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ON THE CALCULUS MCC

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Charles Parsons, in [2], has very nicely tidied up a topic on which I had some correspondence with the late E. J. Lemmon early in 1957, and it may now be of some interest to bring this material into the light of day.

The calculus which Parsons calls MC^+, i.e. Johannson's minimal calculus + $ApNp$, is equivalent to the calculus which H. B. Curry in [1] calls **LD**. When I first learnt of this calculus it occurred to me that one might obtain another calculus between the minimal and the classical by leaving negation as in minimal but making the implicational basis fully classical. This gives the calculus which Parsons calls **MCC**. All that I observed about it was that it was weaker than the full calculus but stronger than Curry's, and I proved it not to be contained in Curry's by the matrix

C	1	2	3	N
*1	1	2	3	2
2	1	1	3	1
3	1	1	1	1

A	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

K	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

(N = (1,1,1) would also do). But Lemmon noticed a good deal more, and I cannot do better than quote the relevant portion of a letter from him of January 18, 1957:

"Let **PL** be positive logic (in C, K, A); **MC** = minimal calculus (C, K, A, 0 or C, K, A, N); **TPL** = classical positive logic (in C, K, A with $CCCpqpp$); **C** = Curry's system (**MC** in C, K, A, 0 or C, K, A, N + $CCNppp$ or $ApNp$); **TMC** = the system you describe (**MC** + $CCCpqpp$, or **TPL** + 0); **IPC** = the full intuitionistic propositional calculus; **PC** = full classical propositional calculus. Then, as you say:

$$\begin{tikzcd}
 & \mathsf{TMC} \arrow[rr] \arrow[dr] & & \mathsf{TPL} \arrow[dr] & \\
\mathsf{PC} \arrow[ur] \arrow[dr] & & \mathsf{C} \arrow[dr] & & \mathsf{PL} \\
 & \mathsf{IPC} \arrow[rr] & & \mathsf{MC} \arrow[ur] &
\end{tikzcd}$$

(where relationships are not given, independence obtains). Then these systems are described by the following equations, using C, K, A and 0

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(where needed) as primitive, and defining $N\alpha$ as $C\alpha 0$. (i) **MC** = **PL** + 0. (ii) **IPC** = **PL** + 0 + $C0p$. These are standard equations. (iii) **C** = **PL** + 0 + $ApCp0$ ($ApCp0 \rightarrow ApNp$). (iv) **TMC** (your system) = **PL** + 0 + $ApCpq$ = **TPL** + 0. By these equations, a cardinal difference between your system and Curry's is that yours has $ApCpq$, from which, by **PL**, $CCCpqpp$ follows (in **PL** $\vdash CApqCCqpp$), whilst Curry's has the weaker $ApCp0$. In Curry, we prove, from $ApNp$ and $CNpCpNq$ (this last in **MC**) $ApCpNq$, and so $CCCpNqpp$; but, as your matrix-test shows, we can't get in Curry $CCCpqpp$. On the basis of the corresponding results for **MC**, we can show that **C** has no finite characteristic matrix and that the addition of $ApNp$ does not strengthen the pure C-content, or even the pure C-K-A content. On the other hand, your system appears to have the following finite characteristic matrix:

C	1	2	3	4	K	1	2	3	4	A	1	2	3	4	N	*or*	N
*1	1	2	3	4	1	1	2	3	4	1	1	1	1	1	2		3
2	1	1	3	3	2	2	2	4	4	2	1	2	1	2	1		3
3	1	2	1	2	3	3	4	3	4	3	1	1	3	3	2		1
4	1	1	1	1	4	4	4	4	4	4	1	2	3	4	1		1

C, K, A tables are the products of the two-valued ones by themselves. N-tables are the product of (0,1) and (1,1) in either way possible. One N comes from the other by interchange of 2 and 3 throughout and consequent reordering. Since your system is **TPL** + 0, this matrix must characterise the system for the matrix satisfies all and only those tautologies in C, K, A, N which remain tautologies whether the N is taken as usual negation or taken as V, i.e. whether 0 is used as 0 proper or as 1 for truth-table tests. And it is just those tautologies which are in **TMC**."

Lemmon's **TMC, C, MC, TPL** and **PL** are respectively Parsons' **MCC, MC$^+$, MC, PCC** and **PC,** and his 4-valued matrix embodies the combination of Parsons' "pseudo-validity" with ordinary validity.

NOTE: H. B. Curry in *Foundations of Mathematical Logic* (1963), p. 260, mentions an unpublished paper by Kripke, of 1958, in which a calculus equivalent to Parsons' **MCC** is studied under the name of **LE**. This calculus is called **HE** by J. Jay Zeman in "Some calculi with strong negation primitive," *The Journal of Symbolic Logic*, vol. 33, no. 1 (March 1968), p. 98.

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[1] Curry, H. B., "The System **LD**," *The Journal of Symbolic Logic*, vol. 17 (1952), pp. 98-104.

[2] Parsons, Charles, "A propositional calculus intermediate between the minimal calculus and the classical," *Notre Dame Journal of Formal Logic*, vol. 7 (1966), pp. 353-58.

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