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MODELS OF $\mathrm{Th}(\langle\omega^\omega, <\rangle)$

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In this paper* we characterize the models of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$. Our main tool will be the game-theoretic characterization of elementary equivalence given by Ehrenfeucht in [2] (*cf.* also Fraissé [3]). In particular our work may be viewed as a generalization of Theorem 13 in [2] which gives a characterization of the standard, i.e., well-ordered, models of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$.

The main result, Theorem 3 of section **2**, is that a model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ consists of an ultrashort model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ followed by at each point of an arbitrary linear order ultrashort models of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ or of $\mathrm{Th}(\langle\ldots + \omega^n + \omega^{n-1} + \ldots + \omega + 1 + \omega^\omega, <\rangle)$, where by an ultrashort model is meant one such that for any two points x, y there is an upper bound on n such that if z is between x and y, z may be a $\lim_n$. In Theorems 1 and 2 of section **2** we characterize ultrashort models of these two theories in terms of models of $\mathrm{Th}(\langle\omega^n, <\rangle)$. In section **1** we characterize models of $\mathrm{Th}(\langle\omega^n, <\rangle)$. In section **3** we discuss short models, namely models having no elements which are $\lim_n$ for every n. In section **4** we briefly discuss how the techniques of section **2** can be used to classify the completions of the theory of well-ordering and the element types of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$.

We will assume the reader is familiar with the results and techniques in Ehrenfeucht [2]. In particular we will freely use these without further reference or mention. Several lemmas, in particular Lemmas 6, 7, 8 essentially appear in [4]. We include them for completeness and self-containment.

Our notation in general will follow that suggested in Addison, Henkin and Tarski [1]. The games $\mathbf{G}_n$ are as denoted in Ehrenfeucht [2]. We now briefly indicate our notation for linearly ordered sets:

Ordinals will be denoted as usual.
Usually if it is clear specific mention of the linear order of a linearly ordered set will be omitted.

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If A, B are linearly ordered by $<_A$, $<_B$ respectively, then $A + B$ denotes $A \cup B$ linearly ordered by $<_{A+B}$ where $x <_{A+B} y \leftrightarrow (x \in A \wedge y \in B) \vee (x, y \in A \wedge x <_A y) \vee (x, y \in B \wedge x <_B y)$. (We assume A, B are disjoint. Otherwise, they should first be made disjoint. Henceforth we will assume as needed that sets are disjoint).

More generally if A is linearly ordered by $<_A$, and if for each $a \in A$, A_a is linearly ordered by $<_a$, then $\sum_{a \in A} A_a$ denotes $\bigcup_{a \in A} A_a$ linearly ordered by $<_{\Sigma A_a}$ where $x <_{\Sigma A_a} y \leftrightarrow (x \in A_a \wedge y \in A_b \wedge a <_A b) \vee (x, y \in A_a \wedge x <_a y)$.

If A, B are linearly ordered by $<_A$, $<_B$ respectively, then $A \times B$ denotes $\sum_{x \in B} A$.

If A is linearly ordered by $<_A$, then A^* denotes A linearly ordered by $<_{A^*}$ where $x <_{A^*} y$ if $y <_A x$.

$$\textstyle\sum^*_{a \in A} A_a = \sum_{a \in A^*} A_a.$$
$$\omega^{\omega^*} = \textstyle\sum^*_{n \in \omega} \omega^n.$$
$$\omega^{\omega^* + \omega} = \omega^{\omega^*} + \omega^\omega$$

If A is a linearly ordered set, a, $b \in A$, then

$$\begin{aligned} [a, b) &= b - a = \{x \in A \mid a \leq x < b\} \\ [0, b) &= b = \{x \in A \mid x < b\} \\ [a, \infty) &= A - a = \{x \in A \mid a \leq x\}. \end{aligned}$$

(a, b), etc. are denoted similarly.

If $a, b \in A, B$, then we write $[a, b)^A$, $[a, b)^B$, etc. to distinguish these intervals in A and B.

$\mathfrak{n}$ = class of all discrete linear ordered sets with first and last elements. We identify order isomorphic elements.

$$\mathfrak{n}_0 = \mathfrak{n} \cup \{\emptyset\}.$$

$\mathfrak{n}_0$ is partially ordered by $<$ given by $A < B$ iff $(\exists f)$ (f: A 1-1 order isomorphically onto an initial segment of B). So ω is an initial segment of $\mathfrak{n}_0$.

If φ is any sentence in the first order language for $<$ and $\psi(x_0)$ is a formula (perhaps with parameters) then $\varphi^{\psi(x_0)}$ is φ relativized to $\psi(x_0)$.

Definition: $\mathrm{lim}_0(x) =_{df} (x = x)$
$\mathrm{lim}_{m+1}(x) =_{df} (\forall y)(y < x \rightarrow (\exists z)(y < z < x \wedge \mathrm{lim}_m(z)))$
$x = 0 =_{df} \neg(\exists y)(y < x)$
$\underset{\sim}{\mathbf{t}} =_{df} \{\mathrm{lim}_n(x_0) \mid n \in \omega\}$
$\tilde{\mathbf{t}} =_{df} \mathbf{t} \cup \{x_0 \neq 0\}$

1 Models of $\mathrm{Th}(\langle\omega^n, <\rangle)$. As is well-known:

Proposition 1. $\langle\eta, <\rangle \models \mathrm{Th}(\langle\omega, <\rangle)$ *iff* $\exists\langle r, <\rangle$ *a linearly ordered set (possibly empty) such that* $\eta = \omega + ({}^*\omega + \omega)\cdot r$.

Proof: Omitted.

Proposition 2. $\langle\eta, <\rangle \models \mathrm{Th}(\langle\omega^{n+1}, <\rangle)$ *iff* $\exists\langle\mu, <\rangle \models \mathrm{Th}(\langle\omega, <\rangle)$, $\forall\alpha \in \mu$, $\exists\langle\mu_\alpha, <\rangle \models \mathrm{Th}(\langle\omega^n, <\rangle)$ *such that* $\eta = \sum_{\alpha\in\mu} \mu_\alpha$.

Proof: Assume $\langle\eta, <\rangle \models \mathrm{Th}(\langle\omega^{n+1}, <\rangle)$.
Now 1) $\langle\omega^{n+1}, <\rangle \models \varphi^{\lim_n(x_0)}$ for each $\varphi \in \mathrm{Th}(\langle\omega, <\rangle)$,
2) $\langle\omega^{n+1}, <\rangle \models \forall x\, \forall y\, ((\lim_n(x) \wedge \lim_n(y) \wedge x < y \wedge (\forall z)(x < z < y \rightarrow \neg \lim_n(z))) \rightarrow \varphi^{x \leq x_0 < y})$ for each $\varphi \in \mathrm{Th}(\langle\omega^n, <\rangle)$,
and 3) $\langle\omega^{n+1}, <\rangle \models \forall x\, \exists y\, (y \leq x \wedge \lim_n(y) \wedge \neg(\exists z)(y < z \leq x \wedge \lim_n(z)))$.

So $\langle\eta, <\rangle \models$ the sentences in 1), 2), 3). Let

$$\mu = \{\alpha \in \eta \mid \langle\eta, <\rangle \models \lim_n(x_0)\ [\alpha]\}.$$

And for each $\alpha \in \mu$, let

$$\mu_\alpha = \{\beta \in \eta \mid \alpha \leq \beta \wedge \langle\eta, <\rangle \models \neg(\exists z)(x_0 < z \leq x_1 \wedge \lim_n(z))\ [\alpha, \beta]\}.$$

Then by 1), $\langle\mu, <\rangle \models \mathrm{Th}(\langle\omega, <\rangle)$ and by 2), $\langle\mu_\alpha, <\rangle \models \mathrm{Th}(\langle\omega^n, <\rangle)$ and clearly $\eta = \sum_{\alpha\in\mu} \mu_\alpha$ by 3).

Conversely, assume the conclusion. So player II has a winning strategy in $\mathbf{G}_m(\langle\omega, <\rangle, \langle\mu, <\rangle)$ and in $\mathbf{G}_m(\langle\omega^m, <\rangle, \langle\mu_\alpha, <\rangle)$, $\forall\alpha \in \mu$, for every $m \geq 0$.

We give a winning strategy for II in $\mathbf{G}_m(\langle\omega^{n+1}, <\rangle, \langle n, <\rangle)$. Given a move of I, II chooses which 'ω^n' segment of model to use by winning strategy in the first game and then which point in it to use by winning strategy in the appropriate latter game.

2 Main Theorems:

Definition: A model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ or of $\mathrm{Th}(\langle\omega^{\omega^*+\omega}, <\rangle)$ which omits $\tilde{\mathbf{t}}$ is called a *short* model.

If $\langle\eta, <\rangle \models \mathrm{Th}(\langle\omega^\omega, <\rangle)$ or $\models \mathrm{Th}(\langle\omega^{\omega^*+\omega}, <\rangle)$, it is called *ultrashort* if $\forall x, y \in \eta, (x < y \rightarrow (\exists n)(\forall z)(x < z \leq y \rightarrow \neg \lim_n(z)))$.

Clearly any ultrashort model is short.

Theorem 1. *$\langle\eta, <\rangle$ is an ultrashort model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ iff $\exists$ for each $n \in \omega$ a model $\langle\eta_n, <\rangle \models \mathrm{Th}(\langle\omega^n, <\rangle)$ such that $\eta = \sum_{n\in\omega} \eta_n$.*

Theorem 2. *$\langle\eta, <\rangle$ is an ultrashort model of $\mathrm{Th}(\langle\omega^{\omega^*+\omega}, <\rangle)$ iff*
$\exists$ 1) for each $n \in \omega$ a $\mu_n \in \mathfrak{n}_0$ such that infinitely many $\mu_n \neq 0$,
2) for each $y \in \mu_n$ a model $\langle\eta_{n,y}, <\rangle \models \mathrm{Th}(\langle\omega^n, <\rangle)$,
3) a η' an ultrashort model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ such that $\eta = \sum^*_{n\in\omega} \sum_{y\in\mu_n} \eta_{n,y} + \eta'$.

Theorem 3. *$\langle\eta, <\rangle$ is a model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ iff*
$\exists$ 1) linearly ordered set μ (possibly empty),
2) for each $y \in \mu$, an ultrashort model $\langle\eta_y, <\rangle$ of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ or of $\mathrm{Th}(\langle\omega^{\omega^*+\omega}, <\rangle)$,
3) an ultrashort model $\langle\eta', <\rangle$ of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ such that $\eta = \eta' + \sum_{y\in\mu} \eta_y$.

The proofs of these three results will be by a sequence of lemmas. We first consider the $\rightarrow$ directions.

Lemma 1. *$\langle\omega^\omega, <\rangle$ and hence any model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ satisfies the following sentences:*

a) $(\exists x)(\forall y)(y \geq x)$
b) $(\forall x)(\exists y)(y > x \wedge \lim_n(y) \wedge \neg(\exists z)(x < z < y \wedge \lim_n(z)))$

c) $(\forall x)(\forall y)((y > x \wedge \lim_n(y) \wedge \neg(\exists z)(x < z < y \wedge \lim_n(z))) \rightarrow \varphi^{x \leq x_0 < y})$ *for every* $\varphi \in \mathrm{Th}(\langle\omega^n, <\rangle)$

d) $(\forall x)(\exists y)(y \leq x \wedge \lim_n(y) \wedge (\forall z)(y < z \leq x \rightarrow \neg\lim_n(z)))$

e) $(\forall x)(\forall y)((x < y \wedge \lim_n(x) \wedge \lim_n(y) \wedge \neg(\exists z)(x < z \leq y \wedge \lim_{n+1}(z))) \rightarrow \varphi^{x \leq x_0 < y \wedge \lim_n(x_0)})$ *for every* $\varphi \in \mathrm{Th}(\mathfrak{n})$.

Lemma 2. $\langle\omega^{\omega^*+\omega}, <\rangle$ *and hence any model of* $\mathrm{Th}(\langle\omega^{\omega^*+\omega}, <\rangle)$ *satisfies the following sentences:*

a) $(\forall x)(\exists y)(y < x)$,

b)-e) of Lemma 1.

Proofs: Routine.

Lemma 3. $\rightarrow$ of Theorem 1.

Proof: Let $\langle\eta, <\rangle$ be an ultrashort model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$. Let $x_0 = 0$. By induction define x_{n+1} = least $\lim_n > x_n$. Such exist by 1b). By 1c), $\langle[x_i, x_{i+1}), <\rangle \models \mathrm{Th}(\langle\omega^i, <\rangle)$. By the definition of ultrashort, $\eta = \sum_{i\in\omega}[x_i, x_{i+1})$.

Lemma 4. $\rightarrow$ of Theorem 2.

Proof: Let $\langle\eta, <\rangle$ be an ultrashort model of $\mathrm{Th}(\langle\omega^{\omega^*+\omega}, <\rangle)$. Let $y_0 = z_0 \in \eta$. By induction define y_{n+1} = greatest $\lim_{n+1} \leq y_n$. Such exist by 2d). By induction define x_{n+1} = least $\lim_n > z_n$. Such exist by 2b). By 2a) infinitely many of y_i are distinct. Let $\mu_n = \{a \mid y_{n+1} \leq a < y_n \wedge \lim_n(a)\}$. So infinitely many $\mu_n \neq 0$ and by 2e), $\mu_n \in \mathfrak{n}_0$. For each $y \in \mu_n$, let $\eta_{n,y} = [y, y')$ where y' is least $\lim_n > y$. By 2c) $\langle\eta_{n,y}, <\rangle \models \mathrm{Th}(\langle\omega^n, <\rangle)$. Also $[y_{n+1}, y_n) = \sum_{y\in\mu_n}\eta_{n,y}$. And $(0, y_0) = \sum^*_{n\in\omega}[y_{n+1}, y_n) = \sum^*_{n\in\omega}\sum_{y\in\mu_n}\eta_{n,y}$. Let $\eta_n = [z_n, z_{n+1})$. By 2c), $\langle\eta_n, <\rangle \models \mathrm{Th}(\langle\omega^n, <\rangle)$. And $[z_0, \infty) = \sum_{n\in\omega} n_{x,n}$. So by Theorem 1, $\eta' = [z_0, \infty)$ is an ultrashort model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$. Now $\eta = (0, y_0) + [z_0, \infty) = \sum^*_{n\in\omega}\sum_{y\in\mu_n}\eta_{n,y} + \eta'$.

Lemma 5. $\rightarrow$ of Theorem 3.

Proof: Let $\langle\eta, <\rangle$ be a model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$. On η define $a \approx b$ if $(\exists n)(\exists x)(a < x \leq b \rightarrow \neg\lim_n(x))$ for $a < b$. If $a > b$, define $a \approx b$ if $b \approx a$. And define $a \approx a$. So $\approx$ is an equivalence relation.

By 1a), $\tilde{\tilde{\eta}}$ has a least element $\tilde{\tilde{0}}$. Let $\mu = \tilde{\tilde{\eta}} - \{\tilde{\tilde{0}}\}$. As in Lemma 3, $\tilde{\tilde{0}} = \sum_{i\in\omega}\eta_i$ where $\langle\eta_i, <\rangle \models \mathrm{Th}(\langle\omega^i, <\rangle)$ and hence is an ultrashort model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ by Theorem 1. If $x \in \mu$ then either x realizes $\mathbf{t}$ or not. If so arguing similarly to Lemma 3 we find $x = \sum_{i\in\omega}\eta_{i,x}$ where $\langle\eta_{i,x}, <\rangle \models \mathrm{Th}(\langle\omega^i, <\rangle)$ and hence x is ultrashort model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$. On the other hand if x does not realize $\mathbf{t}$, x has no least element and arguing similarly to Lemma 4 we find $x = \sum^*_{n\in\omega}\sum_{y\in\mu_{n,x}}\eta_{n,x,y} + \eta'$ where $\mu_{n,x} \in \mathfrak{n}_0$, infinitely many are $\neq 0$, $\langle\eta_{n,x,y}, <\rangle) \models \mathrm{Th}(\langle\omega^n, <\rangle)$, $\langle\eta', <\rangle$ is ultrashort model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$. And hence x is ultrashort model of $\mathrm{Th}(\omega^{\omega^*+\omega}, <\rangle)$ by Theorem 2. As $\eta = \tilde{\tilde{0}} + \sum_{x\in\mu} x$, we are done.

Lemmas 1-5 may be viewed as giving a means of partitioning models of these theories. The theorems assert any model which can be partitioned in such a manner is a model of the theory in question.

Lemma 6. *If player* II *has a winning strategy in* $\mathsf{G}_n(\langle\alpha_x, <\rangle, \langle\beta_x, <\rangle)$ *for every* $x \in \gamma$, *then* II *has a winning strategy in* $\mathsf{G}_n\left(\left\langle\sum_{x\in\gamma}\alpha_x, <\right\rangle, \left\langle\sum_{x\in\gamma}\beta_x, <\right\rangle\right)$.

Proof. Player II's winning strategy is: If on some move player I chooses a point in α_x (or β_x), then player II uses his winning strategy in $\mathsf{G}_n(\langle\alpha_x, <\rangle, \langle\beta_x, <\rangle)$ to give his move.

Corollary. *If* $\alpha_x \equiv \beta_x$, $\forall x \in \gamma$, *then* $\sum_{x\in\gamma}\alpha_x \equiv \sum_{x\in\gamma}\beta_x$.

Lemma 7. *If player* II *has a winning strategy in* $\mathsf{G}_n(\langle\gamma, <\rangle, \langle\delta, <\rangle)$ *and if player* II *has a winning strategy in* $\mathsf{G}_n(\langle\alpha_x, <\rangle, \langle\beta_y, <\rangle)$ *for every* $x \in \gamma$, $y \in \delta$, *then* II *has a winning strategy in*

$$\mathsf{G}_n\left(\left\langle\sum_{x\in\gamma}\alpha_x, <\right\rangle\right), \left\langle\sum_{y\in\delta}\beta_y, <\right\rangle\right).$$

Proof. Player II's winning strategy is: If on some move player I chooses a point $y \in \alpha_x$, then player II uses his winning strategy in $\mathsf{G}_n(\langle\gamma, <\rangle, \langle\delta, <\rangle)$ assuming a move by I of x to give a point $x' \in \delta$ and his winning strategy in $\mathsf{G}_n(\langle\alpha_x, <\rangle, \langle\beta_{x'}, <\rangle)$ assuming a move by I of y to give a point $y' \in \beta_{x'}$. II then plays as his move y'.

Similarly if player I chooses a point $y \in \beta_x$, then player II selects a point $x' \in \gamma$ and then a point $y' \in \alpha_{x'}$. II's move then will be y'.

Corollary. *If* $\gamma \equiv \delta$, $\alpha_x \equiv \beta_y$, $\forall x \in \delta\ \forall y \in \delta$, *then* $\sum_{x\in\gamma}\alpha_x \equiv \sum_{y\in\delta}\beta_y$.

Corollary. *If* $\gamma \equiv \delta$, $\alpha \equiv \beta$, *then* $\alpha \times \gamma \equiv \beta \times \delta$.

Lemma 8. *If player* II *has a winning strategy in* $\mathsf{G}_n(\langle\alpha_i, <\rangle, \langle\beta_i, <\rangle)$ *for* $i =$ 1, 2, *then* II *has a winning strategy in* $\mathsf{G}_{n+1}(\langle\alpha_1 + 1 + \alpha_2, <\rangle, \langle\beta_1 + 1 + \beta_2, <\rangle)$ *after the initial move* $0 \longleftrightarrow 0$. (Note $1 = \{0\}$.)

Proof: Player II's winning strategy is on each segment to use his given winning strategies. I.e., if I chooses a point in an α, player II responds in other α using winning strategy in $\mathsf{G}_n(\langle\alpha_1, <\rangle, \langle\alpha_2, <\rangle)$. And similiarly for β.

Lemma 9. *Player* II *has a winning strategy in* $\mathsf{G}_n(\langle\alpha, <\rangle, \langle\beta, <\rangle)$ *if* $\alpha, \beta \in \mathrm{n}$, $\alpha, \beta \geq 2^n - 1$.

Proof: By induction on n. $n = 1$ is trivial. Assume the result for $n = k$. Let $\alpha, \beta \in \mathrm{n}$, $\alpha, \beta \geq 2^{k+1} - 1$. We give player II's winning strategy for $\mathsf{G}_{k+1}(\langle\alpha, <\rangle, \langle\beta, <\rangle)$. Without loss of generality player I's first move is in α. Say it is x_0.

Case 1: $x_0 < 2^k - 1$. By induction, as $\alpha - x_0$, $\beta - x_0 \geq 2^k - 1$, player II has winning strategy in $\mathsf{G}_k(\langle\alpha - x_0, <\rangle, \langle\beta - x_0, <\rangle)$. Also II has winning strategy in $\mathsf{G}_k(\langle x_0, <\rangle, \langle y_0, <\rangle)$. So by Lemma 8 if II responds with x_0 in β, then II has winning strategy in $\mathsf{G}_{k+1}(\langle\alpha, <\rangle, \langle\beta, <\rangle)$.
Case 2: $\alpha - x_0 < 2^k - 1$. Player II responds with $\beta - (\alpha - x_0)$, i.e., with the point $\alpha - x_0 \leq$ the last element of β. This case is similar to 1.
Case 3: Neither Case 1 nor Case 2. Player II responds with $2^k - 1$ (or any other element $y_0 \in \beta$ such that $y_0 \geq 2^k - 1$, $\beta - y_0 \geq 2^k - 1$). By induction player II has winning strategies in $\mathsf{G}_k(\langle x_0, <\rangle, \langle y_0, <\rangle)$ and $\mathsf{G}_k(\langle\alpha - x_0, <\rangle, \langle\beta - y_0, <\rangle)$. So by Lemma 8 we are done.

Notation: If $\alpha, \beta \in \mathfrak{n}_0$, we write $\alpha \underset{n}{=} \beta$ to denote $\alpha = \beta$ or $\alpha, \beta \geq 2^n - 1$.

Lemma 10: *Player* II *has a winning strategy in*

$$\mathsf{G}_n\left(\left\langle \sum_{x\in\alpha} \alpha_x, <\right\rangle, \left\langle \sum_{x\in\beta} \beta_x, <\right\rangle\right)$$

where

i) $\alpha, \beta \in \mathfrak{n}, \alpha \underset{n}{=} \beta$,
ii) $\langle \alpha_x, <\rangle, \langle \beta_y, <\rangle \models \mathrm{Th}(\langle\omega^m, <\rangle), \ \forall x \in \alpha, \ y \in \beta$.

Proof: By Lemmas 7 and 9.

Remark: By combining the techniques of Lemma 8 and Theorem 12 in [2], one can, in fact, obtain:

Lemma 11. *Player* II *has a winning strategy in*

$$\mathsf{G}_n\left(\left\langle \sum_{x\in\alpha} \alpha_x, <\right\rangle, \left\langle \sum_{x\in\beta} \beta_x, <\right\rangle\right)$$

where

i) $m < n$,
ii) $\alpha, \beta \in \mathfrak{n}, \alpha \underset{n-m}{=} \beta$,
iii) $\langle \alpha_x, <\rangle, \langle \beta_y, <\rangle \models \mathrm{Th}(\langle\omega^m, <\rangle), \ \forall x \in \alpha, \ y \in \beta$.

Lemma 12. *Player* II *has a winning strategy in*

$$\mathsf{G}_n\left(\sum\nolimits^*_{0\leq i\leq k}\omega^i.\ n_i, \sum\nolimits^*_{0\leq i\leq k}\sum\nolimits_{x\in\mu_i} \eta_{x,i}\right)$$

where

i) $k < n$,
ii) $\mu_i \in \mathfrak{n}_0, \ n_i \in \omega$,
iii) $\langle \eta_{x,i}, <\rangle \models \mathrm{Th}(\langle\omega^i, <\rangle)$,
iv) $n_i \underset{n}{=} \mu_i$ ($n_i \underset{n-i}{=} \mu_i$ *with* Lemma 11).

Proof: By Lemmas 6 and 10.

Lemma 13. *Player* II *has a winning strategy in*

$$\mathsf{G}_n\left(\sum\nolimits^*_{0\leq i\leq k}\omega^i.\ n_i, \sum\nolimits^*_{0\leq i\leq k'}\sum\nolimits_{x\in\mu_i} \eta_{x,i}\right)$$

where

i) $k, k' \in \omega, \ k, k' \geq n$,
ii) $n_i \in \omega, \ \mu_i \in \mathfrak{n}_0, \ n_k \neq 0, \ \mu_{k'} \neq 0$,
iii) $\langle \eta_{x,i}, <\rangle \models \mathrm{Th}(\langle\omega^i, <\rangle)$,
iv) $\forall i < n, \ n_i \underset{n}{=} \mu_i$.

Proof: By induction on n.

Case 1: $n = 1$ trivial.
Case 2: Let $n \geq 2$. Assume the result for $n - 1$. Let k, k', etc. be as above. Let

$$\alpha = \sum\nolimits^*_{0\leq i\leq k}\omega^i.\ n_i, \ \beta = \sum\nolimits^*_{0\leq i\leq k'}\sum\nolimits_{x\in\mu_i} \eta_{x,i}$$

Case a: On move 1 player I chooses an element $a \in \alpha$.

Case ai: $a < \omega^n$. Say

$$a = \sum^*_{0\leq i\leq l}\omega^i . k_i \text{ where } l < n,\ k_i < \omega,\ k_l \neq 0.$$

As $\langle \eta_{0,k'}, <\rangle \models \mathrm{Th}(\langle \omega^{k'}, <\rangle)$,

$$\exists b \in \eta_{0,k'} \text{ such that } \langle a, <\rangle \equiv \langle b, <\rangle.$$

Also $\langle \omega^k - (a+1), <\rangle \equiv \langle \omega^k, <\rangle$ and $\langle \eta_{0,k'} - (b+1), <\rangle \equiv \langle \omega^{k'}, <\rangle$.

Let b be II's move. Then $\langle \alpha - (a+1), <\rangle$ and $\langle \beta - (b+1), <\rangle$ meet the conditions of the lemma for $n - 1$. So by Lemma 8, II has winning strategy in $\mathbf{G}_n(\langle \alpha, <\rangle, \langle \beta, <\rangle)$.
Case aii: $\alpha - a < \omega^n$. Say

$$\alpha - a = \sum^*_{0\leq i\leq l}\omega^i . k_i \text{ where } l < n,\ k_i < \omega.$$

By the definition of α, $k_l \leq n_l$, $k_i = n_i$ if $i < l$. Let

$$a' = a - \left(\sum^*_{l<i\leq k}\omega^i . n_i + \omega^l . (n_l - k_l)\right).$$

So $a' < \omega^l$. Say

$$a' = \sum^*_{0\leq i<l}\omega^i . j_i.$$

Let

$$k_l' \in \mu_l \text{ be defined by } k_l' = \begin{cases} n_l - k_l \text{ if } n_l - k_l < 2^n - 1, \\ \mu_l - k_l \text{ if } k_l < 2^n - 1, \\ 2^n - 1 \text{ otherwise.} \end{cases}$$

Let

$$b = \sum^*_{l<i\leq k'}\sum_{x\in\mu_i} \eta_{x,i} + \sum_{x<k_l'}\eta_{x,l} + \sum^*_{0\leq i<l}\sum_{x<j_i} \mu_{i,x}$$

where $\langle \mu_{i,x}, <\rangle \models \mathrm{Th}(\langle \omega^i, <\rangle)$ and $\sum^*_{0\leq i<l}\sum_{x<j_i}\mu_{i,x}$ is an initial segment of $\eta_{k_l',l}$.

Then $\langle a, <\rangle$, $\langle b, <\rangle$ satisfy the conditions of the lemma for $n - 1$, and $\langle \alpha - (a+1), <\rangle$, $\langle \beta - (b+1), <\rangle$ satisfy the conditions of Lemma 12 for $n - 1$. So by Lemma 8, II has winning strategy in $\mathbf{G}_n(\langle \alpha, <\rangle, \langle \beta, <\rangle)$.
Case aiii: Neither case ai nor aii. Say

$$a = \sum^*_{0\leq i\leq l}\omega^i . k_i \text{ where } k_l \neq 0.$$

So $l \geq n$. Let

$$b = \sum_{x<2^{n-1}-1} \mu_{n-1,x} + \sum^*_{i<n-1}\sum_{x<k_i}\mu_{i,x}$$

where b is initial segment of β, $\langle \mu_{i,x}, <\rangle \models \mathrm{Th}(\langle \omega^i, <\rangle)$ for $i < n$. Then $\langle a, <\rangle$, $\langle b, <\rangle$ satisfy the conditions of the lemma for $n - 1$, and $\langle \alpha - (a+1), <\rangle$, $\langle \beta - (b+1), <\rangle$ satisfy the conditions of the lemma for $n - 1$. So by Lemma 8, II has winning strategy in $\mathbf{G}_n(\langle \alpha, <\rangle, \langle \beta, <\rangle)$.
Case b: On move 1 player I chooses an element $b \in \beta$.
Case bi: There is no $\lim_n \leq b$. Say

$$b = \sum^*_{0\leq i\leq l}\sum_{x\in\mu_i} \eta_{x,i} \text{ where } \langle \eta_{x,i}, <\rangle \models \mathrm{Th}(\langle \omega^i, <\rangle),\ l < n.$$

Let

$$n_i = \begin{cases} \mu_i \text{ if } \mu_i < 2^n - 1, \\ 2^n - 1 \text{ otherwise.} \end{cases}$$

Let $a = \sum^*_{0 \leq i \leq l} \omega^i . n_i$. Then if a is II's move, II has winning strategy in $\mathsf{G}_n(\langle \alpha, <\rangle, \langle \beta, <\rangle)$ by induction (as $\langle a, <\rangle$, $\langle b, <\rangle$ satisfy conditions of Lemma 12 for $n - 1$ and $\langle \alpha - (a+1), <\rangle$, $\langle \beta - (b+1), <\rangle$ satisfy conditions of lemma for $n - 1$).

Case bii: There is no $\lim_n \geq b$. Say

$$\beta - b = \sum^*_{0 \leq i \leq l} \sum_{x \epsilon \mu_i'} \eta_{x,i} \text{ where } l < n.$$

By definition of β, $\mu_i' = \mu_i$ if $i < l$, $\mu_l' \leq \mu_l$. Let

$$b' = b - \left(\sum^*_{l < i \leq k'} \sum_{x \epsilon \mu_i} \eta_{x,i} + \sum_{x \epsilon \mu_l - \mu_l'} \eta_{x,l} \right).$$

So b' has no $\lim_l$. Say

$$b' = \sum^*_{0 \leq i < l} \sum_{x \epsilon \tilde{\mu}_i} \tilde{\eta}_{x,i} \text{ where } \tilde{\mu}_i \in \mathfrak{n}_0,\ \langle \tilde{\eta}_{x,i}, <\rangle \models \mathrm{Th}(\langle \omega^i, <\rangle).$$

Let

$$k_l \leq n_l \text{ be defined by } k_l = \begin{cases} \mu_l - \mu_l' \text{ if } \mu_l - \mu_l' < 2^n - 1, \\ n_l - \mu_l' \text{ if } \mu_l' < 2^n - 1, \\ 2^n - 1 \text{ otherwise.} \end{cases}$$

Let

$$a = \sum^*_{l < i \leq k} \omega^i . n_i + \omega^l . k_l + \sum^*_{0 \leq i < l} \omega^i . j_i$$

$$\text{where } j_i = \begin{cases} \tilde{\mu}_i \text{ if } \tilde{\mu}_i < 2^n - 1, \\ 2^n - 1 \text{ otherwise.} \end{cases}$$

Then $\langle a, <\rangle$, $\langle b, <\rangle$ satisfy the conditions of the lemma for $n - 1$, and $\langle \alpha - (a+1), <\rangle$, $\langle \beta - (b+1), <\rangle$ satisfy the conditions of Lemma 12 for $n - 1$. So by Lemma 8, II has winning strategy in $\mathsf{G}_n(\langle \alpha, <\rangle, \langle \beta, <\rangle)$.

Case biii: Neither case bi nor bii. Say

$$b = \sum^*_{0 \leq i \leq l} \sum_{x \epsilon \tilde{\mu}_i} \tilde{\eta}_{i,x} \text{ where } \tilde{\mu}_i \in \mathfrak{n}_0,\ \mu_l \neq 0,\ \langle \tilde{\eta}_{i,x}, <\rangle \models \mathrm{Th}(\langle \omega^i, <\rangle).$$

So $l \geq n$. Let

$$k_i = \begin{cases} \tilde{\mu}_i \text{ if } \tilde{\mu}_i < 2^n - 1 \\ 2^n - 1 \text{ otherwise} \end{cases} \text{ for } i < n - 1.$$

Let

$$a = \omega^{n-1}.(2^{n-1} - 1) + \sum^*_{0 \leq i < n-1} \omega^i . k_i.$$

Then $\langle a, <\rangle$, $\langle b, <\rangle$ satisfy the conditions of the lemma for $n - 1$ and $\langle \alpha - (a+1), <\rangle$, $\langle \beta - (b+1), <\rangle$ satisfy them for $n - 1$. So as usual, II has winning strategy in $\mathsf{G}_n(\langle \alpha, <\rangle, \langle \beta, <\rangle)$.

Proof of Theorem 1: $\leftarrow$ Clearly η is an ultrashort model. We will, thus, be done when we show by induction on n:

Lemma 14. *Player* II *has a winning strategy in*

$$G_n(\langle \omega^\omega, <\rangle, \langle \textstyle\sum_{n\epsilon\omega}\eta_n, <\rangle).$$

Proof:

Case 1: $n = 1$ is trivial.
Case 2: Let $n \geqslant 2$. Assume the result for $n - 1$. Let

$$\alpha = \omega^\omega,\ \beta = \textstyle\sum_{n\epsilon\omega}\eta_n.$$

Case a: On move 1 player I chooses an element $a \epsilon \alpha$. Say

$$a = \textstyle\sum^*_{0\leqslant i\leqslant l}\omega^i.\ k_i \text{ where } \ k_i < \omega,\ k_l \neq 0.$$

Let

$b = \eta_l + \sum_{y<k_l-1}\eta_{l,y} + \sum^*_{0\leqslant i<l}\sum_{y<k_i}\eta_{i,y}$ where $\langle \eta_{i,y}, <\rangle \models \mathrm{Th}(\langle \omega^i, <\rangle)$,
$\sum_{y<k_l-1}\eta_{l,y} + \sum^*_{0\leqslant i<l}\sum_{y<k_i}\eta_{i,y}$ is initial segment of η_{l+1}.

Let b be II's move. Then by Lemma 12 or 13, player II has winning strategy in $G_{n-1}(\langle a, <\rangle, \langle b, <\rangle)$. $\langle \alpha - (a+1), <\rangle$, $\langle \beta - (b+1), <\rangle$ satisfy the induction hypotheses for $n - 1$. So by the lemma, we are done in this case.
Case b): On move 1 player I chooses an element $b \epsilon \beta$. Say

$$b = \eta_l + \textstyle\sum^*_{0\leqslant i\leqslant l}\sum_{y<\mu_i}\eta_{i,y} \text{ where } \mu_i \epsilon \mathfrak{n}_0,\ \langle \eta_{i,y}, <\rangle \models \mathrm{Th}(\langle \omega^i, <\rangle).$$

Let

$$k_l = \begin{cases} 2^n - 1 \text{ if } \mu_i > 2^n - 2, \\ \mu_i + 1 \text{ otherwise;} \end{cases}$$

$$k_i = \begin{cases} 2^n - 1 \text{ if } \mu_i > 2^n - 1 \\ \mu_i \text{ otherwise} \end{cases} \quad \text{for } i < n.$$

Let

$$a = \textstyle\sum^*_{0\leqslant i\leqslant l}\omega^i.\ k_i.$$

Then by Lemma 12 or 13, player II has winning strategy in $G_{n-1}(\langle a, <\rangle, \langle b, <\rangle)$. $\langle \alpha - (a+1), <\rangle, \langle \beta - (b+1), <\rangle$ satisfy the induction hypothesis for $n - 1$. So as usual this case is done.

Lemma 15: *Player* II *has a winning strategy in*

$$G_n\left(\langle \textstyle\sum^*_{i\epsilon\omega}\omega^i.\ n_i, <\rangle, \langle \textstyle\sum^*_{i\epsilon\omega}\sum_{x\epsilon\mu_i}\eta_{x,i}, <\rangle\right)$$

where

i) $n_i \epsilon \omega$,
ii) $\exists m$ *such that* $\forall i \geqslant m,\ n_i = 1$,
iii) $\forall i < n,\ \mu_i \equiv_n n_i$,
iv) $\mu_i \epsilon \mathfrak{n}_0$,
v) *infinitely many* $\mu_i \neq 0$,
vi) $\langle \eta_{x,i}, <\rangle \models \mathrm{Th}(\langle \omega^i, <\rangle)$.

Proof: Similar to Lemma 13.

Lemma 16. *Player* II *has winning strategy in*

$$G_n\left(\langle\omega^{\omega^*+\omega}, <\rangle, \langle\textstyle\sum^*_{i\in\omega}\sum_{x\in\mu_i}\eta_{x,i} + \sum_{i\in\omega}\eta_i, <\rangle\right)$$

where

i) $\mu_i \in \mathfrak{n}_0$,
ii) *infinitely many* $\mu_i \neq 0$,
iii) $\langle\eta_{x,i}, <\rangle \models \mathrm{Th}(\langle\omega^i, <\rangle)$,
iv) $\langle\eta_i, <\rangle \models \mathrm{Th}(\langle\omega^i, <\rangle)$.

Proof: Similar to Lemma 14. It uses primarily Lemmas 14 and 15 and Theorem 1.

The proof of Theorem 2 now follows immediately from Lemma 16 as $\sum^*_{i\in\omega}\sum_{x\in\mu_i}\eta_{x,i} + \sum_{i\in\omega}\eta_i$ is clearly ultrashort.

Lemma 17. $\langle\eta' + \sum_{x\in\mu}\eta_x, <\rangle \equiv \langle\omega^\omega + \sum_{x\in\mu}\mu_x, <\rangle$ *where*

i) $\langle\eta', <\rangle$ *is ultrashort model of* $\mathrm{Th}(\langle\omega^\omega, <\rangle)$,
ii) $\langle\eta_x, <\rangle$ *is ultrashort model of* $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ *or* $\mathrm{Th}(\langle\omega^{\omega^*+\omega}, <\rangle)$,
iii) $\mu_x = \begin{cases}\omega^\omega \textit{ if } \langle\eta_x, <\rangle \models \mathrm{Th}(\langle\omega^\omega, <\rangle),\\ \omega^{\omega^*+\omega} \textit{ if } \langle\eta_x, <\rangle \models \mathrm{Th}(\langle\omega^{\omega^*+\omega}, <\rangle).\end{cases}$

Proof: By Lemma 6.

Lemma 18. *Player* II *has a winning strategy in*

$$G_n\left(\langle\omega^n + \textstyle\sum^*_{l<n}\omega^l.\ m_l, <\rangle, \langle\omega^\omega + \sum_{x\in\mu}\mu_x + \sum^*_{l\leq n}\omega^l.\ r_l + \sum^*_{l<n}\omega^l.\ n_l, <\rangle\right)$$

where

i) $m_l \underset{n}{=} n_l$, *if* $l < n$, $m_l, n_l \in \omega$,
ii) $\mu_x = \omega^\omega$ *or* $\omega^{\omega^*+\omega}$, $\forall x \in \mu$,
iii) μ *is arbitrary linear order (possibly empty)*,
iv) $r_l \in \omega$, $l \geq n$,
v) $\exists m$ *such that* $\forall i \geq m$, $r_l = 0$ *or* $\forall i \geq m$, $r_l = 1$.

Proof: Similar to Lemma 13.

Lemma 19. *Player* II *has a winning strategy in*

$$G_n\left(\langle\omega^\omega, <\rangle, \langle\omega^\omega + \textstyle\sum_{x\in\mu}\mu_x, <\rangle\right)$$

where

i) $\mu_x = \omega^\omega$ *or* $\omega^{\omega^*+\omega}$,
ii) μ *is arbitrary linear order (possibly empty)*.

Proof: Similar to Lemma 14. It uses primarily Lemma 18.

The proof of Theorem 3 now follows immediately from Lemmas 17 and 19.

3 Short models. By techniques similar to those in section 2 one can prove:

Theorem 4. $\langle\eta, <\rangle$ *is a short model of* $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ *iff*

$\exists$ 1) *an ultrashort model* $\langle\eta', <\rangle$ *of* $\mathrm{Th}(\langle\omega^\omega, <\rangle)$,
2) *a linear order* μ *possibly empty*,
3) *for each* $x \in \mu$, *an ultrashort model* $\langle\eta_x, <\rangle$ *of* $\mathrm{Th}(\langle\omega^{\omega^*+\omega}, <\rangle)$ *such that* $\eta = \eta' + \sum_{x\epsilon\mu} \eta_x$.

Theorem 5. $\langle\eta, <\rangle$ *is a short model of* $\mathrm{Th}(\langle\omega^{\omega^*+\omega}, <\rangle)$ *iff*

$\exists$ 1) *a linear order* μ,
2) *for each* $x \in \mu$, *an ultrashort model* η_x *of* $\mathrm{Th}(\langle\omega^{\omega^*+\omega}, <\rangle)$ *such that* $\eta = \sum_{x\epsilon\mu} \eta_x$.

Theorem 6. $(\eta, <\rangle \models \mathrm{Th}(\langle\omega^\omega, <\rangle)$ *iff*

$\exists$ 1) *a short model* $\langle\eta', <\rangle$ *of* $\mathrm{Th}(\langle\omega^\omega, <\rangle)$,
2) *a linear order* μ *possibly empty*,
3) *for each* $x \in \mu$, *a short model* $\langle\eta_x, <\rangle$ *of* $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ *or of* $\mathrm{Th}(\langle\omega^{\omega^*+\omega}, <\rangle)$ (*the latter occurring only if x does not have an immediate predecessor*) *such that* $\eta = \eta' + \sum_{x\epsilon\mu} \eta_x$.

4 Other Results. Using the lemmas of section 2 and similar results one can obtain Ehrenfeucht's classification of the completions of the theory of well-ordered sets. Using the result and the fact that $\langle\alpha, a_1 \ldots a_n <\rangle \equiv \langle\beta, b_1 \ldots b_n <\rangle$ iff $\langle a_{i+1} - a_i, <\rangle \equiv \langle b_{i+1} - b_i, <\rangle, \forall i \leqslant n+1$ if $a_1 < \ldots < a_n$, $b_1 < \ldots < b_n$ and $a_0 = b_0 = 0$, $a_{n+1} = \alpha$, $b_{n+1} = \beta$ one can then classify the element types of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$, or any other completion of the theory of well-ordering.

In particular the following are the distinct completions of the theory of well-ordering:

$$\left\{\mathrm{Th}\left(\left\langle\omega^n.\ m + \textstyle\sum^*_{i<n}\omega^i.\ n_i, <\right\rangle\right) \mid n \in \omega,\ m \in \omega,\ m \neq 0,\ n_i \in \omega \cup \{\omega + \omega^* + \omega\}\right\}$$
$$\bigcup \left\{\mathrm{Th}\left(\left\langle\omega^\omega + \textstyle\sum^*_{i<n}\omega^i.\ n_i, <\right\rangle\right) \mid n_i \in \omega \cup \{\omega + \omega^* + \omega\}\right\}.$$

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