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A NOTE ON "TRANSITIVITY, SUPERTRASITIVITY AND INDUCTION"

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In the review of our paper "Transitivity, Supertransitivity and Induction," [1] that occurred in [2], the reviewer pointed out two apparent errors. We will here clarify the points in mention.

The reviewer first stated that Lemma 9 "seems to be in error." The difficulty, as we see it, is that the transition from step (1) to step (2) was unclear, so we will present a somewhat more complete proof. We will assume

(1) $(y)(y \in \mathsf{Fld}\,\epsilon_S \wedge (x)(x \epsilon\ y \rightarrow \varphi(x)) \rightarrow \varphi(y)) \rightarrow (y)(y \in \mathsf{Fld}\,\epsilon_S \rightarrow \varphi(y))$

for formulas $\varphi(x)$ not containing y or u and show that

(2) $(u)(u \in \mathsf{Fld}\,R \wedge (v)(vRu \rightarrow \varphi(v)) \rightarrow \varphi(u)) \rightarrow (u)(u \in \mathsf{Fld}\,R \rightarrow \varphi(u))$

for formulas $\varphi(x)$ not containing y or u. This would conclude the proof of the lemma. We now suppose the hypothesis of (2); i.e., we assume that

(3) $(u)(u \in \mathsf{Fld}\,R \wedge (v)(vRu \rightarrow \varphi(v)) \rightarrow \varphi(u))$

where $\varphi(v)$ does not contain y or u. It remains to show that

(4) $(u)(u \in \mathsf{Fld}\,R \rightarrow \varphi(u))$.

We now define the formula ψ as follows:

(5) $\psi(x) \equiv x \in \mathsf{Fld}\,\epsilon_S \wedge \varphi(f'x)$.

We will first show that ψ satisfies the hypothesis of (1). Suppose that

(6) $y \in \mathsf{Fld}\,\epsilon_S$

and

(7) $(x)(x \epsilon\ y \rightarrow \psi(x))$.

We must show that $\psi(y)$. It is clear from (6) that the first part of the definition of ψ is satisfied. It remains to show that $\varphi(f'y)$. Since f is an isomorphism, there exists u such that

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(8) $u \in \mathsf{Fld}\,R$

and

(9) $u = f'y$.

Thus, we must show $\varphi(u)$. To do this, we need only show the hypothesis of (3). The first part of the hypothesis is clear from (8). Now suppose that

(10) vRu.

Therefore, $v \in \mathsf{Fld}\,R$, hence there exists x such that $x \in \mathsf{Fld}\,\epsilon_S$ and $f'x = v$. Since f is an isomorphism and by (10), we have that $x \in y$. Therefore, by (7), $\psi(x)$; in particular, $\varphi(f'x)$ and hence $\varphi(v)$. Therefore, we have

(11) $(v)(vRu \rightarrow \varphi(v))$.

By (8), (9), (11), and (3), we have that $\varphi(u)$. This shows that ψ satisfies the hypothesis for (1). Since (1) is assumed true, the conclusion must follow. Therefore, we have

(12) $(y)(y \in \mathsf{Fld}\,\epsilon_S \rightarrow \psi(y))$.

We now return to the proof of (4). Suppose $u \in \mathsf{Fld}\,R$. Therefore, there is a $y \in \mathsf{Fld}\,\epsilon_S$ such that $f'y = u$. By (12), we have that $\psi(y)$. By (5), we have that $\varphi(f'y)$; therefore, we have $\varphi(u)$, which completes the proof of (4) and hence of (2), and so Lemma 9 is proved.

The second remark that the reviewer makes in [2] is that Theorem 19, part (ii) seems to be false. Actually it is vacuously true. Sets A_n^S are defined such that $A = \bigcup_{n=0}^{\infty} A_n^S \cup B^S$. However, $A_n^S = \phi$ for $n \geq 1$, easily seen from Lemma 17, making parts (ii) and (iii) of Theorem 19 redundant.

REFERENCES

[1] Belding, W. R., R. L. Poss, and P. J. Welsh, Jr., "Transitivity, supertransitivity and induction," *The Notre Dame Journal of Formal Logic*, vol. XIII (1972), pp. 177-190.

[2] Mendelson, E., The review of [1] in *Mathematical Reviews*, vol. 45 (1973), pp. 587-588.

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