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α-MODELS AND THE SYSTEMS T AND T*

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This paper[1] is the fourth (and last) of a series in which we study two systems of set theory, **T** and **T***, which were designed to serve as foundations for category theory (*cf.* [3], [4], and [5]). It is divided into two parts; in the first, we develop and make more precise certain results of section **3** of [5]. As the subject matter of the first part is of independent interest, it is treated here in some detail; nonetheless, the full development of the subject will appear elsewhere. In the second part, we only outline how **T** and **T*** can be used as foundations for the notions of category and functor, since it is not difficult to work out the details.

I. α-Models

1 Introduction Let us consider a first-order language $\mathcal{L}$ containing the family $(t)_{i\epsilon\alpha}$ of (distinct) constant terms, where α is an ordinal (in the sense of von Neumann) greater than 0, and a set Γ of sentences of $\mathcal{L}$ (in particular, Γ can be a first-order theory). An α-model of Γ is a model in the ordinary sense, such that every element of it is denoted by at least one term t_i, $i \epsilon \alpha$. A sentence F of $\mathcal{L}$ is said to be a semantic α-consequence of Γ if it is true in every α-model of Γ. Then, it seems natural to ask the following question: Is it possible to strengthen the first-order predicate calculus, with or without equality, in such a manner as to assure that if F is a semantic α-consequence of Γ, then F is also a syntactic consequence of Γ in the new strengthened calculus? A strengthened version $\mathcal{C}$ of the predicate calculus is called α-complete if, and only if, whenever a sentence F is a semantic α-consequence of a set Γ of sentences, we also have that F is a syntactic consequence of Γ in $\mathcal{C}$.

The concept of α-model appears when we consider theories in which it is natural to suppose that their intuitive models must satisfy the condition

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that every element belonging to such a model has a "name", as in the case, for example, of first-order arithmetic, and is related to the underlying logic of $\mathbf{T}$ and $\mathbf{T}^*$ (see [5]).

Our purpose, in this first part of the paper, is to study some methods which can be used to modify the first-order predicate calculus in order to ensure its α-completeness.

2 α-*Models* $\mathscr{C}_\alpha(\mathscr{C}_\alpha^=)$ is a first-order predicate calculus (with equality) having the following primitive symbols: (1) propositional connectives: $\vee$ (or) and $\neg$ (not); (2) the universal quantifier: $\forall$ (for all); (3) individual variables: a denumerable set of individual variables; (4) individual constants: a family $(\mathbf{c}_i)_{i\in\alpha}$ of (distinct) individual constants, where α is an ordinal greater than 0; (5) predicate symbols: a family $(R_j)_{j\in\beta}$ of (distinct) predicate symbols, where β is an ordinal greater than 0 (for every $j \in \beta$, R_j has a finite rank; in the case of $\mathscr{C}_\alpha^=$, one of the predicate symbols is the symbol of equality); (6) auxiliary symbols: parentheses and comma. The notions of formula, of free variable, of sentence or statement (formula without free variables), etc. are defined as usual. The connectives $\supset$ (implies), & (and), and $\equiv$ (equivalent), and the existential quantifier $\exists$ are introduced by the common definitions. The usual metalinguistic conventions and notations are employed without explicit mention.

Let Γ be a set of sentences of $\mathscr{C}_\alpha(\mathscr{C}_\alpha^=)$; an α-model of Γ is a model $\mathbf{M}$ of this set of sentences which satisfies the following condition: For every element m of $\mathbf{M}$ there is a constant $\mathbf{c}_i$, $i \in \alpha$, which denotes m ($\mathbf{c}_i$ is the "name" of m). If F is a sentence of $\mathscr{C}_\alpha(\mathscr{C}_\alpha^=)$, F is called a semantic α-consequence of Γ if it is true in all α-models of Γ. In this case, we write: $\Gamma \models_{\overline{\alpha}} F$; if $\Gamma = \emptyset$, we employ the notation $\models_{\overline{\alpha}} F$ to express that $\emptyset \models_{\overline{\alpha}} F$.

The postulates (axiom schemata and rules of deduction) of $\mathscr{C}_\alpha(\mathscr{C}_\alpha^=)$ are the ordinary ones, and the notions of deduction, of theorem, etc. are defined as in [5], with clear modifications.[2] If Γ is a set of formulas and F is a formula, we say that F is a syntactic consequence of Γ if $\Gamma \vdash F$ ($\vdash F$ is an abbreviation of $\emptyset \vdash F$). In the sequel, the Greek letters Σ and Δ will denote sets of statements and the Latin letters G and H will stand for sentences.

3 *The Calculi* $\mathscr{C}_\alpha^*$ *and* $\mathscr{C}_\alpha^{=*}$. We shall call rule (α) the following rule

$$(\alpha) \qquad \frac{A(\mathbf{c}_0), A(\mathbf{c}_1), A(\mathbf{c}_2), \ldots}{\forall x A(x)},$$

where $A(x)$ is a formula containing the free variable x, and $A(\mathbf{c}_i)$ is the

2. The principal modification is in the notion of deduction. The rule of generalization (or, more precisely, the rule we have employed instead of it) is subject to the following restriction: when the rule of generalization is applied to a formula of a deduction $\mathscr{D}$, it is supposed that there exists a subsequence $\mathscr{D}'$ of $\mathscr{D}$, where $\mathscr{D}'$ is a deduction having $\emptyset$ as its set of hypotheses, such that the last term of $\mathscr{D}'$ is the premise of the application of generalization. For our purposes, this restriction is better than other common ones, for instance that using the concept of variation of variables (*cf.* [12]).

formula resulting from $A(x)$ by substituting the constant $\mathbf{c}_i$ for each free occurrence of the variable x.[3]

If we add rule (α) to $\mathcal{C}_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}})$, we obtain $\mathcal{C}^*_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$. It is easy to define the concepts of deduction, of theorem, etc. for these calculi (*cf.* [5]); the new concepts are called respectively α-deduction, α-theorem, etc.[4] If Γ is a set of formulas, F is a formula and F is a syntactic α-consequence of Γ in $\mathcal{C}^*_\alpha$ or in $\mathcal{C}_{\bar{\bar{\alpha}}}^*$, we write $\Gamma \vdash_\alpha F$ ($\vdash_\alpha F$, if $\Gamma = \emptyset$). All the usual deductive rules of $\mathcal{C}_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}})$ are valid for $\mathcal{C}^*_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$, with obvious modifications; for example, we have, where θ is any set of formulas and E and F are any formulas whatsoever: if $\theta \cup \{E\} \vdash_\alpha F$, then $\theta \vdash_\alpha E \supset F$, and if $\theta \vdash_\alpha E$ and x is a variable which does not occur free in the formulas of θ, then $\theta \vdash_\alpha \forall x E$.

Theorem 1 *In $\mathcal{C}^*_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$, whenever $\Sigma \vdash_\alpha H$, we also have $\Sigma \models_\alpha H$.*

Theorem 2 *Suppose $\alpha \geqslant \omega$. Then: $\vdash_\alpha H$ in $\mathcal{C}^*_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}^*) \Longleftrightarrow \vdash H$ in $\mathcal{C}_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}})$.*[5]

Theorem 3 *Suppose that either $\bar{\bar{\alpha}} = \aleph_0$ and $\bar{\bar{\beta}} \leqslant \aleph_0$ or $0 < \alpha < \omega$; if $\Gamma \cup \{F\}$ is a set of sentences of $\mathcal{C}^*_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$ and $\Gamma \models_\alpha F$, then $\Gamma \vdash_\alpha F$ in $\mathcal{C}^*_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$, that is to say, $\mathcal{C}^*_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$ is α-complete.*

Proof: By a modification of the proof of Gödel's generalized theorem, also known as the Gödel-Malcev-Henkin theorem, in Cohen [2], pp. 13-16. The modification is the following: Let us assume that a consistent (in $\mathcal{C}^*_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$) set of sentences Γ is given. To begin with, we prove that it is possible to obtain a set of sentences $\overline{\Gamma}$, consistent in $\mathcal{C}_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}})$, such that $\Gamma \subset \overline{\Gamma}$ and, for each sentence of the form $\exists x A(x)$ of $\mathcal{C}^*_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$, $\overline{\Gamma}$ has an element of the form $\exists x A(x) \supset A(\mathbf{c}_A)$, where $\mathbf{c}_A$ is a convenient constant of the family $(\mathbf{c}_i)_{i \in \alpha}$. We consider the following cases in order to prove this fact:

(i) If $\alpha < \omega$, we may suppose, without loss of generality (using the axiom of choice if required), that the sentences of $\mathcal{C}^*_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$ of the form $\exists x A(x)$ are disposed in a sequence $\exists x A_0(x)$, $\exists x A_1(x)$, $\exists x A_2(x)$, . . . As Γ is consistent in $\mathcal{C}^*_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$, it is clear that corresponding to $A_0(x)$ there is a constant $\mathbf{c}_{A_0}$, of the family $(\mathbf{c}_i)_{i\in\alpha}$, such that $\exists x A_0(x) \supset A(\mathbf{c}_{A_0})$ is consistent with Γ. In fact, if this were not true, $\neg(\exists x A_0(x) \supset A(\mathbf{c}_i))$ would be a syntactic α-consequence of Γ for every $i \in \alpha$. Hence, $\Gamma \vdash_\alpha \forall x \neg(\exists x A_0(x) \supset A_0(x))$ and $\Gamma \vdash_\alpha \neg(\exists x A_0(x) \supset \exists x A_0(x))$; then, Γ would not be consistent in $\mathcal{C}^*_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$. Therefore, if we adjoin $\exists x A_0(x) \supset A_0(\mathbf{c}_{A_0})$ to Γ, the resulting set of sentences, Γ^0, is consistent. Similarly, there is a constant $\mathbf{c}_{A_1}$ such that $\exists x A_1(x) \supset A_1(\mathbf{c}_{A_1})$ is consistent with Γ^0; if this last sentence is adjoined to Γ^0, the resulting set Γ^1 is also consistent, etc. Having constructed Γ^n for

3. (α) is a generalization of the so-called Carnap's rule (*cf.* [1]).

4. The notion of deduction is subject to restrictions similar to those of footnote 2, and this remark applies to all calculi that we shall consider in the present paper.

5. In the sequel, the symbols $\Longrightarrow$ (implies) and $\Longleftrightarrow$ (equivalent) are employed as metalinguistic abbreviations.

every $n \in \omega$, the set $\Gamma^{\omega} = \bigcup_{n\in\omega} \Gamma^{n}$ is consistent in $\mathcal{C}^{*}_{\alpha}(\mathcal{C}^{*}_{\bar{\bar{\alpha}}})$, since α is finite and if any contradiction is derivable from Γ^{ω}, then it would be derivable from Γ^{n}, for a convenient $n \in \omega$, and this is absurd. Next, we proceed to construct the sets $\Gamma^{\omega+1}$, $\Gamma^{\omega+2}$, $\Gamma^{\omega+3}$, . . ., until all formulas of the sequence $\exists x A_0(x)$, $\exists x A_1(x)$, $\exists x A_2(x)$, . . . are considered. Therefore, if $\alpha < \omega$, given a consistent set of sentences Γ, there exists a set $\overline{\Gamma}$ of sentences which is consistent in $\mathcal{C}^{*}_{\alpha}(\mathcal{C}^{*}_{\bar{\bar{\alpha}}})$ (and a fortiori in $\mathcal{C}_{\alpha}(\mathcal{C}_{\bar{\bar{\alpha}}})$), contains Γ, and satisfies the following condition: For every sentence of $\mathcal{C}^{*}_{\alpha}(\mathcal{C}^{*}_{\bar{\bar{\alpha}}})$ having the form $\exists x A(x)$, $\overline{\Gamma}$ has an element of the type $\exists x A(x) \supset A(\mathsf{c}_A)$, where c_A is an appropriate constant of the family $(\mathsf{c}_i)_{i\in\alpha}$.

(ii) If $\bar{\bar{\alpha}} = \aleph_0$ and $\bar{\bar{\beta}} \leqslant \aleph_0$, the reasoning is analogous; nevertheless, there is a denumerable set of sentences of the form $\exists x A(x)$, whose elements belong to $\mathcal{C}^{*}_{\alpha}(\mathcal{C}^{*}_{\bar{\bar{\alpha}}})$, and the resulting set, $\overline{\Gamma}$, is consistent in $\mathcal{C}_{\alpha}(\mathcal{C}_{\bar{\bar{\alpha}}})$, but is not in general in $\mathcal{C}^{*}_{\alpha}(\mathcal{C}^{*}_{\bar{\bar{\alpha}}})$. (It deserves to be noted that if α is infinite, the reasoning is valid only if the set of formulas of the calculus is denumerable.)

Now, it is easy to verify that in Cohen's proof we can use only constants of the family $(\mathsf{c}_i)_{i\in\alpha}$ in order to construct a model for Γ in $\mathcal{C}_{\alpha}(\mathcal{C}_{\bar{\bar{\alpha}}})$; that is, no new constants are necessary. Indeed, the model so obtained is an α-model, and the theorem is proved.

Remark The above theorem is essentially the ω-completeness theorem for ω-logic (*cf.* [8] and [13]).

Assuming the axiom of choice, by Zermelo's theorem there are well orderings on the open interval $(0,1)$; let δ denote the least ordinal number of such orderings. Since there is a bijection between $(0,1)$ and the set of constants of $\mathcal{C}_{\bar{\bar{\delta}}}$, $\bigcup_{i\in\delta} \{\mathsf{c}_i\}$, this last set can be indexed in $(0,1)$, and represented as a family $(v_r)_{r\in(0,1)}$, where, for all $r_1, r_2 \in (0,1)$, $v_{r_1} \neq v_{r_2}$ if $r_1 \neq r_2$.

We describe now a theory F_δ having $\mathcal{C}_{\bar{\bar{\delta}}}$ as its underlying logic, and whose specific symbols are a single binary predicate symbol L and three unary predicate symbols D, T, and B. The set A_δ of non-logical axioms of F_δ is the following (aside from the postulates of $\mathcal{C}_{\bar{\bar{\delta}}}$):

$$\forall x \neg \mathsf{L}(x,x),$$
$$\forall x \forall y [\mathsf{L}(x,y) \vee x = y \vee \mathsf{L}(y,x)],$$
$$\forall x \forall y \forall z [\mathsf{L}(x,y) \wedge \mathsf{L}(y,z) \supset \mathsf{L}(x,z)],$$
$$\forall x \forall z [\mathsf{L}(x,z) \supset \exists y (\mathsf{L}(x,y) \;\&\; \mathsf{L}(y,z))],$$
$$\forall x \exists y \mathsf{L}(z,x) \;\&\; \forall y \exists z \mathsf{L}(y,z),$$
$$\exists x \mathsf{D}(x) \;\&\; \exists y \neg \mathsf{D}(y),$$
$$\forall x \forall y [\mathsf{D}(x) \;\&\; \neg \mathsf{D}(y) \supset \mathsf{L}(x,y)],$$
$$\forall x [\mathsf{T}(x) \equiv \mathsf{D}(x) \;\&\; \forall z (\mathsf{D}(z) \;\&\; z \neq x \supset \mathsf{L}(z,x))],$$
$$\forall y [\mathsf{B}(y) \equiv \neg \mathsf{D}(y) \;\&\; \forall z (\neg \mathsf{D}(z) \;\&\; z \neq y \supset \mathsf{L}(y,z))],$$
$$\mathsf{L}(v_r, v_s) \text{ for all } r < s;\ r, s \in (0,1).$$

Lemma (Henkin) (i) *The set of theorems of F_δ is closed under rule* (δ) (that is, Carnap's rule for the family of constants of F_δ); (ii) $A_\delta \models_{\bar{\bar{\delta}}} \exists x \mathsf{B}(x) \vee \exists x \mathsf{T}(x)$; (iii) $\exists x \mathsf{B}(x) \vee \exists x \mathsf{T}(x)$ *is not a formal theorem of F_δ.*

For the proof of this lemma, the reader may consult Henkin's paper [7], pp. 5-14.

Applying the preceding result, it is not difficult to see that the calculi $\mathcal{C}_\alpha^*(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$ are in general not α-complete. This is made precise by the next theorem.

Theorem 4 *If* $\alpha \geqslant 2^{\aleph_0}$ *and* $\mathcal{C}_\alpha^*(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$ *contains at least one unary and two binary predicate symbols, then* $\mathcal{C}_\alpha^*(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$ *is not* α*-complete.*

Proof: Consequence of the above lemma.

Definition 1 In $\mathcal{C}_\alpha^*$ and $\mathcal{C}_{\bar{\bar{\alpha}}}^*$, Σ is said to be (simply) complete if for every sentence H we have either $\Sigma \vdash_{\bar{\alpha}} H$ or $\Sigma \vdash_{\bar{\alpha}} \neg H$.

Theorem 5 *In* $\mathcal{C}_\alpha^*(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$, *if* Σ *is complete and* $\Sigma \models_{\bar{\alpha}} H$, *then* $\Sigma \vdash_{\bar{\alpha}} H$.

Definition 2 Σ is atomic if for every formula of the form $R_n(k_1, k_2, \ldots, k_p)$, where R_n is a predicate symbol of rank p and $k_1, k_2, \ldots, k_p$ are any constants, either $\Sigma \vdash_{\bar{\alpha}} R_n(k_1, k_2, \ldots, k_p)$ or $\Sigma \vdash_{\bar{\alpha}} \neg R_n(k_1, k_2, \ldots, k_p)$.

Theorem 6 *Let* Σ *be an atomic set of sentences of* $\mathcal{C}_\alpha^*(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$. *If* $\Sigma \models_{\bar{\alpha}} H$, *then* $\Sigma \vdash_{\bar{\alpha}} H$ *in* $\mathcal{C}_\alpha^*(\mathcal{C}_{\bar{\bar{\alpha}}}^*)$.

Remark Rule (α) is equivalent, in $\mathcal{C}_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}})$, to the following one:

$$(\alpha_1) \quad \frac{\neg A(\mathbf{c}_0),\ \neg A(\mathbf{c}_1),\ \neg A(\mathbf{c}_2),\ \ldots}{\neg \exists x A(x)},$$

where the meaning of the notations is clear. In case $\alpha < \omega$, (α) is also equivalent to the schemata $\forall x A(x) \equiv A(\mathbf{c}_0)\ \&\ A(\mathbf{c}_1)\ \&\ \ldots\ \&\ A(\mathbf{c}_{\alpha-1})$ and $\exists x A(x) \equiv A(\mathbf{c}_0) \vee A(\mathbf{c}_1) \vee \ldots \vee A(\mathbf{c}_{\alpha-1})$.

4 The Calculi $\mathcal{C}_{\alpha f}$ and $\mathcal{C}_{\bar{\bar{\alpha}} f}$ $\mathcal{C}_{\alpha f}(\mathcal{C}_{\bar{\bar{\alpha}} f})$ is the calculus having the same primitive symbols and the same postulates as $\mathcal{C}_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}})$, together with new axioms, as follows: There exists a function f which associates with each formula $A(x)$, containing x as the only free variable, a constant $f(A(x))$ (or, simply, $f(A)$), of the family $(\mathbf{c}_i)_{i\epsilon\alpha}$, such that $\exists x A(x) \supset A(f(A))$ is an axiom.

It is easy to define the notions of α_f-deduction, of α_f-theorem, of syntactic α_f-consequence of a set of formulas, etc. for $\mathcal{C}_{\alpha f}$ and $\mathcal{C}_{\bar{\bar{\alpha}} f}$. Hence, if $\Sigma \cup \{F\}$ is a set of formulas of $\mathcal{C}_{\alpha f}(\mathcal{C}_{\bar{\bar{\alpha}} f})$, the notations $\Gamma \vdash_{\bar{\alpha} f} F$ and $\vdash_{\bar{\alpha} f} F$ have clear meanings. The usual meta-theorems of $\mathcal{C}_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}})$ are valid for $\mathcal{C}_{\alpha f}(\mathcal{C}_{\bar{\bar{\alpha}} f})$; for instance, if θ is a set of formulas and E and F are formulas, we have: (1) $\theta \cup \{E\} \vdash_{\bar{\alpha} f} F \Rightarrow \theta \vdash_{\bar{\alpha} f} E \supset F$; (2) $\theta \vdash_{\bar{\alpha} f} E$ and $\theta \vdash_{\bar{\alpha} f} E \supset F \Rightarrow \theta \vdash_{\bar{\alpha} f} F$; (3) If the variable x does not occur free in the formulas of θ and $\theta \vdash_{\bar{\alpha} f} E$, then $\theta \vdash_{\bar{\alpha} f} \forall x E$.

Theorem 7 *For every formula* $A(x)$, *in which* x *is the only free variable,* $\vdash_{\bar{\alpha} f} A(f(\neg A)) \supset \forall x A(x)$, *where* $f(\neg A)$ *is the constant of the family* $(\mathbf{c}_i)_{i\epsilon\alpha}$ *associated with* $\neg A$.

Theorem 8 *In* $\mathcal{C}_{\alpha f}(\mathcal{C}_{\bar{\bar{\alpha}} f})$ *the following weak form of rule* (α) *is valid: If* $A(x)$ *is a formula in which* x *is the sole free variable, and* $A(\mathbf{c}_i)$ *is obtained from* $A(x)$ *by replacing every free occurrence of* x *by* $\mathbf{c}_i$, *then we have:* $\{A(\mathbf{c}_0)\} \cup \{A(\mathbf{c}_1)\} \cup \{A(\mathbf{c}_2)\} \cup \ldots \vdash_{\bar{\alpha} f} \forall x A(x)$.

Definition 3 In $\mathcal{C}_{\alpha f}(\mathcal{C}^{=}_{\alpha f})$, an α_f-model of a set of sentences Σ is an α-model of Σ which satisfies also the axioms $\exists x A(x) \supset A(f(A))$, for every formula containing x as its only free variable. If the sentence H is true in every α_f-model of Σ, we write $\Sigma \models_{\alpha f} H (\models_{\alpha f} H$, if $\Sigma = \emptyset)$.

Theorem 9 *In* $\mathcal{C}_{\alpha f}(\mathcal{C}^{=}_{\alpha f})$: $\Sigma \models_{\alpha f} H \Leftrightarrow \Sigma \vdash_{\alpha f} H$.

Theorem 10 *Let H be a syntactic consequence of Σ in* $\mathcal{C}^{*}_{\alpha}(\mathcal{C}^{=*}_{\alpha})$; *then,* $\Sigma \vdash_{\alpha f} H$ *in* $\mathcal{C}_{\alpha f}(\mathcal{C}^{=}_{\alpha f})$, *for every function f (which associates with each formula $A(x)$ the constant $f(A(x))$, under the conditions of the definition of* $\mathcal{C}_{\alpha f}(\mathcal{C}^{=}_{\alpha f})$).

Theorem 11 $\Sigma \models_{\alpha} H$ *if, and only if, for every f,* $\Sigma \models_{\alpha f} H$.[6]

Corollary $\Sigma \models_{\alpha} H$ *if, and only if,* $\Sigma \vdash_{\alpha f} H$ *for every f.*

5 Rule (α') The aim of this section is to present an α-complete version of the first-order predicate calculus, with or without equality. To attain our objective, we use and make precise a new kind of rule, called here rule (α'), introduced in [5], p. 8.

Definition 4 Let λ be an ordinal different from 0. A family $(\mathbf{k}_l)_{l \in \lambda}$, where $\mathbf{k}_l$, for every $l \in \lambda$, is a constant of the family $(\mathbf{c}_i)_{i \in \alpha}$, is called a λ-family of constants.

$\mathcal{C}'_{\alpha}(\mathcal{C}^{='}_{\alpha})$ is the calculus obtained from $\mathcal{C}^{*}_{\alpha}(\mathcal{C}^{=*}_{\alpha})$, by changing the notion of deduction as follows:

Definition 5 In $\mathcal{C}'_{\alpha}(\mathcal{C}^{='}_{\alpha})$, a formula F is said to be a syntactic α'-consequence of a set of formulas Γ if: (i) F is an axiom; or (ii) F is an element of Γ, or (iii) F is an immediate consequence of syntactic α'-consequences of Γ by one of the rules of $\mathcal{C}^{*}_{\alpha}$; or (iv) there exists a family of sentences $(\exists x A_l(x))_{l \in \lambda}$, where λ is an ordinal greater than 0, such that: (iv′) for every λ-family of constants $(\mathbf{k}_l)_{l \in \lambda}$, F is a syntactic α'-consequence of $\Gamma \cup \bigcup_{l \in \lambda} \{A_l(\mathbf{k}_l)\}$; and (iv″) for every $l \in \lambda$, $\exists x A_l(x)$ is a syntactic α'-consequence of Γ. If F is a syntactic α'-consequence of Γ, we denote this fact by $\Gamma \vdash_{\alpha'} F (\vdash_{\alpha'} F$ when $\Gamma = \emptyset)$.

Clause (iv) of the above definition, whose intuitive meaning is clear, is named rule (α') and represented in the following manner:

$$(\alpha') \quad \frac{\exists x A(x)}{A(\mathbf{c}_0) \vee A(\mathbf{c}_1) \vee A(\mathbf{c}_2) \vee \ldots} .$$

Thus, $\mathcal{C}'_{\alpha}(\mathcal{C}^{='}_{\alpha})$ is the calculus resulting (in a certain sense) from $\mathcal{C}^{*}_{\alpha}(\mathcal{C}^{=*}_{\alpha})$, when we adjoin to this last calculus rule (α').

6. The expression "for every f" is an abbreviation of "for every function which associates constants of the family $(\mathbf{c}_i)_{i \in \alpha}$ to formulas of $\mathcal{C}_{\alpha f}(\mathcal{C}^{=}_{\alpha f})$, according to the definition of this calculus".

Theorem 12 *If Γ and θ are sets of formulas and E and F are formulas of* $\mathcal{C}'_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}')$, we have:

(i) $\Gamma \vdash_{\alpha'} E$ *for any axiom E or for any* $E \in \Gamma$;

(ii) $\Gamma \cup \{E\} \vdash_{\alpha'} F \Rightarrow \Gamma \vdash_{\alpha'} E \supset F$;

(iii) $\Gamma \vdash_{\alpha'} E$ *and* $\Gamma \vdash_{\alpha'} E \supset F \Rightarrow \Gamma \vdash_{\alpha'} F$;

(iv) $\Gamma \vdash_{\alpha'} E \Rightarrow \Gamma \cup \theta \vdash_{\alpha'} E$;

(v) *If $\Gamma \vdash_{\alpha'} E$, and if x is a variable which does not occur free in any formula of Γ, then $\Gamma \vdash_{\alpha'} \forall x E$;*

(vi) *If $\Gamma \vdash_{\alpha'} E$ for all $E \in \theta$, and $\theta \vdash_{\alpha'} F$, then $\Gamma \vdash_{\alpha'} F$.*

Proof of part (ii): The proof of the deduction theorem for $\mathcal{C}'_\alpha$ and $\mathcal{C}_{\bar{\bar{\alpha}}}'$ is rather lengthy, though not difficult. An outline of it is as follows. In $\mathcal{C}'_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}')$, we may define the notion of deduction from a set of formulas (hypotheses) Γ as a (finite or infinite) sequence of formulas, Λ, each of which, F, is an axiom, or is an element of Γ, or is an immediate consequence of preceding formulas of Λ by one of the primitive rules of $\mathcal{C}^*_\alpha$, or is obtained from *immediate subsidiary deductions of* Λ of the forms $\Gamma \cup \bigcup_{l \in \lambda} \{A_l(\mathbf{k}_l)\} \vdash_{\alpha'} F$, for every λ-family of constants $(\mathbf{k}_l)_{l \in \lambda}$, and $\Gamma \vdash_{\alpha'} \exists x A_l(x)$, for every $l \in \lambda$. An immediate subsidiary deduction of Λ is said to be a subsidiary deduction of Λ of grade 1; an immediate subsidiary deduction of grade 1 is said to be a subsidiary deduction of Λ of grade 2; etc. It is easy to define rigorously the notions of subsidiary deduction of a given deduction and of grade of a subsidiary deduction. The grade of a subsidiary deduction Λ is always finite. Taking all this into account, it is possible to prove that all subsidiary deductions of Λ which are of the form $\Gamma \cup \bigcup_{l \in \lambda} \{A_l(\mathbf{k}_l)\} \vdash_{\alpha'} F$ can be changed to deductions of the form $\Gamma_1 \cup \{A\} \cup \bigcup_{l \in \lambda} \{A \supset A_l(\mathbf{k}_l)\} \vdash_{\alpha'} F$, where A is any formula of Γ and $\Gamma_1 = \Gamma - \{A\}$; next, by (finite or transfinite) induction, we complete the proof of the deduction theorem, by noting that it is true for deductions having no subsidiary deductions.

Theorem 13 *If Σ is consistent in $\mathcal{C}'_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}')$, then Σ has an α-model.*

Proof: The following is a consequence of rule (α'): Let Γ be a set of formulas, λ as ordinal number greater than 0, and $(A_l(x))_{l \in \lambda}$ a family of formulas, such that, for every $l \in \lambda$, $A_l(x)$ has only one free variable x; if $\Gamma \cup \bigcup_{l \in \lambda} \{A_l(\mathbf{k}_l)\} \vdash_{\alpha'} C \,\&\, \neg C$ for every λ-family of constants $(\mathbf{k}_l)_{l \in \lambda}$, then $\Gamma \cup \bigcup_{l \in \lambda} \{\exists x A_l(x)\} \vdash_{\alpha'} C \,\&\, \neg C$.

Hence, the already cited proof of Cohen's book [2] can be adapted to the present case: in the construction of the final model, we can employ the constants of the family $(\mathbf{c}_i)_{i \in \alpha}$ instead of using new constants. As the model so obtained is really an α-model, the proposition is proved.

Theorem 14 *In* $\mathcal{C}'_\alpha(\mathcal{C}_{\bar{\bar{\alpha}}}')$: $\Sigma \models_\alpha H \Leftrightarrow \Sigma \vdash_{\alpha'} H$.

Theorem 15 *$\Sigma \vdash_{\bar{\alpha}'} H$ in $\mathcal{C}'_\alpha(\mathcal{C}^{=\prime}_{\bar{\alpha}})$ if, and only if, for every f, $\Sigma \vdash_{\bar{\alpha}_f} H$ in $\mathcal{C}_{\alpha_f}(\mathcal{C}^{=}_{\bar{\alpha}_f})$.*

Theorem 16 *If $\alpha \geqslant \omega$ and $\vdash_{\bar{\alpha}'} F$ in $\mathcal{C}'_\alpha(\mathcal{C}^{=\prime}_{\bar{\alpha}})$, then $\vdash F$ in $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$.*

6 α-Saturation In this section we list, without proof, some consequences of the foregoing exposition.

Definition 5 A basic sequence of formulas of $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$ is a sequence $\exists x A_0(x) \supset A_0(\mathbf{k}_0)$, $\exists x A_1(x) \supset A_1(\mathbf{k}_1)$, . . ., where $A_0(x), A_1(x)$, . . . are all (distinct) formulas of $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$ in which x is the only free variable, and $\mathbf{k}_0, \mathbf{k}_1$, . . . are any not necessarily distinct constants of the family $(\mathbf{c}_i)_{i \epsilon \alpha}$.

Theorem 17 *The two propositions* (a) *and* (b) *below are equivalent:*

(a) $\Delta \vdash_{\bar{\alpha}'} H$ *in* $\mathcal{C}'_\alpha(\mathcal{C}^{=\prime}_{\bar{\alpha}})$;
(b) $\Delta \cup \{\exists x A_0(x) \supset A_0(\mathbf{k}_0), \exists x A_1(x) \supset A_1(\mathbf{k}_1), \ldots\} \vdash H$ *in* $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$, *for every basic sequence of formulas of* $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$.

Definition 6 Let Δ be a theory having $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$ as its underlying logic, that is, a set of sentences of $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$ such that if $\Delta \vdash E$, then $E \epsilon \Delta$. Δ is said to be α-saturated if, for every sentence H, $\Delta \models_{\bar{\alpha}} H$ implies $\Delta \vdash H$.

Theorem 18 *A theory Δ having $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$ as its underlying logic is α-saturated if, and only if, we have: for every sentence H, $\Delta \vdash H$ is equivalent to $\Delta \vdash_{\bar{\alpha}'} H$.*

Definition 7 In $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$, a theory Δ is called a Henkin theory if, for every formula $A(x)$ in which x is the only free variable, $\Delta \vdash \exists x A(x) \supset A(\mathbf{c}_A)$, where $\mathbf{c}_A$ is a constant of the family $(\mathbf{c}_i)_{i \epsilon \alpha}$.

Theorem 19 *Every Henkin theory in $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$ is α-saturated.*

Theorem 20 *A theory having $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$ as its underlying logic is α-saturated if, and only if, it is the intersection of all Henkin theories (in $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$) containing it.*

Definition 8 In $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$, a theory Δ is said to be α-complete if, for every formula $A(x)$, in which x is the sole free variable, $A(\mathbf{c}_0) \epsilon \Delta$, $A(\mathbf{c}_1) \epsilon \Delta$, . . . imply $\forall x A(x) \epsilon \Delta$.

Theorem 21 *If the set of primitive symbols of $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$ is denumerable or if $\alpha < \omega$, then a theory Δ whose underlying logic is $\mathcal{C}_\alpha(\mathcal{C}^{=}_{\bar{\alpha}})$, is α-complete if, and only if, Δ is α-saturated.*

7 Generalizations The above results can be generalized to cover cases in which we add operation symbols to $\mathcal{C}_\alpha$ and to $\mathcal{C}^{=}_{\bar{\alpha}}$, and in which the constants are introduced by definition (there are terms different from individual constants serving to "name" the elements of the model). Moreover, they can be suitably adapted to apply to the many-sorted first-order predicate calculus, with or without equality; as a consequence, the underlying logic of **T** and **T*** is complete in a precise sense (we have shown this in [5]). Clearly, other extensions are possible. For instance, it is not difficult to define the notion of α-model for sets of sentences of the predicate calculus of order ω and to extend to this calculus the most important of the previous results (*cf.* [7] and [13]).

II. The Systems T and T*

1 Introduction In this second part of the paper it is shown how the systems T and T* can be used as bases for category theory. Since the formal structures of T and T* are very similar, it is sufficient to consider only the case of T.

For the sake of clarity, we briefly describe T, whose underlying logic is a two-sorted predicate calculus with equality, conveniently strengthened (see [5] for details). T is a combination of type theory with Kelley-Morse system of set theory ([9], appendix).

Symbols of T: (1) logical symbols: $\supset$, &, $\vee$, $\neg$, $\equiv$, $\forall$, $\exists$, $=$, and variables; (2) auxiliary symbols: (and); (3) specific symbols: ϵ, $\{ : \}_n$, $n \geqslant 2$, (classifiers) and the individual constants V_1, V_2, V_3, . . .

n and p will denote integral indices greater than 0. The symbols of T with analogous definitions in the Kelley-Morse theory will be employed without any explanation.

Specific Postulates of T:

Postulate of extent:

(P1) $\forall z(z \epsilon x \equiv z \epsilon y) \supset x = y$.

Structural postulates:

(P2) $x \epsilon V_n \supset x \subset V_n$,
(P3) $x \subset V_n \supset x \epsilon V_{n+1}$.

Definition 1

x is a class of type $p =_{def} x \epsilon V_p$,
x is a set $=_{def} x \epsilon V_1$,
x is a class of order strictly $p =_{def} x \epsilon V_p$ and $x \notin V_n$, $n < p$.

Postulate of classification:

(P4) *If $F(x)$ is a formula and x and y are variables satisfying certain conditions* (*cf.* [4]), *then:*

$$y \epsilon \{x : F(x)\}_n \equiv y \epsilon V_{n-1} \;\&\; F(y).$$

Definition 2

$x \cup_n y =_{def} \{z : z \epsilon x \vee z \epsilon y\}_n$,
$x \cap_n y =_{def} \{z : z \epsilon x \;\&\; z \epsilon y\}_n$,
$0 =_{def} \{x : x \notin x\}_2$.

Postulate of subclasses:

(P5) $x \epsilon V_n \supset \exists y(y \epsilon V_n \;\&\; \forall z(z \subset x \supset z \epsilon y))$.

Definition 3 $x_n =_{def} \{z : x \epsilon V_{n-1} \supset x = z\}_n$.

Postulate of union:

(P6) $x \epsilon V_{n-1} \;\&\; y \epsilon V_{n-1} \supset x \cup_n y \epsilon V_{n-1}$.

Taking into account the preceding definitions, it is easy to see how we have to define the concepts of $\sim_n x$, $\bigcup_n x$, n-relation, n-function, domain$_n f$, range $_n f$, etc.

Postulate of substitution:

(P7) *f is a n-function & domain$_n f \in V_{n-1} \supset$ range$_n f \in V_{n-1}$.*

Postulate of amalgamation:

(P8) $x \in V_{n-1} \supset \bigcup_n x \in V_{n-1}$.

Postulate of regularity:

(P9) $x \neq 0$ & $x \in V_n \supset \exists y(y \in x$ & $x \cap_n y = 0)$.

Postulate of characterization:

(P10) *For each x there exists V_n such that $x \in V_n$.*

Postulate of infinity:

(P11) $\exists y(y \in V_1$ & $0 \in y$ & $\forall x(x \in y \supset x \cup_2 \{x\}_2 \in y))$.

Definition 4 $x \sim_n y =_{def} x \cap_n (\sim_n y)$.

Postulate of choice:

(P12) *There is a n-choice function f such that domain$_n f = V_{n-1} \sim_n \{0\}_n$.*

To say that we are *working in* certain V_n will mean that all classes we are talking about are subclasses of V_n and that all terms we are considering, distinct from variables, have the form $\{x : F(x)\}_{n+1}$. Hence, whenever we are working in V_n, a statement like "Let f be a function such that . . ." is to signify "Let f be a $(n + 1)$-function such that . . .", etc.

2 Universes and Categories in T In this section it is supposed that we are working in V_n.

Definition 5 F is a formula of Kelley-Morse set theory (**KM**) and t is a term whose free variables have to satisfy clear conditions. The t-transform of F, F^t, is the formula obtained from F by replacing each quantification $\forall y(. . .)$ by $\forall y(y \subset t \supset . . .)$, each quantification $\exists y(. . .)$ by $\exists y(y \subset t$ & $. . .)$, and each term $\{y : g(y)\}$ by $\{y : (g(y))^t\}_{n+1}$.

Definition 6 A class x is called transitive if $\forall y(y \in x \supset y \subset x)$.

Definition 7 A class $x \in V_n$ is said to be a model of **KM** if the x-transforms of the closures of all axioms of **KM** hold in **T**.

Theorem 1 *If $p < n$, V_p is a model of* **KM**.

Proof: Immediate consequence of the manner in which the axioms of **T** were chosen.

Definition 8 A class x is said to be extensional if $\forall u \forall v[u, v \in x \supset (\forall z(z \in x \supset (z \in u \equiv z \in v)) \supset u = v)]$.

Theorem 2 *If $\mathcal{P}(x)$ is extensional, then x is transitive.*

Theorem 3 *Every model of* **KM** *is transitive.*

Proof: By the axiom of extent, any model of **KM** is such that $\mathcal{P}(x)$ is extensional. Hence, x is transitive.

Theorem 4 *No V_n is a set; but V_n is a class of order strictly $n + 1$.*

Definition 9 Let x be an element of V_n. x is called a universe of Sonner-Grothendieck if it satisfies the following conditions:

(U1) $y \in x \supset y \subset x$,
(U2) $y \in x \supset \mathcal{P}(y) \in x$,
(U3) $y \in x \;\&\; f \in x^y \supset \bigcup \text{range } f \in x$.

Definition 10 A universe t is said to be normal if $\exists y(y \in x \;\&\; y$ is infinite).

Theorem 5 *Suppose that $x \in \mathsf{V}_n$; x is a normal universe if, and only if, x is a model of* **KM**.

Theorem 6 *If $p < n$, $\sup\{y : y$ is a cardinal number $\&\; \exists x(x \in \mathsf{V}_p \;\&\; y = \bar{\bar{x}})\}_{n+1}$ is a strongly inaccessible cardinal.*

Theorem 7 *If $p < n$, V_p is a normal universe.*

Theorem 8 *If* **T** *is consistent, then* **KM** *is also consistent.*

In order to have an idea of the proofs of Theorems 5 and 6, the reader may consult, for instance, Kruse's paper [11].

Theorem 7 shows that it is possible to solve in **T** the foundational problems of category theory in the ordinary way, via the systematic use of universes (*cf.* [14]). Though such problems can apparently be solved in a set theory with only one normal universe, a system of set theory having a hierarchy of universes seems to be logically more satisfactory. From this point of view, **T**, being one of the most interesting forms of the theory of types, is a very intuitive system; and to base extant mathematics (including category theory) in **T** is indeed a very natural matter (this is really what specialists in categories implicitly do).[7]

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7. **T** and **T*** are intimately related to a system of set theory proposed by Klaua (see, for example, [10]). We intend to study the relations holding between Klaua's system and a strong version of **T**, in which the set of constants $\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3, \ldots$ is not denumerable and rule (α') is used in the place of (α).

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