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A SYSTEM OF MODALITY

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1. *Introduction* In this paper I develop a system of modality designed to answer to one intuition one might have about possibility and necessity which has so far been overlooked by other, common systems of modal logic.

Before expressing this intuition as a general principle, perhaps I can elicit a sense for it by two examples. Given, what we all know, that the earth revolves about the sun, suppose one is asked what would be the case if the earth did not so revolve. Out of context this is a strange question; after a moment's hesitation, however, one might be inclined to reply that if the earth did not revolve about the sun, then *anything* is possible. Again: given that Caesar crossed the Rubicon, if one is asked what would have happened if he had not, one might be inclined to respond that in that case anything could have happened.[1]

One might be inclined to make this kind of reply because one adheres to something like the following general principle:

If a proposition, p, is true, then if p were not the case, then anything, q, would be possible.

Or, to express it in more convenient symbols:

(P) $$p \rightarrow . \sim p \rightarrow Mq\ ,$$

where, as usual, the arrow represents an appropriate implication, '$\sim$' represents negation, and 'M' possibility. I am not now concerned with the philosophical credentials of this principle. Rather I am interested in what results when (P) is combined with other, obvious assumptions about modality and implication.

2. *Implication* Before we can evaluate the principle (P) from a formal point of view, we must interpret the implication ('if . . . then ___') operative

1. The thrust of these replies is not just that anything regarding astronomical phenomena or regarding the later history of Rome is possible, but that anything *whatsoever* is possible. I recognize that, on first reading, these answers will strike many as counter-intuitive; others may be more receptive to them. In Section 6 I will suggest one sort of support which might be offered in their defense.

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in it. If we take the arrow to represent material implication, then (P) is a trivial instance of the tautology $A \rightarrow . \sim A \rightarrow B$. But this is not the intuition, that if A is true, then not-A implies that anything, B, would happen, but that anything *could* happen.

Suppose then we interpret the connective in (P) as strict implication, in the sense of one of the normal Lewis modal systems M, S4, or S5. (P) is provable in none of these systems; in particular, it is not a consequence of the paradoxes of strict implication, e.g. $NA \rightarrow . B \rightarrow A$, which it might be thought to resemble. This can be verified by means of Lewis' matrix group III which satisfies all of these systems;[2] it also follows from the result below.

The addition of (P) to any of these three systems has the unacceptable consequence of reducing all proper modalities to non-modal assertions—i.e. both $NA \rightarrow A$ (or $A \rightarrow MA$) and $A \rightarrow NA$ ($MA \rightarrow A$) are provable in the extended systems—and hence of reducing these systems to classical logic. $NA \rightarrow A$ is, of course, provable in any alethic modal logic. That $A \rightarrow NA$ is provable can be demonstrated as follows:

(P) is equivalent to[3]

(1) $A \rightarrow . NB \rightarrow A$

which, in these systems, is equivalent to

(2) $A \rightarrow . N(NB \supset A)$.

(3) $A \rightarrow . NNB \supset NA$

follows from (2) together with the appropriate instance of the theorem (schema) $N(A \supset B) \rightarrow . NA \supset NB$. From (3)

(4) $NNB \rightarrow . A \supset NA$

is derivable by permutation. Let B be any proposition such that its double necessity is provable; e.g. let B be $p \rightarrow p$. It follows that

(5) $A \supset NA$

is provable, and hence so is

(6) $A \rightarrow NA$

by the rule that if A is provable then so is NA, for any A.[4]

The proof above applies whenever the implications in (P) carry the force of necessity, i.e. whenever $A \rightarrow B$ says that B follows *necessarily* from A. It does not depend on the paradoxes of strict implication. Thus,

2. Lewis and Langford [11], p. 493. When A takes the value 3 and B takes the value 4, $A \rightarrow . \sim A \rightarrow MB$ takes the undesignated value 4.
3. By contraposition and the equivalence of NB with $\sim M \sim B$. For convenience I will hereafter work with (P) in the form of (1).
4. This proof will not, of course, go through in systems, such as S3, in which no proposition of the form NNB is provable. However, since these systems have little else to recommend them, I will not consider them further.

the proof goes through in the system E of entailment developed by Anderson and Belnap (see, e.g. [2], [3], or [4]), replacing the horseshoes with arrows throughout and letting necessity be defined so that NA is equivalent to $A \rightarrow A \rightarrow A$. As a result, E too is unsuited to provide a background theory of implication for (P).

It thus appears that if the principle (P) is to have any interest or plausibility at all, what is required is a theory which (i) is free of the paradoxes of implication, and (ii) formalizes a non-apodictic implication. These conditions are met by the system R of relevant implication, also developed by Anderson and Belnap (e.g. in [2] and [3]). R, like E, is free of the paradoxes but, unlike E, it makes no modal distinctions. One might say that relevant implication stands to entailment as material implication stands to strict implication.

Much of the main motivation behind the development of R is in the thought that A implies B only if A is *relevant* to B, in the sense that they share some element of meaning (however that be explicated). Failure to meet this condition is what makes $B \rightarrow .A \rightarrow A$ a paradox. In the same regard, (P) should be paradoxical, for it would permit true, even provable, propositions of the form $NB \rightarrow A$ where B and A have nothing to do with each other. Whatever dissatisfaction this produces may, however, be assuaged somewhat if one takes necessitative (possibilitive) propositions—propositions which are provably equivalent to propositions of the form NA (MA)—to be relevant to *any* proposition. Be that as it may, it turns out that whatever irrelevance is introduced with the addition of (P) to R is kept under some control; it is always the product of the application of necessity and does not infect the underlying pure theory of implication.

The system formed by the addition of (P), and other, obvious assumptions concerning modality, to R I shall call the system G.

3. *Formulation of* G The system G is formulated in a language possessing the usual equipment and grammar for a propositional calculus containing modal operators. It is defined by the following axiom schemata and rules:[5]

A1 $A \rightarrow A$
A2 $A \rightarrow B \rightarrow .B \rightarrow C \rightarrow .A \rightarrow C$
A3 $(A \rightarrow .B \rightarrow C) \rightarrow .B \rightarrow .A \rightarrow C$
A4 $(A \rightarrow .A \rightarrow B) \rightarrow .A \rightarrow B$
R1 *Modus ponens*: *from A and A $\rightarrow$ B, to infer B*

A5 $\sim\sim A \rightarrow A$
A6 $A \rightarrow \sim\sim A$
A7 $A \rightarrow B \rightarrow .\sim B \rightarrow \sim A$
A8 $A \rightarrow \sim A . \rightarrow \sim A$

A9 $A \,\&\, B \rightarrow A$
A10 $A \,\&\, B \rightarrow B$

5. 'A', 'B', 'C', etc. are used as variables for well formed formulas in the language of G. Church's conventions for the elimination of parentheses are used throughout. (Church [7], pp. 74-75.)

A11 $(A \to B) \& (A \to C) \to .A \to (B \& C)$
R2 *Adjunction: from A and B, to infer A & B*

A12 $A \to A \vee B$
A13 $B \to A \vee B$
A14 $(A \to B) \& (C \to B) \to .(A \vee C) \to B$
A15 $A \& (B \vee C) \to (A \& B) \vee C$

A16 $NA \to A$
A17 $N(A \to B) \to .NA \to NB$
A18 $NA \& NB \to N(A \& B)$
R3 *Necessitation: from A, to infer NA*

A19 $A \to .NB \to A$ A20 $NA \to NNA$

A1 - A15, with R1 and R2, define the system R of relevant implication. I regard A16 - A18, with R3, to be a *sine qua non* of any reasonable theory of necessity. A19 is, of course, the principle (P) (with possibility defined in the usual way in terms of necessity and negation). A20 is an optional axiom; it effects S4-like reductions among modalities, whereas the system without A20 has an *M*-like theory of modality. In what follows I shall consider the two systems, obtained by including A20 or not, together referring to them indifferently as G, except where mentioned.

The addition of the further postulate

A21 $\sim NA \to N \sim NA$

to produce S5-like reductions among modalities has the unfortunate effect of reducing relevant implication to material implication: $A \to .B \to A$ is provable; thus

1. $A \to .N \sim N \sim B \to A$	A19
2. $\sim N \sim B \to N \sim N \sim B$	A21
3. $A \to .\sim N \sim B \to A$	1,2 transitivity
4. $B \to \sim N \sim B$	A16, etc.
5. $A \to .B \to A$	3,4 transitivity.

Since the system with A21 thus collapses to S5 itself, I shall not consider it further.

Since what is of primary interest is the interplay between (P) and the other assumptions about implication and modality, in what follows I will consider only the implication-negation-necessity fragment of G, defined by A1 - A8, A16 - A19 (A20) with R1 and R3.[6] I shall call this fragment $G_{\bar{I}N}$. No new difficulties are expected when the results given below for $G_{\bar{I}N}$ are extended to the full system G with conuunction and disjunction, except for the results of section 5.

Since $G_{\bar{I}N}$ is an extension of $R_{\bar{I}}$ (the implication-negation fragment of R), the deduction theorem for that system applies to $G_{\bar{I}N}$ as well.

6. Since R3 is not an elementary rule, it is often desirable (e.g., for the deduction theorem) to eliminate it in favor of a principle for defining axioms such that if *A* is an axiom, then so is *NA*. (Cf. Anderson [1].)

T1. *If there exists a proof of* B *from* $A_1, \ldots, A_n$ *in which all of* $A_1, \ldots, A_n$ *are used in arriving at* B *and* R1 *is the only rule applied, then there exists a proof of* $A_n \rightarrow B$ *from* $A_1, \ldots, A_{n-1}$ *satisfying the same conditions.*

(From [3], p. 36, where an explication of 'use' is also given.) Furthermore, with the principles for necessity we have

T2. *If there exists a proof of* B *from* $A_1, \ldots, A_n$ *in which all of* $A_1, \ldots, A_n$ *are used in arriving at* B *and* R1 *is the only rule applied, then there exists a proof of* $N(A_n \rightarrow B)$ *from* $NA_1, \ldots, NA_{n-1}$ *satisfying the same conditions.*

And, given A20, T2 may be strengthened to

T2′ *If there is a proof of* B *from* $NA_1, \ldots, NA_{n-1}, A_n$, *satisfying the same conditions as in* T2, *then there is a proof of* $N(A_n \rightarrow B)$ *from* $NA_1, \ldots, NA_{n-1}$ *satisfying the same conditions.*

A Replacement Theorem of the customary sort also holds for $G_{\bar{T}N}$ which can be proved in the usual way:

T3. *If* $\vdash A \rightarrow B$ *and* $\vdash B \rightarrow A$, *then* $\vdash C \rightarrow C'$ *and* $\vdash C' \rightarrow C$, *where* C' *is the result of replacing one or more occurrences of* A *in* C *by* B.

The consistency of $G_{\bar{T}N}$ is established by the following matrices (adapted from [4], p. 15):

$\rightarrow$	1	2	3	4	5	6	7	8	$\sim$	N
1	8	8	8	8	8	8	8	8	8	1
2	1	7	1	7	1	1	7	8	7	1
3	1	1	6	6	1	6	1	8	6	1
4	1	1	1	5	1	1	1	8	5	1
*5	1	2	3	4	5	6	7	8	4	5
*6	1	1	3	3	1	6	1	8	3	5
*7	1	2	1	2	1	1	7	8	2	5
*8	1	1	1	1	1	1	1	8	1	5

which have starred values designated. It can be easily, if tediously, verified that all the axioms of $G_{\bar{T}N}$ are satisfied by this group and that the rules preserve this property.

In contrast to the (extended) systems briefly considered in section 2, $G_{\bar{T}N}$ is free from the paradoxes of material and strict implication, and it preserves modal distinctions. That is to say, none of $A \rightarrow .B \rightarrow A$, $NA \rightarrow .B \rightarrow A$, $A \rightarrow NA$, etc. are provable in $G_{\bar{T}N}$. This may be checked against the matrices above. (When A takes the value 6 and B takes the value 2, then each of these formulas assumes the undesignated value 1). This fact follows from these two more general theorems:

T4. *If* A *contains no* N, *then for no* B *is* $A \rightarrow NA$ *provable.*

Assign the value 6 to all the variables in A; since A contains no N, A assumes either the value 3 or the value 6. NB assumes the value 1 or 5 for any assignment to its values. But $3 \rightarrow 1 = 3 \rightarrow 5 = 6 \rightarrow 1 = 6 \rightarrow 5 = 1$ which is undesignated; hence $A \rightarrow NB$ is unprovable.

T5. *If A and B contain no N then $A \to B$ is provable only if A and B share a variable.*

Suppose A and B do not share a variable; assign all the variables in A the value 6 and all the variables in B the value 7, then A assumes either 3 or 6 as values while B takes either 2 or 7. But $3 \to 2 = 3 \to 7 = 6 \to 2 = 6 \to 7 = 1$. Hence $A \to B$ is unprovable. (Cf. [4], p. 16.)

T5 is also a consequence of the fact that G_{TN} is a conservative extension of R_{T} (proved in section 5), and the fact that the theorem holds for that system.

One might be tempted to think that stronger results would hold for G_{TN}, that no non-necessitative proposition implies a necessitative, and that if A is non-necessitative then $A \to B$ is provable only if A and B share a variable. Both of these claims are shown to be false, however, by the fact that $C \to C \to NA \to NB$ is provable whenever NB is provable, even though A, C and B may share no variables.

1. $\vdash NB$	hypothesis
2. $\vdash NA \to NB$	1, A19, R1
3. $\vdash C \to C \to NA \to NA$	theorem of R
4. $\vdash C \to C \to NA \to NB$	2,3 transitivity

$C \to C \to NA$ is not, in general, a necessitative proposition; NB, of course, is. Nevertheless, T4 and T5 do demonstrate that whatever fallacies of modality and fallacies of relevance (see [3], pp. 42 ff.) occur in G_{TN} must be the result of the operation of necessity.

4. *A natural deduction system* One feature which makes a formal system interesting and which gives it an air of naturalness is the ability to present it in a variety of different forms. In the preceding section I considered G_{TN} as an axiomatic system; in this section I formulate it as a Fitch-style natural deduction system with subordinate proofs. In the next I will present it in the form of a Gentzen consecution calculus.[7]

We adopt Fitch's notions of subordinate proof (subproof) and strict subordinate proof (see [9]), and deploy the method, due to Anderson and Belnap ([3]), of indexing hypotheses to keep track of what is relevant to what. The system $\mathsf{G}_{\mathsf{TN}}^*$ is then defined by the following rules:

(Hyp) Any wff, A, may be introduced as the hypothesis of a new subproof (or strict subproof); each new hypothesis receives a unit class $\{k\}$ of numerical subscripts, *except* that where A is of the form NB, it receives the subscript $\{\wedge\}$, where $\wedge$ is understood to be a (tacit) member of every class of subscripts.[8]

7. Strictly speaking these should be considered three different systems, which can all be proved equivalent in an appropriate sense.

8. I.e. inferences from $\left|\begin{array}{l} A_{\{\wedge\}} \\ \vdots \\ B_a \end{array}\right.$ to $(A \to B)_a$ by $(\to \mathrm{I})$

are admissible (similarly for $(\sim \mathrm{I})$).

(Rep) A_a may be repeated in a subproof (or strict subproof), retaining relevance index a.

(Reit) A_a may be reiterated into a regular subproof, without restriction, retaining a.

(Reit*)$_1$ NA_a may be reiterated into a strict subproof as A_a.

(Reit*)$_2$ NA_a may be reiterated into a strict subproof (as NA_a).[9]

($\rightarrow$ E) From A_a and $(A \rightarrow B)_b$ to infer $B_{a \cup b}$.

($\rightarrow$ I) From a proof of B_a with hypothesis $A_{\{k\}}$ to infer $(A \rightarrow B)_{a-\{k\}}$, provided that k is in a (or $k = \wedge$, see note 8).

($\sim\sim$ E) From $\sim\sim A_a$ to infer A_a.

($\sim\sim$ I) From A_a to infer $\sim\sim A_a$.

($\sim$ E) From $\sim B_a$ and $(A \rightarrow B)_b$, to infer $\sim A_{a \cup b}$.

($\sim$ I) From a proof of $\sim A_a$ on the hypothesis $A_{\{k\}}$, to infer $\sim A_{a-\{k\}}$, provided that k is in a (or $k = \wedge$).

(NE) From NA_a, to infer A_a.

(NI) From a strict proof of A_a with no hypotheses, to infer NA_a.

A proof in $G_{\bar{T}N}$* is *categorical* if all its hypotheses have been discharged by means of ($\rightarrow$ I) or ($\sim$ I). A formula, A, is a *theorem* of $G_{\bar{T}N}$* just in case it is the last step of a categorical proof.

The great advantage of $G_{\bar{T}N}$* is the ease it affords the construction of proofs of theorems. This advantage extends to $G_{\bar{T}N}$ since the two systems are equivalent:

T6. *A is a theorem of* $G_{\bar{T}N}$* *if and only if A is provable in* $G_{\bar{T}N}$.

To show that $G_{\bar{T}N}$ is contained in $G_{\bar{T}N}$* is a small matter. All axioms of the former are easily proved in the latter. (Verification is left to the reader.) R1 of $G_{\bar{T}N}$ is just the rule ($\rightarrow$ E). To see that R3 is admissible in $G_{\bar{T}N}$* one need only observe that any categorical proof of A can be converted into a categorical proof of NA simply by making it subordinate to a strict proof without hypotheses.

The proof that all theorems of $G_{\bar{T}N}$* are provable in $G_{\bar{T}N}$ follows closely that given by Anderson and Belnap in [3] for the equivalence of $E_{\bar{T}}$* and $E_{\bar{T}}''$ so, rather than reconstruct their proof here, the reader is referred to that work, or to [2]. (The only interesting modification of their method for constructing axiomatic proofs from given natural deductions is that we must allow for the case in which a reiterated formula shares an index with the hypothesis of the subproof into which it is reiterated; this occurs when the hypothesis is of the form NA and so bears the subscript $\{\wedge\}$. This case is easily accommodated through the insertion of the necessary form of A19 into the quasi-proof.)

9. Two alternate versions of the rule (Reit*) are given. (Reit*)$_1$ assures an M-like theory of modality; (Reit*)$_2$ an S4-like theory. As with the axiom A20, I shall not specify which version is to be adopted. The system with (Reit*)$_1$ is equivalent to $G_{\bar{T}N}$ without A20; while the system with (Reit*)$_2$ is equivalent to $G_{\bar{T}N}$ with A20.

The device of subscripting each hypothesis with a relevance index, with the restrictions on the rules ($\rightarrow$ I) and ($\sim$ I), blocks the proofs of the paradoxes of implication in the Anderson-Belnap systems R_T^* and E_T^* and also in G_{TN}^*. Thus the proof of $A \rightarrow .B \rightarrow A$ would require discharging an hypothesis by ($\rightarrow$ I) which was not used in the derivation of the consequent:

1.	$A_{\{1\}}$	Hyp
2.	$B_{\{2\}}$	Hyp
3.	$A_{\{1\}}$	1, Reit
4.	$(B \rightarrow A)_{\{1\}}$	2,3 $\rightarrow$ I (Invalid)
5.	$A \rightarrow .B \rightarrow A$	1,4 $\rightarrow$ I.

However, when the hypothesis is a necessitative proposition, *in effect*, the restrictions on the rules no longer apply. Thus, compare the 'proof' just given with the proof of A19 in G_{TN}^*:

1.	$A_{\{1\}}$	Hyp
2.	$NB_{\{\wedge\}}$	Hyp
3.	$A_{\{1\}}$	1, Reit
4.	$(NB \rightarrow A)_{\{1\}}$	2,3 $\rightarrow$ I
5.	$A \rightarrow . NB \rightarrow A$	1,4 $\rightarrow$ I.

In this case step 4 is admissible because of the peculiarities of $\wedge$. By subscripting all hypotheses of the form NA with $\{\wedge\}$, and supposing that $\wedge$ belongs to all subscripts on all formulas in a proof, we, in effect, render the relevance index irrelevant when working with necessitative propositions. (This reflects the remark made at the end of section 2, that one might regard necessitatives as relevant to all propositions.)

But if the relevance indices are irrelevant for necessitative propositions and it is the use of these indices which distinguishes relevant implication from material implication, then one might expect necessitatives to satisfy classical laws. This is indeed the case.

T7. *If A, B, and C are necessitative propositions, then all of the following are all derivable in* G_{TN}:

(1) $A \rightarrow . B \rightarrow A$
(2) $(A \rightarrow .B \rightarrow C) \rightarrow .A \rightarrow B \rightarrow .A \rightarrow C$
(3) $\sim B \rightarrow \sim A \rightarrow .A \rightarrow B$
(4) *From A and A $\rightarrow$ B, to infer B.*

(2), (3), and (4) are all instances of theorems (or rules) of R_T and so of G_{TN}. Where A is equivalent to ND and B is equivalent to NE, (1) is equivalent to $ND \rightarrow .NE \rightarrow ND$, which is an instance of A19. ((1) - (4) are, of course, a complete set of postulates for the classical propositional calculus.) Thus, if all propositions were necessitative, G_{TN} would reduce to the classical calculus. G_{TN} shares this feature with Lewis' system S4, though not with E.

5. *A Gentzen-style system* In this section I present a Gentzen-style consecution calculus, LG_{TN}, which is equivalent to G_{TN}. It should be remarked at this point that it is not yet known how to construct the full theory of rele-

vant implication (with positive truth functions) as a consecution calculus satisfying the elimination theorem. In the absence of such a system, the results given below for $G_{\bar{T}N}$ cannot yet be extended to the full system G.

'α', 'β', 'γ', etc. range over finite (perhaps null) sequences of well formed formulas. 'A', 'B', 'C', etc. range over wff's as before. Elementary statements, consecutions, in $LG_{\bar{T}N}$ are of the form $\alpha \Vdash \beta$. The system $LG_{\bar{T}N}$ is then defined by:

Primes: (where A is a propositional variable)

$$(\mathrm{Pr})\ A \Vdash A$$

Structural Rules (where $\alpha'(\beta')$ is some permutation of $\alpha(\beta)$):

$$(\mathrm{C}\Vdash)\ \frac{\alpha \Vdash \beta}{\alpha' \Vdash \beta} \qquad (\Vdash\mathrm{C})\ \frac{\alpha \Vdash \beta}{\alpha \Vdash \beta'}$$

$$(\mathrm{W}\Vdash)\ \frac{\alpha, A, A \Vdash \beta}{\alpha, A \Vdash \beta} \qquad (\Vdash\mathrm{W})\ \frac{\alpha \Vdash A, A, \beta}{\alpha \Vdash A, \beta} \qquad (\mathrm{K}_N\Vdash)\ \frac{\alpha \Vdash \beta}{\alpha, NA \Vdash \beta}$$

Logical Rules (where $N\alpha$ is the result of prefixing every member of α with N)[10]:

$$(\rightarrow\Vdash)\ \frac{\alpha \Vdash A, \gamma \quad \beta, B \Vdash \delta}{\alpha, \beta, A \rightarrow B \Vdash \gamma, \delta} \qquad (\Vdash\rightarrow)\ \frac{\alpha, A \Vdash B, \beta}{\alpha \Vdash A \rightarrow B, \beta}$$

$$(\sim\Vdash)\ \frac{\alpha \Vdash A, \beta}{\alpha, \sim A \Vdash \beta} \qquad (\Vdash\sim)\ \frac{\alpha, A \Vdash \beta}{\alpha \Vdash \sim A, \beta}$$

$$(\mathrm{N}\Vdash)\ \frac{\alpha, A \Vdash \beta}{\alpha, NA \Vdash \beta} \qquad (\Vdash\mathrm{N})_1\ \frac{\alpha \Vdash A}{N\alpha \Vdash NA} \qquad (\Vdash\mathrm{N})_2\ \frac{N\alpha \Vdash A}{N\alpha \Vdash NA}$$

The rule $(\mathrm{K}_N\Vdash)$ deserves special mention. In the Fitch varients of natural deduction, the restrictions on the rules $(\rightarrow \mathrm{I})$ and $(\sim \mathrm{I})$ distinguish relevant implication from material implication. As remarked, these restrictions are (really) irrelevant to necessitative propositions in $G_{\bar{T}N}$*, and so, as a connection between necessitatives, relevant implication is indistinguishable from material implication. In the Gentzen systems it is the absence of an admissible rule of weakening

$$(\mathrm{K}\Vdash)\ \frac{\alpha \Vdash \beta}{\alpha, A \Vdash \beta}$$

which makes relevant implication distinct from material implication. With the inclusion of the weaker rule of weakening $(\mathrm{K}_N\Vdash)$ in $LG_{\bar{T}N}$ we, in effect, apply classical rules to necessitative propositions while preserving the relevance preserving rules for the rest.

The Elimination Theorem (E.T.) holds for $LG_{\bar{T}N}$; that is,

T8 *The following rule is admissible in* $LG_{\bar{T}N}$:

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10. As before, there is an option for principles governing necessity on the right. $(\Vdash \mathrm{N})_1$ determines an M-like theory, $(\Vdash \mathrm{N})_2$ an S4-like theory.

$$\frac{\alpha \Vdash A, \gamma \quad \beta, A \Vdash \delta}{\alpha, \beta \Vdash \gamma, \delta}$$

This can be proved along the lines of Gentzen [10]. Given E.T., $LG_{\bar{T}N}$ can be shown to be equivalent to $G_{\bar{T}N}$.

T9. *If A is provable in $G_{\bar{T}N}$, then $\Vdash A$ is provable in* $LG_{\bar{T}N}$.

It is a small matter to show that the analogues of all the axioms of $G_{\bar{T}N}$ are provable in $LG_{\bar{T}N}$. R1, *modus ponens*, is a special case of the elimination rule. R3, *necessitation,* is a special case of the rule ($\Vdash$N). Hence, $G_{\bar{T}N}$ is contained in $LG_{\bar{T}N}$.

To show that $LG_{\bar{T}N}$ is contained in $G_{\bar{T}N}$, the consecutions of the one are interpreted in the wff's of the other as follows.
Where Γ (in $LG_{\bar{T}N}$) has the form $A_1, \ldots, A_n \Vdash B_1, \ldots, B_m$, let Γ' (in $G_{\bar{T}N}$) have the form

$$A_1 \to . \cdots \to . A_n \to . \sim B_1 \to . \cdots \to . \sim B_{m-1} \to B_m$$

or, in case $m = 0, A_1 \to . \cdots \to . A_{n-1} \to \sim A_n$.[11] With Γ and Γ' so defined, then

T10. *If Γ is provable in* $LG_{\bar{T}N}$, *then Γ' is provable in* $G_{\bar{T}N}$.

The proof is by induction on the proof of Γ in $LG_{\bar{T}N}$.

Basis. If Γ is prime, then Γ' has the form $A \to A$, an axiom of $G_{\bar{T}N}$.

Induction. Suppose T10 holds for the premisses of each of the rules; we show it holds for the conclusion by cases:

(a) if Γ is from Γ_1 by (C$\Vdash$) or ($\Vdash$C), then Γ' follows from Γ_1' by *Generalized Permutation*[12] (and, if $m = 0$, by contraposition as well).
(b) if Γ is from Γ_1 by (W$\Vdash$) or ($\Vdash$W), then Γ' is from Γ_1' by *Generalized Contraction.*[12]
(c) If Γ is from Γ_1 by ($K_N\Vdash$), then Γ' is from Γ_1' and A19 by transitivity (and, if $m = 0$, contraposition).
(d) If Γ is from Γ_1 by ($\Vdash \to$), then Γ' is from Γ_1' by permutation (and contraposition as necessary).
(e) in case Γ is from Γ_1 and Γ_2 by ($\to \Vdash$), then Γ' is from Γ_1' and Γ_2' by *Generalized Transitivity*[12] (and permutation and contraction as necessary).
(f) When Γ is from Γ_1 by ($\sim\Vdash$) then $\Gamma' = \Gamma_1'$ (or, if $m = 0$, Γ' follows from Γ_1' and A6 by transitivity).
(g) if Γ is from Γ_1 by ($\Vdash\sim$), then Γ' follows from Γ_1' and A6.
(h) When Γ is from Γ_1 by (N$\Vdash$), Γ' is from Γ_1' and A16.
(i) If Γ is from Γ_1 by $(\Vdash\text{N})_1$, then Γ' follows from Γ_1' thus:

1. $\vdash A_1 \to . \cdots \to . A_n \to \sim B$ $(= \Gamma_1')$ inductive hypothesis

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11. No interpretation is given in case both $n = 0$ and $m = 0$ since the consecution, $\Vdash$, is not provable in $LG_{\bar{T}N}$.
12. See Belnap and Wallace [5], p. 280 for a statement of these principles. (It should be noted that for $G_{\bar{T}N}$, and $R_{\bar{T}}$, permutation is not restricted to implicative propositions as it is for $E_{\bar{T}}$.)

2. $\vdash N(A_1 \rightarrow . \cdots \rightarrow . A_n \rightarrow \sim B)$ R3
3. $\vdash N(A_1 \rightarrow . \cdots \rightarrow . A_n \rightarrow \sim B) \rightarrow$
$. NA_1 \rightarrow . \cdots \rightarrow . NA_n \rightarrow N \sim B$ A17, generalized
4. $\vdash NA_1 \rightarrow . \cdots \rightarrow . NA_n \rightarrow N \sim B$ 2,3 R1
5. $\vdash N \sim B \rightarrow \sim NB$ from A16
6. $\vdash NA_1 \rightarrow . \cdots \rightarrow . NA_n \rightarrow \sim NB$ (= Γ'). 4,5 transitivity

(i′) If $(\Vdash\mathrm{N})_2$ is used instead of $(\Vdash\mathrm{N})_1$, the derivation of Γ' from Γ_1' is similar to that for (i), except that the equivalence of NA_i with NNA_i (guaranteed by A20) must be invoked as well. This completes the proof of T10.

An immediate corollary of T10 is that if $\Vdash A$ is provable in $\mathrm{LG_{\bar{T}N}}$, then A is provable in $\mathrm{G_{\bar{T}N}}$, establishing the equivalence of the two systems. This result constitutes, in a sense, a completeness theorem of $\mathrm{G_{\bar{T}N}}$.

$\mathrm{LG_{\bar{T}N}}$ has the sub-formula property, where

(i) A is a sub-formula of A;
(ii) if C is a sub-formula of A or B, then C is a sub-formula of $A \rightarrow B$, $\sim A$, and NA; and
(iii) all the sub-formulas of A are defined by (i) and (ii).

T11. *All constituents of any Γ_i in a proof of Γ in* $\mathrm{LG_{\bar{T}N}}$ *are constructed only out of sub-formulas of constituents of* Γ.

That this is so can be seen by examination of the rules of $\mathrm{LG_{\bar{T}N}}$.

$\mathrm{LG_{\bar{T}N}}$ has the separation property:

T12. *If an operation, ϕ, does not occur in any constituent in a consecution, Γ, then if Γ is provable in* $\mathrm{LG_{\bar{T}N}}$, *it can be proved without using rules governing ϕ.*

This follows from T11. (*Cf.* Curry [8], p. 226).

From T12 it follows that $\mathrm{G_{\bar{T}N}}$ is a conservative extension of $\mathrm{R_{\bar{T}}}$:

T13. *If a wff, A, contains no N, then A is provable in* $\mathrm{G_{\bar{T}N}}$ *if and only if A is provable in* $\mathrm{R_{\bar{T}}}$.

By T9, if A is provable in $\mathrm{G_{\bar{T}N}}$, $\Vdash A$ is provable in $\mathrm{LG_{\bar{T}N}}$, and by T12, if A contains no N, $\Vdash A$ is provable without applying $(\mathrm{N}\Vdash)$, $(\Vdash\mathrm{N})$ or $(K_N\Vdash)$. But the remaining postulates define a system equivalent to $\mathrm{R_{\bar{T}}}$. Hence, if A, without N, is provable in $\mathrm{G_{\bar{T}N}}$, it is provable in $\mathrm{R_{\bar{T}}}$. The converse is obvious.

T14. $\mathrm{LG_{\bar{T}N}}$ *is decidable.*

The decision procedure given by Belnap and Wallace in [5] applies to the system $\mathrm{LG_{\bar{T}N}}'$ got from $\mathrm{LG_{\bar{T}N}}$ by (i) letting consecutions of the form $\mathrm{N}\alpha, A, N\beta \Vdash A$ be prime (when A is a propositional variable); (ii) modifying each of the rules so that the principle constituent and its sub-alterns need not be the rightmost (leftmost) constituents on the left (right) sides; and (iii) deleting $(\mathrm{K_N}\Vdash)$. $\mathrm{LG_{\bar{T}N}}'$ is readily shown to be equivalent to $\mathrm{LG_{\bar{T}N}}$. Hence, $\mathrm{LG_{\bar{T}N}}$ is decidable, and so is $\mathrm{G_{\bar{T}N}}$, by the equivalence of those systems.

6. *Concluding remarks* In the preceding sections I did not attempt to give

philosophical grounds for the acceptance of the principle that, if p is true, then if not-p then anything, q, is possible. It might be worthwhile, however, to consider briefly what might underlie the adoption of this principle (P). (This discussion is independent of the foregoing formal development.)

One basic view behind the acceptance of the principle (P) is a picture of the (material) world as such an integrated, coherent system of facts that any change in one of its parts may effect changes elsewhere throughout the structure. This picture is reminiscient of an Idealist metaphysic, according to which every fact is 'relevant' to every other (Cf. [6], chapter 9). Nevertheless, acceptance of the principle does not commit one to a doctrine of internal relations, and it certainly does not commit one to an Idealist ontology.

Indeed, the principle might instead be based on a simple sort of determinism. Thus, if one supposes, contrary to fact, that a given proposition is false, that an event (or series of events) which did did not occur, then one must also suppose that whatever configuration of events which cause the given event(s) must be other than it is, either in whole or in part. But if this configuration were different, then whatever caused the events in the configuration which are supposed to be changed must likewise be supposed to be different from what it, in fact, is; and so on throughout the causal chain. Furthermore, events which are otherwise unrelated to the original given event might be effects of groups of events which now must be imagined not to obtain; so these events too could be supposed not to occur. They could also be preserved, provided that other causes could be found for them. As this process ramifies, it becomes appropriate to say that anything is possible.

This picture can, perhaps, be dramatized by considering the first illustration given in section 1. The earth revolves about the sun, but suppose that it did not; what would follow if the earth's position relative to the sun were fixed? Amongst other things, one might say, there could no longer be seasonal change, and to the extent that certain meterological conditions depend on these changes they too could no longer obtain, etc. But the story is not that simple.

The earth does not casually gambol around the sun. Its revolution is ruled by the laws of mechanics and set by certain initial conditions which are, in turn, determined by further laws and conditions. Hence, if one supposes that the earth does not revolve about the sun, one must also suppose that the laws do not hold or else that the antecedent conditions were other than they, in fact, are (or that some cataclysmic event has happened to freeze the earth in its orbit.) One cannot abandon the laws of mechanics, etc. without making extensive readjustments throughout the rest of the framework of theories to preserve the order, not to mention consistency, of one's system. Similarly, unless one believes in miracles, one cannot suppose the antecedent conditions to be other than they are (or that such a cataclysm has occurred) without re-evaluating points elsewhere in the structure. In either case, the world would be constituted quite otherwise than it is. It could be that in the new theoretical frame the effects of seasonal change would be preserved. Perhaps not. The original supposition does not determine what alterations must be made in the scientific theories; it does not determine what facts should be preserved and what should be abandoned. In this sense, anything is possible.

This line of thought may be extended to other kinds of cases, such as the historical example of section 1, merely by regarding all such events to be similarly law governed (though the relevant laws might be less readily identified.) It might be further extended to any domain in which it is appropriate to speak of one fact or proposition being a consequence of others.

Finally, one might not care to accept (P) with its unrestricted generality. Thus it is perverse to think, in common sense, that if Caesar had not crossed the Rubicon, given that he did, then it would be possible for Uranus to travel in a square orbit. Accordingly, one might prefer to restrict the range of the variables in (P) to range only over propositions of a certain kind (e.g. astronomical propositions, historical propositions, etc.) or to range only over propositions of the same kind (however these kinds be identified). Nevertheless, within the restricted range (P) should possess the properties I have described, and perhaps more besides.

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