In particular, if $\mathfrak{A}$ is a model structure, we denote the set on which $\mathfrak{A}$ is defined by $|\mathfrak{A}|$, and the element of $|\mathfrak{A}|$ to which a constant d is assigned by $\mathfrak{A}$ we denote by $d^{\mathfrak{A}}$. We write $\mathfrak{A} \models \varphi[a_1, \ldots, a_m]$ if the assignment of $a_i \in |\mathfrak{A}|$ to v_i satisfies φ, where m is the greatest integer such that v_m occurs free in φ. If there is a model structure $\mathfrak{A}$ and $a_1, \ldots, a_m \in |\mathfrak{A}|$ such that $\mathfrak{A} \models \varphi[a_1, \ldots, a_m]$, then φ is *satisfiable*, and if $|\mathfrak{A}|$ has exactly k elements, φ is *k-satisfiable*. A formula is *finitely satisfiable* if it is k-satisfiable for some positive integer k.

1. A description of this system was first presented at the IV'th International Congress for Logic, Methodology and Philosophy of Science; see [2].

Received July 6, 1972

For a formula φ, let $x_1, \ldots, x_n$ be the subsequence of $v_1, v_2, \ldots$ containing all variables occurring free in φ, let $y_1, y_2, \ldots$ be the subsequence containing all variables not occurring in φ, and let $d_1, \ldots, d_p$ be the subsequence of $c_1, c_2, \ldots$ containing all constants occurring in φ. Then the *closure* of φ is defined by:

$$\varphi^* = \exists x_1 \ldots \exists x_n \exists y_1 \ldots \exists y_p\, \varphi[d_1/y_1, \ldots, d_p/y_p].$$

The closure of a formula has no free occurrences of terms, and has the following easily proved property.

Proposition 1: *For any formula φ and any positive integer k, φ is k-satisfiable iff φ^* is k-satisfiable.*

The proofs of the following two propositions are omitted because they involve routine but tedious constructions of models having precisely the required properties.

Proposition 2: *If φ is a satisfiable, quantifier-free formula containing no terms other than $c_1, \ldots, c_k$ $(k > 0)$ then there is a model structure $\mathfrak{A}$ such that $\mathfrak{A} \vDash \varphi$ and $|\mathfrak{A}| = \{c_1^{\mathfrak{A}}, \ldots, c_k^{\mathfrak{A}}\}$.*

Proposition 3: *If φ is a k-satisfiable and quantifier-free formula, and ψ is an atomic formula of the form $p_i^n t_1 \ldots t_n$ not occurring in φ, then both $(\varphi \wedge \psi)$ and $(\varphi \wedge \neg\psi)$ are $(k+n)$-satisfiable. Hence if φ is a finitely satisfiable formula which is quantifier-free and ψ is an atomic formula not occurring in φ, then $(\varphi \wedge \psi)$ as well as $(\varphi \wedge \neg\psi)$ are finitely satisfiable formulas.*

The *k-transform* φ^k of a formula φ is defined inductively by:

(1) If φ is an atomic formula, then $\varphi^k = \varphi$.
(2) If $\varphi = \neg\psi$, then $\varphi^k = \neg(\psi^k)$.
(3) If $\varphi = (\psi_1 \wedge \psi_2)$, then $\varphi^k = (\psi_1^k \wedge \psi_2^k)$.
(4) If $\varphi = \forall x\psi$, then $\varphi^k = ((\psi[x/c_1])^k \wedge \ldots \wedge (\psi[x/c_k])^k)$.

We omit the proof of the following proposition which is basically a straightforward induction on the length of a formula.

Proposition 4: *If φ is a formula with no free variables, k a positive integer and $\mathfrak{A}$ is such that $|\mathfrak{A}| = \{c_1^{\mathfrak{A}}, \ldots, c_k^{\mathfrak{A}}\}$, then $\mathfrak{A} \vDash \varphi^k$ iff $\mathfrak{A} \vDash \varphi$.*

Theorem 1: *For any formula φ and any positive integer k, φ is k-satisfiable iff φ^{*k} is satisfiable.*

Proof: Assume φ is k-satisfiable, so φ^* is k-satisfiable by Proposition 1. Hence there is a model structure $\mathfrak{A}$ such that $\mathfrak{A} \vDash \varphi^*$ and $|\mathfrak{A}| = \{a_1, \ldots, a_k\}$ is a set of k elements. Since φ^* has no constants, we can assume $\mathfrak{A}$ does not assign constants, and then extend $\mathfrak{A}$ by defining $c_i^{\mathfrak{A}} = a_i$ for $i = 1, \ldots, k$. Thus $\mathfrak{A}$ is such that $|\mathfrak{A}| = \{c_1^{\mathfrak{A}}, \ldots, c_k^{\mathfrak{A}}\}$ and $\mathfrak{A} \vDash \varphi^*$, so $\mathfrak{A} \vDash \varphi^{*k}$ by Proposition 4. Conversely, assume φ^{*k} is satisfiable. Since φ^{*k} is a satisfiable, quantifier-free formula containing no terms other than $c_1, \ldots, c_k$, there is a model structure $\mathfrak{A}$ such that $\mathfrak{A} \vDash \varphi^{*k}$ and $|\mathfrak{A}| = \{c_1^{\mathfrak{A}}, \ldots, c_k^{\mathfrak{A}}\}$ by

Proposition 2, so $\mathfrak{A} \models \varphi^*$ by Proposition 4. It clearly follows that φ^* is k-satisfiable, and hence φ is k-satisfiable by Proposition 1.

Let $\mathcal{S}$ be the set of formulas generated by the following rules:

(S1) If φ is an atomic formula, then $\varphi \in \mathcal{S}$.
(S2) If $\varphi \in \mathcal{S}$, φ quantifier-free, and ψ is an atomic formula not occurring in φ, then $(\varphi \wedge \psi) \in \mathcal{S}$ and $(\varphi \wedge \neg\psi) \in \mathcal{S}$.
(S3) If $\varphi \in \mathcal{S}$ and $\varphi \to \psi$ is a tautology, then $\psi \in \mathcal{S}$.
(S4) If $\varphi^{*k} \in \mathcal{S}$ for some positive integer k, then $\varphi \in \mathcal{S}$.

These rules are concerned only with syntactical properties of formulas, except for the condition "$\varphi \to \psi$ is a tautology" in (S3). Since there are well-known syntactical equivalents of the notion of a tautology, we can regard these rules as entirely syntactical, so $\mathcal{S}$ is recursively generated. We now show that these rules are sound and complete in the sense that $\mathcal{S}$ contains only and all those formulas which are finitely satisfiable.

Theorem 2 (Soundness): *If $\varphi \in \mathcal{S}$, then φ is finitely satisfiable.*

Proof: It suffices to show the soundness of each of the four rules.

(1) Formulas introduced by (S1) are atomic, and obviously finitely satisfiable.
(2) Any formula introduced by rule (S2) is finitely satisfiable in view of Proposition 3.
(3) Any formula ψ introduced by (S3) is such that $\varphi \to \psi$ is a tautology for some $\varphi \in \mathcal{S}$. If φ is finitely satisfiable, there is a model structure $\mathfrak{A}$ such that $\mathfrak{A} \models \varphi$ and $|\mathfrak{A}|$ is finite. Since $\varphi \to \psi$ is a tautology, we have also $\mathfrak{A} \models \varphi \to \psi$, and therefore $\mathfrak{A} \models \psi$; thus ψ is finitely satisfiable.
(4) Any formula φ introduced by (S4) is such that $\varphi^{*k} \in \mathcal{S}$. If φ^{*k} is finitely satisfiable, then φ is k-satisfiable by Theorem 1, and hence φ is finitely satisfiable.

Theorem 3: *If φ is a satisfiable, quantifier-free formula, then $\varphi \in \mathcal{S}$.*

Proof: Let φ be a quantifier-free formula and $\mathfrak{A}$ a model structure such that $\mathfrak{A} \models \varphi$. Let $\psi_1, \ldots, \psi_n$ be the distinct atomic formulas occurring in φ. Let φ_0 be an atomic formula other than $\psi_1, \ldots, \psi_n$, and for $i = 1, \ldots, n$ let $\varphi_i = \psi_i$ if $\mathfrak{A} \models \psi_i$, and $\varphi_i = \neg\psi_i$ otherwise. φ_0 is an atomic formula, so $\varphi_0 \in \mathcal{S}$ by (S1). If $(\varphi_0 \wedge \ldots \wedge \varphi_j) \in \mathcal{S}$ for $j < n$, then φ_{j+1} is either ψ_{j+1} or $\neg\psi_{j+1}$ where ψ_{j+1} is an atomic formula not occurring in the quantifier-free formula $(\varphi_0 \wedge \ldots \wedge \varphi_j)$; hence $(\varphi_0 \wedge \ldots \wedge \varphi_{j+1}) \in \mathcal{S}$ by (S2). By induction, it follows that $(\varphi_0 \wedge \ldots \wedge \varphi_n) \in \mathcal{S}$. We now observe that $(\varphi_0 \wedge \ldots \wedge \varphi_n) \to \varphi$ is a tautology as can be shown by a truth-table argument. Since $(\varphi_0 \wedge \ldots \wedge \varphi_n) \in \mathcal{S}$ and $(\varphi_0 \wedge \ldots \wedge \varphi_n) \to \varphi$ is a tautology, $\varphi \in \mathcal{S}$ by (S3).

Theorem 4 (Completeness): *If φ is finitely satisfiable, then $\varphi \in \mathcal{S}$.*

Proof: If φ is finitely satisfiable, φ is k-satisfiable for some positive integer k, and hence φ^{*k} is satisfiable by Theorem 1. Since φ^{*k} is a satisfiable quantifier-free formula, $\varphi^{*k} \in \mathcal{S}$ by Theorem 3. By (S4) it follows that $\varphi \in \mathcal{S}$.

REFERENCES

[1] Bell, J. L., and A. B. Slomson, *Models and Ultraproducts: an Introduction*, North-Holland, Amsterdam (1969).

[2] Bullock, A. M., and H. H. Schneider, "A calculus for finitely satisfiable formulas," *Abstracts of the IV'th International Congress for Logic, Methodology and Philosophy of Science*, Bucharest (1971), pp. 14-15.

[3] Hailperin, Th., "A complete set of axioms for logical formulas invalid in some finite domain," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 7 (1961), pp. 84-96.

[4] Trahtenbrot, B. A., "Impossibility of an algorithm for the decision problem in finite classes," (Russian), *Doklady Akadêmii Nauk SSSR*, vol. 70 (1950), pp. 569-572.

The University of Nebraska
Lincoln, Nebraska