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CONCERNING THE QUANTIFIER ALGEBRAS IN THE SENSE OF PINTER

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1 In [3],[1] *cf.* especially p. 362, section **2.1** and p. 365, section **4.1**, C. C. Pinter formulated and investigated an algebraic system which he called the quantifier algebras and which, in conformity with the style used in [6], pp. 529-530, [5], p. 111, section **1**, and [4], is defined here as follows:

(A) *Any algebraic structure*

$$\mathfrak{A} = \langle A, +, \times, -, 0, 1, \mathsf{S}^{\kappa}_{\lambda}, \exists_{\kappa}\rangle_{\kappa,\lambda<\alpha}$$

where α *is any ordinal number,* $+$ *and* $\times$ *are two binary operations, and* $-$, $\mathsf{S}^{\kappa}_{\lambda}$ *(for any ordinal numbers* $\kappa, \lambda < \alpha$*) and* $\exists_{\kappa}$ *(for any ordinal number* $\kappa < \alpha$*) are the unary operations defined on the carrier set* A, *and* 0 *and* 1 *are two constant elements belonging to* A, *is a quantifier algebra of dimension* α, *if it satisfies the following postulates:*

C0	*the structure* $\langle A, +, \times, -, 0, 1\rangle$ *is a Boolean algebra*
Q1	$[a\kappa\lambda] : a \in A \,.\, \kappa, \lambda < \alpha \,.\supset.\, \mathsf{S}^{\kappa}_{\lambda}(-a) = -\mathsf{S}^{\kappa}_{\lambda}a$
Q2	$[ab\kappa\lambda] : a, b \in A \,.\, \kappa, \lambda < \alpha \,.\supset.\, \mathsf{S}^{\kappa}_{\lambda}(a + b) = \mathsf{S}^{\kappa}_{\lambda}a + \mathsf{S}^{\kappa}_{\lambda}b$
Q3	$[a\kappa] : a \in A \,.\, \kappa < \alpha \,.\supset.\, \mathsf{S}^{\kappa}_{\kappa}a = a$
Q4	$[a\kappa\lambda\mu] : a \in A \,.\, \kappa, \lambda, \mu < \alpha \,.\supset.\, \mathsf{S}^{\kappa}_{\lambda}\mathsf{S}^{\mu}_{\kappa}a = \mathsf{S}^{\kappa}_{\lambda}\mathsf{S}^{\mu}_{\lambda}a$
Q5	$[ab\kappa] : a, b \in A \,.\, \kappa < \alpha \,.\supset.\, \exists_{\kappa}(a + b) = \exists_{\kappa}a + \exists_{\kappa}b$
Q6	$[a\kappa] : a \in A \,.\, \kappa < \alpha \,.\supset.\, a \leq \exists_{\kappa}a$
Q7	$[a\kappa\lambda] : a \in A \,.\, \kappa, \lambda < \alpha \,.\supset.\, \mathsf{S}^{\kappa}_{\lambda}\exists_{\kappa}a = \exists_{\kappa}a$
Q8	$[a\kappa\lambda] : a \in A.\, \kappa, \lambda < \alpha \,.\, \kappa \neq \lambda \,.\supset.\, \exists_{\kappa}\mathsf{S}^{\kappa}_{\lambda}a = \mathsf{S}^{\kappa}_{\lambda}a$
Q9	$[a\kappa\lambda\mu] : a \in A \,.\, \kappa, \lambda, \mu < \alpha \,.\, \mu \neq \kappa, \lambda \,.\supset.\, \mathsf{S}^{\kappa}_{\lambda}\exists_{\mu}a = \exists_{\mu}\mathsf{S}^{\kappa}_{\lambda}a$

Moreover, if, besides 0 *and* 1, *the carrier set* A *of the structure* $\mathfrak{A}$ *contains also the constant elements* $\mathsf{e}_{\kappa\lambda}$ *(for any ordinal numbers* $\kappa, \lambda < \alpha$*) such that* $\mathfrak{A}$ *satisfies the following two additional postulates:*

1. An acquaintance with papers [3], [6] and [4] and with the symbolism used in [6] is presupposed.

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Q10 $[\kappa\lambda]: \kappa, \lambda < \alpha . \supset . \mathbf{S}_{\lambda}^{\kappa}\mathbf{e}_{\kappa\lambda} = 1$
Q11 $[a\kappa\lambda]: a \epsilon A . \kappa, \lambda < \alpha . \supset . a \times \mathbf{e}_{\kappa\lambda} = \mathbf{S}_{\lambda}^{\kappa}a$

then the structure $\mathfrak{A}$ *is a quantifier algebra with equality of dimension* α.

In this note: 1) a problem will be discussed concerning the quantifier algebras of dimension $\alpha \leqslant 1$ (this problem was not investigated in [3]) and 2) it will be proved that in the postulate-system of quantifier algebras of dimension $\alpha \geqslant 1$ the postulate *C0* can be replaced by each of the following assumptions:

*C0** *the structure* $\langle A, +, \times, -, 0, 1\rangle$ *is a non-associative Newman algebra*

and

*C0*** *the structure* $\langle A, +, \times, -, 0, 1\rangle$ *is a dual non-associative Newman algebra*

i.e., by a postulate which is weaker than *C0*.

1.1 In connection with the formalization of system $\mathfrak{A}$, *cf*. [3], p. 362, section **2.1**, and definition (**A**) given above, it should be noted that since in $\mathfrak{A}$ we have *C0* and "$\leqslant$" is not a primitive notion of the investigated system, we have obviously two inferentially equivalent forms of the postulate *Q6*, viz:

*Q6** $[a\,\kappa]: a \epsilon A . \kappa < \alpha . \supset . a + \exists_{\kappa}a = \exists_{\kappa}a$

and

*Q6*** $[a\,\kappa]: a \epsilon A . \kappa < \alpha . \supset . a \times \exists_{\kappa}a = a$

2 The deductions which follow will be used in the considerations presented in the next sections.[2]

2.1 Let us assume *C0*** and the formulas *Q1*, *Q6***, *Q7* and *Q8*, each of them of dimension $\alpha \geqslant 2$. Hence, due to *C0***, we have the formulas *Df1* and *N19* given in [6], p. 537. Then:

*Q0** $[a\,\kappa]: a \epsilon A . \kappa < \alpha . \supset . \exists_{\kappa} - \exists_{\kappa}a = -\exists_{\kappa}a$
PR $[a\,\kappa]: \mathrm{Hp}(2) . \supset .$
$[\exists\lambda] .$
3. $\lambda < \alpha .$
4. $\kappa \neq \lambda .$ [2, since $\alpha \geqslant 2$]
5. $-\exists_{\kappa}a = -\mathbf{S}_{\lambda}^{\kappa}\exists_{\kappa}a .$ [1; 2; 3; *Q7*]
6. $-\exists_{\kappa}a = -\mathbf{S}_{\lambda}^{\kappa}\exists_{\kappa}a = \mathbf{S}_{\lambda}^{\kappa} - \exists_{\kappa}a$ [1; 2; 3; 4; 5; *Q1*]
$= \exists_{\kappa}\mathbf{S}_{\lambda}^{\kappa} - \exists_{\kappa}a = \exists_{\kappa} - \exists_{\kappa}a .$ [*Q8*; *Q1*; *Q7*]
$\exists_{\kappa} - \exists_{\kappa}a = -\exists_{\kappa}a$ [6]
Z1 $[\alpha]: \kappa < \alpha . \supset . 1 = \exists_{\kappa}1$ [*Q6***; *N19*]

2. The deductions presented in the sections **2.2** and **2.3** below are analogous in some respect to the proofs which are given by Halmos in [1], pp. 38-44.

$Q0^{**}$ $[\kappa]:\kappa < \alpha \,.\supset.\, 0 = \exists_\kappa 0$

PR $[\kappa]:\mathrm{Hp}(1)\,.\supset.$

2. $0 = 1 \times - 1 = -1\,.$ [$Df1$; $N17$]

$0 = -1 = -\exists_\kappa 1 = \exists_\kappa - \exists_\kappa 1 = \exists_\kappa - 1 = \exists_\kappa 0$ [1; 2; $Z1$; $Q0^*$; $Z1$; 2]

Thus, $\{C0^{**}, Q1, Q6^{**}, Q7, Q8\} \rightarrow \{Q0^*, Q0^{**}\}$.

2.2 Now, let us assume $C0$ and the formulas $Q1, Q5, Q6, Q7$ and $Q8$, each of them of dimension $\alpha \geq 2$. Since we have $C0$, both forms of $Q6$, i.e., $Q6^*$ and $Q6^{**}$, are valid, and since $C0^{**}$ is a proper subsystem of $C0$, *cf.* [6], p. 536, section **2**, we have $Q0^*$ and $Q0^{**}$, *cf.* section **2.1** above. Then:

$Z2$ $[a\,\kappa]:a \in A\,.\,\kappa < \alpha \,.\supset.\, \exists_\kappa a = \exists_\kappa \exists_\kappa a$

PR $[a\,\kappa]:\mathrm{Hp}(2)\,.\supset.$

$[\exists \lambda]\,.$

3. $\lambda < \alpha\,.$ [2, since $\alpha \geq 2$]

4. $\kappa \neq \lambda\,.$

5. $\exists_\kappa a = \mathsf{S}^\kappa_\lambda \exists_\kappa a = \exists_\kappa \mathsf{S}^\kappa_\lambda \exists_\kappa a = \exists_\kappa \exists_\kappa a\,.$ [1; 2; 3; 4; $Q7$; $Q8$; $Q7$]

$\exists_\kappa a = \exists_\kappa \exists_\kappa a$ [5]

$Z3$ $[a\,b\,\kappa]:a, b \in A\,.\,\kappa < \alpha\,.\,a \leq b \,.\supset.\, \exists_\kappa a \leq \exists_\kappa b$

PR $[a\,b\,\kappa]:\mathrm{Hp}(3)\,.\supset.$

4. $a \leq \exists_\kappa b\,.$ [1; 2; 3; $Q6$; **BA**]

5. $a + \exists_\kappa b = \exists_\kappa b\,.$ [1; 2; 4; **BA**]

6. $\exists_\kappa b = \exists_\kappa \exists_\kappa b = \exists_\kappa(a + \exists_\kappa b) = \exists_\kappa a + \exists_\kappa \exists_\kappa b$ [1; 2; $Z2$; 5; $Q5$]

$= \exists_\kappa a + \exists_\kappa b\,.$ [$Z2$]

$\exists_\kappa a \leq \exists_\kappa b$ [1; 2; 6; **BA**]

$Q5^*$ $[a\,b\,\kappa]:a, b \in A\,.\,\kappa < \alpha \,.\supset.\, \exists_\kappa(a \times \exists_\kappa b) = \exists_\kappa a \times \exists_\kappa b$

PR $[a\,b\,\kappa]:\mathrm{Hp}(2)\,.\supset.$

3. $a \times \exists_\kappa b \leq \exists_\kappa a \times \exists_\kappa b\,.$ [1; 2; $Q6$; **BA**]

4. $\exists_\kappa(a \times \exists_\kappa b) \leq \exists_\kappa(\exists_\kappa a \times \exists_\kappa b) = \exists_\kappa - (-\exists_\kappa a + - \exists_\kappa b)$ [1; 2; 3; $Z3$, **BA**]

$= \exists_\kappa - ((\exists_\kappa - \exists_\kappa a) + (\exists_\kappa - \exists_\kappa b))$ [$Q0^*$]

$= \exists_\kappa - \exists_\kappa (-\exists_\kappa a + - \exists_\kappa b)$ [$Q5$]

$= -\exists_\kappa(-\exists_\kappa a + - \exists_\kappa b)$ [$Q0^*$]

$= -((\exists_\kappa - \exists_\kappa a) + (\exists_\kappa - \exists_\kappa b))$ [$Q5$]

$= -(-\exists_\kappa a + - \exists_\kappa b) = \exists_\kappa a \times \exists_\kappa b\,.$ [$Q0^*$; **BA**]

5. $a = (a \times \exists_\kappa b) + (a \times - \exists_\kappa b) \leq (a + \exists_\kappa b) + - \exists_\kappa b\,.$ [1; 2; **BA**]

6. $\exists_\kappa a \leq \exists_\kappa((a \times \exists_\kappa b) + - \exists_\kappa b)$ [1; 2; 5; $Z3$]

$= \exists_\kappa(a \times \exists_\kappa b) + (\exists_\kappa - \exists_\kappa b)$ [$Q5$]

$= \exists_\kappa(a \times \exists_\kappa b) + - \exists_\kappa b\,.$ [$Q0^*$]

7. $\exists_\kappa a \times \exists_\kappa b \leq (\exists_\kappa(a \times \exists_\kappa b) + - \exists_\kappa b) \times \exists_\kappa b$ [1; 2; 6; **BA**]

$= (\exists_\kappa(a \times \exists_\kappa b) \times \exists_\kappa b) + (-\exists_\kappa b \times \exists_\kappa b)$ [**BA**]

$\leq \exists_\kappa(a \times \exists_\kappa b)\,.$ [**BA**]

$\exists_\kappa(a \times \exists_\kappa b) = \exists_\kappa a \times \exists_\kappa b$ [1; 2; 4; 7; **BA**]

Thus, $\{C0, Q1, Q5, Q6, Q7, Q8\} \rightarrow \{Q0^*, Q0^{**}, Q5^*\}$.

2.3 Finally, let us assume $C0$ and the formulas $Q1$, $Q5^*$, $Q6$, $Q7$ and $Q8$, each of them of dimension $\alpha \geq 2$. Since we have $C0$ both forms of $Q6$, i.e., $Q6^*$ and $Q6^{**}$, are valid and since $C0^{**}$ is a proper subsystem of $C0$, *cf.*

section **2.2** above, and in the field of $C0^{**}$ the formulas $Q0^*$ and $Q0^{**}$ follow from $Q1$, $Q6^{**}$, $Q7$ and $Q8$, we have these formulas at our disposal. Then:

$Z2$ $\quad [a\,\kappa]: a \in A \,.\, \kappa < \alpha \,.\supset .\, \exists_\kappa a = \exists_\kappa \exists_\kappa a$

[$C0$; $Q7$; $Q8$, since $\alpha \geq 2$. *Cf.* a proof of $Z2$ in section **2.2**]

$Z3$ $\quad [a\,b\,\kappa]: a, b \in A \,.\, \kappa < \alpha \,.\, a \leq b \,.\supset .\, \exists_\kappa a \leq \exists_\kappa b$

PR $\quad [a\,b\,\kappa]: \mathrm{Hp}(3) \,.\supset .$

4. $\quad a \times b = a \,.$ $\qquad$ [1; 3; **BA**]

5. $\quad b \times \exists_\kappa b = b \,.$ $\qquad$ [1; 2; $Q6^{**}$; **BA**]

6. $\quad \exists_\kappa a \times \exists_\kappa b = \exists_\kappa(a \times \exists_\kappa b) = \exists_\kappa((a \times b) \times \exists_\kappa b)$ $\qquad$ [1; 2; $Q5^*$; 4]

$\qquad = \exists_\kappa(a \times (b \times \exists_\kappa b)) = \exists_\kappa(a \times b) = \exists_\kappa a \,.$ $\qquad$ [**BA**; 5; 4]

$\quad \exists_\kappa a \leq \exists_\kappa b$ $\qquad$ [1; 2; 6; **BA**]

$Q5$ $\quad [a\,b\,\kappa]: a, b \in A \,.\, \kappa < \alpha \,.\supset .\, \exists_\kappa(a + b) = \exists_\kappa a + \exists_\kappa b$

PR $\quad [a\,b\,\kappa]: \mathrm{Hp}(2) \,.\supset .$

3. $\quad \exists_\kappa a + \exists_\kappa b = -(-\exists_\kappa a \times -\exists_\kappa b)$ $\qquad$ [1; 2; **BA**]

$\qquad = -((\exists_\kappa - \exists_\kappa a) \times (\exists_\kappa - \exists_\kappa b))$ $\qquad$ [$Q0^*$]

$\qquad = -\exists_\kappa(-\exists_\kappa a \times (\exists_\kappa - \exists_\kappa b))$ $\qquad$ [$Q5^*$]

$\qquad = \exists_\kappa - \exists_\kappa(-\exists_\kappa a \times (\exists_\kappa - \exists_\kappa b))$ $\qquad$ [$Q0^*$]

$\qquad = \exists_\kappa - ((\exists_\kappa - \exists_\kappa a) \times (\exists_\kappa - \exists_\kappa b))$ $\qquad$ [$Q5^*$]

$\qquad = \exists_\kappa - (-\exists_\kappa a \times -\exists_\kappa b)$ $\qquad$ [$Q0^*$]

$\qquad = \exists_\kappa(\exists_\kappa a + \exists_\kappa b) \,.$ $\qquad$ [**BA**]

4. $\quad a + b \leq \exists_\kappa a + \exists_\kappa b \,.$ $\qquad$ [1; 2; $Q6$; **BA**]

5. $\quad \exists_\kappa(a + b) \leq \exists_\kappa(\exists_\kappa a + \exists_\kappa b) = \exists_\kappa a + \exists_\kappa b \,.$ $\qquad$ [1; 2; 4; $Z3$; 3]

6. $\quad \exists_\kappa a \leq \exists_\kappa(a + b) \,.$ $\qquad$ [1; 2; **BA**; $Z3$]

7. $\quad \exists_\kappa b \leq \exists_\kappa(a + b) \,.$ $\qquad$ [1; 2; **BA**; $Z3$]

$\quad \exists_\kappa(a + b) = \exists_\kappa a + \exists_\kappa b$ $\qquad$ [1; 2; 5; 6; 7; **BA**]

Thus, $\{C0, Q1, Q5^*, Q6, Q7, Q8\} \rightarrow \{Q0^*, Q0^{**}, Q5\}$.

2.4 The deductions presented in sections **2.2** and **2.3** above show at once that if system $\mathfrak{A}$ has dimension $\alpha \geq 2$, then its set of postulates $\{C0, Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9\}$ is inferentially equivalent to the set $\{C0, Q1, Q2, Q3, Q4, Q5^*, Q6, Q7, Q8, Q9\}$.

3 An analysis of the arguments, *cf.* [3], pp. 361-362, sections 1 and 2, which led Pinter to construct the quantifier algebras shows clearly that for dimensions 0 and 1 these algebras should be reducible to the Boolean algebras and to the monadic algebras in the sense of Halmos, respectively.

3.1 It is self-evident that in the case $\alpha = 0$ the formulas $Q1$, $Q2$, $Q3$, $Q4$, $Q5$, $Q5^*$, $Q6$, $Q7$, $Q8$, $Q9$, $Q10$ and $Q11$ become void, and therefore, the quantifier algebras (even with equality) of dimension $\alpha = 0$ are reduced to the postulate $C0$, i.e., to the Boolean algebras.

3.2 On the other hand, in the case $\alpha = 1$, the postulates $Q8$ and $Q9$ are void and the operations **S** and **e** disappear, i.e., the postulates $Q1$, $Q2$, $Q3$, $Q7$, $Q10$ and $Q11$ are reduced to the valid Boolean formulas automatically. Thus there remains only the postulate $C0$, i.e., a Boolean algebra, and the postulates $Q5$ (or $Q5^*$) and $Q6$ of the following forms in which the index 0, i.e., the ordinal number <1, is omitted:

$Q5^\circ$ $[a]:a \epsilon A\,.\supset.\,\exists(a+b) = \exists a + \exists b$
$Q5*^\circ$ $[a]:a \epsilon A\,.\supset.\,\exists(a \times \exists b) = \exists a \times \exists b$
$Q6^\circ$ $[a]:a \epsilon A\,.\supset.\,a \leqslant \exists a$

Obviously, the formulas $Q5^\circ$, $Q5*^\circ$, and $Q6^\circ$ can be considered as well-known formulas from the field of the monadic algebras of Halmos, *cf.* [1], p. 21, but neither the set $\{C0, Q5^\circ, Q6^\circ\}$ nor $\{C0, Q5*^\circ, Q6^\circ\}$ constitutes an adequate postulate system for those algebra. Namely, the example given in [1], p. 41, which is presented here in the form of an algebraic table:

𝔐1

+	0	1
0	0	1
1	1	1

×	0	1
0	0	0
1	0	1

a	$-a$
0	1
1	0

a	$\exists a$
0	1
1	1

verifies $C0$, $Q5$, $Q5*$ and $Q6$, but obviously falsifies the formulas

$Q0*^\circ$ $[a]:a \epsilon A\,.\supset.\,\exists - \exists a = -\exists a$

and

$Q0**^\circ$ $0 = \exists 0$

which are valid in monadic algebras. From this it follows that, in order to establish a postulate-system for quantifier algebras which would be adequate for dimension $\alpha \geqslant 0$, some additional postulates should be added to the postulate-system given in [3], *cf.* definition (**A**) above, and also to the postulate-system which was established in section **2.4** of this paper. Obviously, since both foregoing postulate-systems are adequate for quantifier algebras of dimension $\alpha \geqslant 2$, the additional postulates have to be independent in the case of dimension $\alpha = 1$, but superfluous, if a quantifier algebra is of dimension $\alpha \geqslant 2$. As we know, a similar situation exists in the field of cylindric algebras in regard to their postulate $C1$, *cf.* [2], p. 178, and [6], p. 530.

3.3 In section **2.2** above it has been proved that in quantifier algebras of dimension $\alpha \geqslant 2$ the formulas $Q0*$ and $Q0**$ are the consequences of the original postulate-system of Pinter. But we cannot complement this set of postulates by adding, as a new axiom, $Q0**$ to it, since the set $\{C0, Q5^\circ, Q6^\circ, Q0**^\circ\}$ does not constitute an adequate postulate-system for the monadic algebras. The following algebraic table:

𝔐2

+	0	α	β	1
0	0	α	β	1
α	α	α	1	1
β	β	1	β	1
1	1	1	1	1

×	0	α	β	1
0	0	0	0	0
α	0	α	0	α
β	0	0	β	β
1	0	α	β	1

a	$-a$
0	1
α	β
β	α
1	0

a	$\exists a$
0	0
α	1
β	β
1	1

verifies $C0$, $Q5^\circ$, $Q6^\circ$ and $Q0**^\circ$, but falsifies $Q0*^\circ$ for a/β: (i) $\exists - \exists\beta = \exists - \beta = \exists\alpha = 1$, (ii) $-\exists\beta = -\beta = \alpha$; and falsifies $Q5*^\circ$ for a/α and b/β: (i) $\exists(\alpha \times \exists\beta) = \exists(\alpha \times \beta) = \exists 0 = 0$, (ii) $\exists\alpha \times \exists\beta = 1 \times \beta = \beta$. On the other hand, it has been proved in [4], p. 421, section **2.1**, that the set $\{C0,\ Q5^\circ,\ Q6^\circ,\ Q0^{*\circ}\}$ constitutes an adequate postulate-system for the monadic algebras. Therefore, in order to make the first postulate-system mentioned in section **2.4** above sufficient for $\alpha \geqslant 0$ we have to add $Q0^*$ to it as an additional postulate. Thus, the structure $\mathfrak{A}$ is a quantifier algebra of dimension $\alpha \geqslant 0$, if it satisfies the set $\{C0,\ Q0^*,\ Q1,\ Q2,\ Q3,\ Q4,\ Q5,\ Q6,\ Q7,\ Q8,\ Q9\}$ of postulates. Clearly, with such a postulate-system for $\alpha = 0$, $\mathfrak{A}$ is reduced to a Boolean algebra, for $\alpha = 1$ to a monadic algebra, and in the case $\alpha \geqslant 2$ postulate $Q0^*$ is a consequence of the remaining axioms, *cf.* section **2.1** above.

3.4 Since the set $\{C0,\ Q0^{**\circ},\ Q5^{*\circ},\ Q6^\circ\}$ is the standard postulate system for monadic algebras, *cf.* [1], p. 40, it is obvious that, in order to obtain the same result for the second postulate system given in section **2.4**, it is sufficient to add $Q0^{**}$ to that system. But there is also another possibility.

3.4.1 Namely, it is easy to prove that not only in the field of $C0$, but also in the field of $C0^{**}$ or $C0^*$ $\{Q0^{**\circ},\ Q5^{*\circ},\ Q6^{*\circ}\} \rightleftarrows \{Q0^{*\circ},\ Q5^{*\circ},\ Q6^{*\circ}\}$.

Proof: Obviously, it is sufficient to prove that in the field of $C0^{**}$ $\{Q0^{*\circ},\ Q5^{*\circ},\ Q6^{**\circ}\}$ implies $Q0^{**\circ}$. (If instead of $C0^{**}$ we accept $C0^*$, $Q6^{*\circ}$ should be assumed, *cf.* section **1.1** above.) Since we accepted $C0^{**}$, we have $Df1$ and $N19$, *cf.* [6], p. 537. Then:

$Q0^{**}$ $\quad 0 = \exists 0$

PR

1. $\quad 1 = \exists 1.$ $\qquad$ [$Q6^{**\circ}$; $N17$]

2. $\quad 0 = 1 \times -1 = -1.$ $\qquad$ [$Df1$; $N17$]

$\quad 0 = -1 = -\exists 1 = \exists - \exists 1 = \exists - 1 = \exists 0$ $\qquad$ [2; 1; $Q0^{*\circ}$; 1; 2]

If instead of $C0^{**}$ system $C0^*$ is accepted, the proof that in the field of $C0^*$ $\{Q0^{*\circ},\ Q6^{*\circ}\} \rightarrow \{Q0^{**}\}$ is analogous. And since $C0^*$ and $C0^{**}$ are the proper subsystems of $C0$, the foregoing deductions hold in the field of a Boolean algebra. Thus, it is obvious that we can reach our aim by adding $Q0^*$, as a new postulate, to the second postulate-system of the quantifier algebras.

3.5 Recapitulating the discussions presented in this section, we can state that, in order to obtain an adequate postulate-system for quantifier algebras of dimension $\alpha \geqslant 0$, it is sufficient to accept one of the following sets of axioms:

(a) $\quad \{C0,\ Q0^*,\ Q1,\ Q2,\ Q3,\ Q4,\ Q5,\ Q6,\ Q7,\ Q8,\ Q9\}$

(b) $\quad \{C0,\ Q0^*,\ Q1,\ Q2,\ Q3,\ Q4,\ Q5^*,\ Q6,\ Q7,\ Q8,\ Q9\}$

(c) $\quad \{C0,\ Q0^{**},\ Q1,\ Q2,\ Q3,\ Q4,\ Q5^*,\ Q6,\ Q7,\ Q8,\ Q9\}$

Each of these sets of postulates is such that in the case of $\alpha = 0$ it gives a Boolean algebra, in the case of $\alpha = 1$ a monadic algebra, and in the case of $\alpha \geqslant 2$ the postulates $Q0^*$ and $Q0^{**}$ are the consequences of the remaining axioms belonging to the appropriate set of postulates.

4 Since in the field of $C0$ the postulate $Q6$ is either $Q6^*$ or $Q6^{**}$, *cf.* section **1.1** above, it follows automatically from Lemma I, proven in [4], pp. 419-420, section **1**, that if $Q6^*$ is accepted as a form of $Q6$, then in the sets (a), (b) and (c) for $\alpha \geqslant 1$ the postulate $C0$ can be replaced by a weaker postulate, viz. $C0^*$; and if $Q6^{**}$ is accepted as a form of $Q6$, then in the set (c) for $\alpha \geqslant 1$ the postulate $C0$ can be replaced by $C0^{**}$. Moreover, it follows immediately from the deductions presented in sections **2.1** and **3.4.1** above, and in [4], pp. 421-422, section **2.2**, that if $Q6^{**}$ is accepted as a form of $Q6$, then $Q0^{**}$ is a consequence of the postulate system (a) or of (b) in each of which $\alpha \geqslant 1$ and in each of which the postulate $C0$ is replaced by $C0^{**}$. Hence, again due to Lemma I, we know that if $Q6^{**}$ is chosen as a form of $Q6$, then in sets (a) and (b) for $\alpha \geqslant 1$, the postulate $C0$ can be replaced by a weaker assumption, namely $C0^{**}$. Naturally, such replacements cannot be done in a system of quantifier algebras whose dimension is $\alpha = 0$, since they would reduce such a system automatically either to a non-associative Newman algebra or to a dual non-associative Newman algebra, but not to a Boolean algebra, as required.

4.1 It is self-evident that all conclusions established in this paper are applicable *a fortiori* to the quantifier algebras with equality.

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