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NOTE ABOUT THE BOOLEAN PARTS OF THE EXTENDED BOOLEAN ALGEBRAS

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Throughout this note[1] the Boolean algebras extended by the additional extra-Boolean operations and postulates and containing the so-called Boolean part, in short **BA**, i.e., a postulate

C0 *the structure* $\langle A, +, \times, -, 0, 1\rangle$ *is a Boolean algebra*

will be called the extended Boolean algebras. In [3] and [2] it has been proved that in several systems of the extended Boolean algebras the postulate *C0* can be substituted for the postulates weaker than *C0*, namely either by

*C0** *the structure* $\langle A, +, \times, -, 0, 1\rangle$ *is a non-associative Newman algebra*

or by

*C0*** *the structure* $\langle A, +, \times, -, 0, 1\rangle$ *is a dual non-associative Newman algebra.*

1 An inspection of the deductions presented in [3] and [2] suggests the following elementary, but general lemma:

Lemma I. *Let* $\mathfrak{M}$ *be an arbitrary extended Boolean algebra,* M *be the carrier set of* $\mathfrak{M}$, $\mathcal{A}$ *be the set of all primitive extra-Boolean operations occurring in the definition of* $\mathfrak{M}$, *and* $\mathcal{B}$ *be the set of all extra-Boolean postulates accepted in* $\mathfrak{M}$. *Let* Z *be a unary extra-Boolean operation which either belongs to* $\mathcal{A}$ *or is definable in the field of the postulates of* $\mathfrak{M}$. *Then:*

(i) *if* Z *either belongs to* $\mathcal{A}$ *or is syntactically definable in the field of C0*, extended by the postulates belonging to* $\mathcal{B}$, *and in that field a formula*

A1 $[a]:a \in M .\supset. a + Za = Za$

1. An acquaintance with [3] and [2] is presupposed.

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is provable, then in the postulate-system of $\mathfrak{M}$, *the axiom* $C0$ *can be replaced by* $C0^*$;

and

(ii) *if* Z *either belongs to* $\mathcal{A}$ *or is syntactically definable in the field of* $C0^{**}$, *extended by the postulates belonging to* $\mathcal{B}$, *and in that field the formulas*

$B1 \quad [a]:a \in A \,.\supset.\, a \times Za = a$

$B2 \quad 0 = Z0$

are provable, then in the postulate-system of $\mathfrak{M}$, *the axiom* $C0$ *can be replaced by* $C0^{**}$.

Proof: Let us assume that Z is an operation which satisfies the assumptions of Lemma I and the antecedent of its point (i). Hence, we have formula *A1* and, due to $C0^*$, the Theorems *M1, M4, D1, M7* and *M25* presented in [3], pp. 532-533. Then:

$R1 \quad [a]:a \in M \,.\supset.\, a = a + a$

$$\begin{array}{lll} \textbf{PR} & [a]:\mathrm{Hp}(1)\,.\supset. & \\ & a = a \times (Z1 + - Z1) = a \times ((1 + Z1) + - Z1) & [1; M4; A1] \\ & \quad = a \times (1 + (Z1 + - Z1)) = a \times (1 + 1) = a + a & [M25; D1; M1; M7] \end{array}$$

On the other hand, if Z is an operation which satisfies the assumptions of Lemma I and the antecedent of its point (ii), then we have the formulas *B1* and *B2* and, due to $C0^{**}$, the Theorems *N1* and *N7* presented in [3], pp. 536-537. Then:

$T1 \quad [a]a \in M \,.\supset.\, a = a \times a$

$$\begin{array}{lll} \textbf{PR} & [a]:\mathrm{Hp}(1)\,.\supset. & \\ & a = a + 0 = a + (0 \times Z0) = a + (0 \times 0) = a \times a & [1; N7; B1; B2; N1; N7] \end{array}$$

Since the additions of *R1* to $C0^*$ and of *T1* to $C0^{**}$ yield Boolean algebras in both cases, *cf.* [5], pp. 533-534, section **1.2**, and pp. 538-539, section **2.2**, the proof is complete.

2 As an example, we shall discuss here an application of Lemma I to the monadic algebras of Halmos, *cf.* [1]. In the style which is used for the definitions of the algebraic systems in [3] and [2], these algebras are presented here as follows:

Any algebraic structure

$$\mathfrak{A} = \langle A, +, \times, -, 0, 1, \exists \rangle$$

where + *and* × *are two binary operations, and* - *and* ∃ *are two unary operations defined on the carrier set* A, *and* 0 *and* 1 *are two constant elements belonging to* A, *is a monadic algebra, if it satisfies the following postulates:* $C0$ *and*

$V1 \quad [a]:a \in A \,.\supset.\, a \leq \exists a$

$V2$ $\exists 0 = 0$
$V3$ $[ab]: a, b \epsilon A . \supset . \exists(a \times \exists b) = \exists a \times \exists b$

Cf. [1], p. 21 and p. 40. Since in $\mathfrak{A}$ we have the postulate $C0$ and "$\leqslant$" is not a primitive notion of the investigated system, obviously we have two inferentially equivalent forms of $V1$, viz.

$V1^*$ $[a]: a \epsilon A . \supset . a + \exists a = \exists a$

and

$V1^{**}$ $[a]: a \epsilon A . \supset . a \times \exists a = a.$

Therefore, there are two versions which are inferentially equivalent to the postulate system of $\mathfrak{A}$, namely $\{C0, V1^*, V2, V3\}$ and $\{C0, V1^{**}, V2, V3\}$. It follows automatically from Lemma I that in the first version $C0$ can be replaced by $C0^*$ and in the second version $C0$ can be replaced by $C0^{**}$.

2.1 In [1], p. 21, it is stated that in the field of $C0$ the set of postulates $V1$, $V2$ and $V3$ is inferentially equivalent to the following set of axioms: $V1$, $V2$ and

$W1$ $[ab]: a, b \epsilon A . \supset . \exists(a + b) = \exists a + \exists b$
$W2$ $[a]: a \epsilon A . \supset . \exists - \exists a = - \exists a$
$W3$ $[a]: a \epsilon A . \supset . \exists\exists a = \exists a.$

As far as I know, it was not mentioned in the literature that, in this second postulate-system of the monadic algebras, the axioms $V1$ and $W3$ are superfluous.

Proof: Assume $C0$ and the axioms $V1$, $W1$ and $W2$. Then:

$V2$ $\exists 0 = 0$
PR
1. $\exists 1 = 1$ [$V1$; **BA**]
$0 = -1 = - \exists 1 = \exists - \exists 1 = \exists - 1 = \exists 0$ [**BA**; 1; $W2$; 1; **BA**]
$W3$ $[a]: a \epsilon A . \supset . \exists\exists a = \exists a$
PR $[a]: \mathrm{Hp}(1) . \supset .$
2. $\exists a = - \exists - \exists a$ [1; $W2$, **BA**]
$\exists a = - \exists - \exists a = \exists - \exists - \exists a = \exists\exists a$ [1; 2; $W2$; 2]

Thus, in the field of $C0$, $V1$ and $W2$ imply $V2$ and $W3$ and, therefore, due to the deductions given in [1], pp. 40-44, we can establish that

$$\{C0, V1, V2, V3\} \rightleftarrows \{C0, V1, W1, W2\}$$

2.2 Now, it follows from Lemma I at once that $\{C0, V1, W1, W2\} \rightleftarrows \{C0^*, V1^*, W1, W2\}$. On the other hand, a proof that the equivalence

(a) $$\{C0, V1, W1, W2\} \rightleftarrows \{C0^{**}, V1^{**}, W1, W2\}$$

holds is more elaborate, since we have to prove that in the field of $C0^{**}$, $V1^{**}$, $W1$ and $W2$ imply $V2$ and, therefore, in virtue of Lemma I, $C0$. It will be shown here that in the case of the equivalence (a) such deduction is possible.

Proof: Let us assume *C0***, *V1***, *W1* and *W2*. Hence, due to *C0*** we have at our disposal the Theorems *N1, Df1, N7, N20, N24* and *N25*, *cf.* [3], pp. 536-538, sections 2 and 2.1. Then:

W3 $[a] : a \epsilon A . \supset . \exists a = \exists\exists a$

PR $[a] : \mathrm{Hp}(1) . \supset .$

2. $\exists a = - \exists - \exists a$ [1; *N20*; *W2*]

$\exists a = - \exists - \exists a = \exists - \exists - \exists a = \exists\exists a$ [1; 2; *W2*; 2]

W4 $[a] : a \epsilon A . \supset . \exists a = \exists a \times \exists a$

PR $[a] : \mathrm{Hp}(1) . \supset .$

$\exists a = \exists a \times \exists\exists a = \exists a \times \exists a$ [1; *V1***, *W3*]

W5 $[a] : a \epsilon A . \supset . -\exists a = -\exists a \times -\exists a$

PR $[a] : \mathrm{Hp}(1) . \supset .$

$-\exists a = -\exists a \times \exists - \exists a = -\exists a \times -\exists a$ [1; *V1***, *W2*]

T1 $[a] : a \epsilon A . \supset . a = a \times a$

PR $[a] : \mathrm{Hp}(1) . \supset .$

$a = a + 0 = a + (\exists a \times -\exists a) = a + ((\exists a \times \exists a) \times (-\exists a \times -\exists a))$ [1; *N7*; *Df1*; *W4*; *W5*]

$= a + ((\exists a \times -\exists a) \times (\exists a \times -\exists a))$ [*N24*; *N25*]

$= a + (0 \times 0) = (a + 0) \times (a + 0) = a \times a$ [*Df1*; *N1*; *N7*]

Since the addition of *T1* to *C0*** generates *C0* and since, in the field of *C0*, *V1*** implies *V1*, we have *V2*, *cf.* section 2.1 above. Therefore, in virtue of Lemma I, *V1*** and *V2*, the proof is complete.

REFERENCES

[1] Halmos, P. R., *Algebraic Logic*, Chelsea Publishing Co., New York (1962).

[2] Sobociński, B., "Remark about the Boolean parts in the postulate-systems of closure, derivative and projective algebras," *Notre Dame Journal of Formal Logic*, vol. XIV (1973), pp. 111-117.

[3] Sobociński, B., "Solution to the problem concerning the Boolean bases for cylindric algebras," *Notre Dame Journal of Formal Logic*, vol. XIII (1972), pp. 529-545.

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