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MATRIX SATISFIABILITY AND AXIOMATIZATION

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The appearance of *Polish Logic 1920-1939* (edited by Storrs McCall) is an event of considerable importance for logicians interested in the development of modern symbolic logic.[1] In conjunction with Tarski's *Logic, Semantics, and Metamathematics,* this collection of papers makes the central early source material from the Polish school of logicians available in English translation.[2] There are, however, a few matters of fit between the volumes which have escaped scrutiny. This is in no way intended to be a criticism of McCall's editorial decisions. Within the limits of a single volume of source papers, his choices seem uniformly excellent. In this paper I would like to discuss one theorem which is stated in [3] without proof, and no proof for which occurs in the papers which were chosen for inclusion in [2]. This theorem seems worthy of discussion because of the interesting connection which it establishes between matrix characterizations of propositional calculi and equivalent axiomatic systems.

In their paper "Investigations into the Sentential Calculus," J. Łukasiewicz and A. Tarski state the following theorem about the arbitrary calculus L_n $(2 \leqslant n < \aleph_0)$:[3]

> *Let $\mathfrak{M} = \langle A, B, f, g\rangle$ be a normal matrix in which the set $A \cup B$ is finite. If the sentences 'CCpqCCqrCpr', 'CCqrCCpqCpr', 'CCqrCpp', 'CCpqCNqNp', 'CNqCCpqNp' are satisfied by this matrix, then the set of sentences satisfying $\mathfrak{M}$ may be finitely axiomatized.*

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1. See [2].
2. See [3].
3. See [1], p. 50. A normal matrix in which B is $\{1\}$ defines the calculus L_n when the number of values n is identical with the number of values A in the matrix. Strictly, a normal matrix could have more than one designated value, so that Wajsberg's theorem applies to a larger class of calculi than the calculi L_n. As only the calculi L_n have assumed an important role in the literature, we will ignore this complication in what follows except for one remark preceding Lemma 10.

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I will assume here that the terminology is known from [3], so that L_n is an arbitrary many-valued calculus with C and N as primitive logical signs of the kind investigated extensively by Łukasiewicz and Tarski. This theorem clearly establishes the existence of a finite axiom set for each such calculus. The proof of this theorem referred to in [3] appears in a paper of M. Wajsberg which is not among those included in McCall's collection.[4] My purpose in this paper is to fill in the lacuna by presenting Wajsberg's proof.

The following presentation is *not a translation* of Wajsberg's proof in the ordinary sense, but a close paraphrase containing some additional clarifying material. Further, I am not concerned to deal with Wajsberg's entire paper, which overlaps at many points with material available in [2] and [3]. The relevant proof by Wajsberg appears in section §3 of his article. I here present the exact *outline* of Wajsberg's proof in that Lemmas 1-23 of this paper correspond exactly in content with Wajsberg's Saetze 1-23. In each case, the proof I give for a Lemma will follow Wajsberg's proof strategy closely, but I have taken some liberties in restatement, and I correct some obvious misprints and a few minor errors in proof. My object is to sketch the proof in sufficient detail to exhibit that it is correct to readers whose knowledge of the related literature is assumed to be restricted to some familiarity with allied papers in [2] and [3]. In line with this object, some material embedded in Wajsberg's proof which is not essential to the development of the proof is also omitted. The terminology and symbolism of this paper is taken from [2] and [3].

Apart from notational differences, the Hauptsatz of Wajsberg's paper differs in one respect from the theorem stated by Łukasiewicz and Tarski in [3]. Where [2] includes *CCqrCpp* among the sentences assumed to satisfy the matrix M. Wajsberg includes *CCqqCpp*. Since the latter is an immediate consequence of the former, Wajsberg's proof can be regarded as establishing both theorems. We now state the Hauptsatz of Wajsberg's paper:

Hauptsatz. The set of sentences satisfying a finite normal matrix $\mathfrak{M}$ *can be finitely axiomatized if the following sentences satisfy the matrix*:

CCpqCCqrCpr	(Syl_1),
CCqrCCpqCpr	(Syl_2),
CCpqCNqNp	(Transp_1),
CNqCCpqNp	(Transp_2),
CCqqCpp	(Id*).

4. The paper is [5]. At the time of appearance of [2], Geoffrey Keene of Exeter University (England) and I had been working on a similar volume. [2] put paid to that idea, but I am very grateful to Keene for hard work on translating articles from the original Polish, work that now appears to have had no consequence except improving the quality of his translation. I am also grateful to my wife Inge for help in translating some articles from the original German. The three of us can attest to the high quality exhibited by the translations included in [2].

In the statement of Lemmas and their proofs, we use small greek letters (except ρ and σ) to stand for arbitrary sentences of the calculi L_n. An expression containing greek letters is not really a sentence unless the greek letters are replaced by sentences, but we will use *sentence* to refer either to sentences or to sentential expressions containing greek letters. The introduction of sufficient terminology to be completely rigorous would be tedious, and no obscurity results from this decision. Similarly with quotation. We will rely entirely on context to mark relevant use-mention distinctions involving sentences or symbols of any of the calculi. The sign $\vdash$ is used to express deducibility in the relevant calculus, where deducibility is defined as the sole use of correct substitution and modus ponens. Some properties of deducibility are assumed throughout, in particular

(a) $\alpha \vdash \alpha$,
(b) *If* $\alpha \vdash \beta$ *and* $\beta \vdash \gamma$, *then* $\alpha \vdash \gamma$,
(c) *If* $\vdash \alpha$ *and* $\alpha \vdash \beta$, *then* $\vdash \beta$,
(d) *If* $\vdash C\alpha\beta$, *then* $\alpha \vdash \beta$,
(e) *If* $\alpha \vdash \beta$ *and* $\beta,\gamma \vdash \delta$, *then* $\alpha,\gamma \vdash \delta$.

An expression like $C(1)CNqNr$ is used to stand for the conditional sentence with (1) as antecedent sentence and $CNqNr$ as consequent, with (1) given as an explicit sentence in the context.

Wajsberg's proof can be looked at as having two parts. Lemmas 1-9 establish meta-theorems concerning deducibility for all propositional calculi defined by matrix characterization. Lemmas 10-23 construct an axiom set for an arbitrary but fixed L_n and establish that any sentences satisfying the matrix $\mathfrak{M}$ defining the calculus L_n is deducible from this axiom set. It might be noted that proof of some of the Lemmas depends on some simple theorems of algebra or number theory, which could in principle be eliminated. The remainder of this paper consists of a statement of Lemmas 1-23 and a proof or proof sketch of each.

Lemma 1. $C\alpha\beta, C\beta\gamma \vdash C\alpha\gamma$.

Proof. Lemma 1 follows from (Syl_1) by substitution ($\mathbf{S}$: p/α, q/β, r/γ) and two uses of modus ponens. (This illustrates the substitution notation to be used throughout, except that obvious substitutions will not always be explicitly cited.)

Lemma 2. $C\alpha\beta, C\gamma C\beta\delta \vdash C\gamma C\alpha\delta$.

Proof. $C\alpha\beta \vdash CC\beta\delta C\alpha\delta$ can be obtained by substitution into (Syl_1), and $C\gamma C\beta\delta, CC\beta\delta C\alpha\delta \vdash C\gamma C\alpha\delta$ by substitution into Lemma 1. Lemma 2 follows from property (e) of $\vdash$.

We now introduce some symbolism to be used in connection with later Lemmas. The first is a device which can be used to refer to any sentence which appears as a proper part of another sentence. If $C\alpha\beta$ is a sentence, we set $(C\alpha\beta)^1 = \alpha$ and $(C\alpha\beta)^0 = \beta$. Further, we set $(N\alpha)^1 = \alpha$. By an obvious recursive procedure, we can thus refer to any sentence α which is a proper part of a sentence β by enclosing β in parentheses and following the parentheses with a suitable string of 1's and 0's, such a string to be known as a *place index*. The place index is read from left to right. *A place index is even if it has an even number of 1's in it, otherwise it is odd.* Clearly, each sentential variable occurring as a proper part of some sentence α can be uniquely referred to by enclosing α in parentheses and following it with a suitable place index. For example, $p = (CCpqr)^{11}$, $q = (CCpqr)^{10}$, $q = (CCpNqr)^{101}$, and so on. Using the notion of a place index, we let the formula $\mathbf{S}(\alpha, \beta, \gamma, \delta, x)$ express the fact that a sentence β is obtained from a sentence α when the sentence (or variable) γ defined as the proper part of α with place index x is substituted for by the sentence (or variable) δ. Thus we have $\mathbf{S}(Cpp,Cpq,p,q,0)$ and $\mathbf{S}(Cqp,CNqp,q,Nq,1)$. Using this new symbolism, we state several additional Lemmas.

Lemma 3. If $\mathbf{S}(\alpha,\beta,\gamma,\delta,x)$, *then if* x *is even,* $\vdash C\alpha CC\gamma\delta\beta$ *and* $\vdash CC\gamma\delta C\alpha\beta$, *and if* x *is odd,* $\vdash C\alpha CC\delta\gamma\beta$ *and* $\vdash CC\delta\gamma C\alpha\beta$.

Proof. The proof is by induction on the length of the place index x. For place index of length one, $x = 0$ or $x = 1$. In the former case, α and β are of the form $C\eta\gamma$ and $C\eta\delta$, respectively. Then $CC\gamma\delta C\alpha\beta$ is provable by substitution into (Syl_2). Similarly, $C\alpha CC\gamma\delta\beta$ is provable by substitution into (Syl_1). If $x = 1$, then α and β are either of the form $C\gamma\eta$ and $C\delta\eta$ respectively, or they are of the form $N\gamma$ and $N\delta$. In the former case, we prove $C\alpha CC\delta\gamma\beta$ and $CC\delta\gamma C\alpha\beta$ by substitution into (Syl_2) and (Syl_1) as in the case $x = 0$. In the latter case, we prove $C\alpha CC\delta\gamma\beta$ and $CC\delta\gamma C\alpha\beta$ by substitution into (Transp_2) and (Transp_1) respectively. Notice that (Syl_1), (Syl_2), (Transp_1), and (Transp_2) are all used to establish this important Lemma. To complete the proof by induction, we must show that if the proof holds when the place index x has λ numerals, it also holds when the place index x has $\lambda + 1$ numerals. In this case x has the form $\lambda 0$ or $\lambda 1$. We must consider in each case whether λ is odd or even, giving four cases of $\lambda + 1$ all together. We discuss just the case where λ is even and x is of the form $\lambda 0$. By the assumption that the proof holds for $x = \lambda$, we have $\mathbf{S}(\alpha, \beta, \gamma', \delta', \lambda)$ and the provability of $C\alpha CC\gamma'\delta'\beta$ and $CC\gamma'\delta' C\alpha\beta$, where γ' and δ' are of the forms $C\eta\gamma$ and $C\eta\delta$ in view of the place index $\lambda 0$. Substituting into Lemma 2, we obtain the deducibility of $C\alpha CC\gamma\delta\beta$ from $CC\gamma\delta CC\eta\gamma C\eta\delta$ and $C\alpha CCC\eta\gamma C\eta\delta\beta$. But $CC\gamma\delta CC\eta\gamma C\eta\delta$ is a substitution instance of (Syl_2) and $C\alpha CCC\eta\gamma C\eta\delta\beta$ is equivalent to $C\alpha CC\gamma'\delta'\beta$ which is already known to be provable. Similarly, we establish that $CC\gamma\delta C\alpha\beta$ can be obtained by substitution into Lemma 1 along with the established provability of $CC\gamma\delta CC\eta\gamma C\eta\delta$ and $CC\gamma'\delta' C\alpha\beta$. Proof of the cases where λ is even and x is of the form $\lambda 1$ and where λ is odd and x is of the form $\lambda 0$ or $\lambda 1$ proceed similarly.

Lemma 4. $C\alpha_1 C\alpha_2 \ldots C\alpha_k\gamma$, $C\gamma\delta \vdash C\alpha_1 C\alpha_2 \ldots C\alpha_k\delta$.

Proof. Let α be $C\alpha_1 C\alpha_2 \ldots C\alpha_k\gamma$ and β be $C\alpha_1 C\alpha_2 \ldots C\alpha_k\delta$. By Lemma 3, we have $\mathbf{S}(\alpha, \beta, \gamma, \delta, x)$ for an even x, and the provability of $CC\gamma\delta C\alpha\beta$. Using modus ponens twice with $C\gamma\delta$ and α, we have the provability of β.

Lemma 5. Suppose we have $\mathbf{S}(\alpha_i, \alpha_{i+1}, \gamma_i, \delta_i, x_i)$ *for* $i = 1, 2, \ldots, k$. *Then we have as provable sentences* $C\alpha_1 C\beta_1 C\beta_2 \ldots C\beta_k\alpha_{k+1}$ and $C\beta_1 C\beta_2 \ldots$

$C\beta_k C\alpha_1\alpha_{k+1}$. *$\beta_i$ is of the form $C\gamma_i\delta_i$ or $C\delta_i\gamma_i$, depending on whether i is even or odd.*

Proof. Use induction on k. For $k = 1$, Lemma 3 suffices. The induction step is easy to work out for various cases using Lemmas 3 and 4. Detailed proofs of Lemmas 3 and 5 can be found on pp. 270-272 of [2].

To state the next few Lemmas, we give a recursive definition of a superscripted C:

$C^0pq = q$,

and

$C^{l+1}pq = CpC^lpq$.

These identities allow us to replace one sentence by its notational variant in doing proofs.

Lemma 6. If β follows from α by substitution of δ for γ at k places with even place index and l places with odd place index, then the sentences $C\alpha C^kC\gamma\delta C^lC\delta\gamma\beta$ and $C\alpha C^lC\delta\gamma C^kC\gamma\delta\beta$ are provable. (Neither k nor l is 0.)

Proof. If the notation is read correctly, this Lemma is a special case of one half of Lemma 5. Notice, for example, that $C\alpha C^3C\delta\gamma C^2C\gamma\delta\beta$ is equivalent to $C\alpha CC\delta\gamma CC\delta\gamma CC\delta\gamma CC\gamma\delta CC\gamma\delta\beta$.

Lemma 7. If β follows from α by substitution of δ for γ at k places with even place index and l places with odd place index, then the following hold:

(a) $\alpha \vdash C^kC\gamma\delta C^lC\delta\gamma\beta$,
(b) $\alpha \vdash C^lC\delta\gamma C^kC\gamma\delta\beta$,
(c) $\alpha,\ C\gamma\delta \vdash C^lC\delta\gamma\beta$,
(d) $\alpha,\ C\delta\gamma \vdash C^kC\gamma\delta\beta$,
(e) $\alpha,\ C\gamma\delta,\ C\delta\gamma \vdash \beta$.

Proof. Obvious consequence of Lemma 6 and properties of $\vdash$.

Lemma 8. If α contains the variable p as a proper part, then $C\alpha CCpqCCqp\alpha$ and $C\alpha CCqpCCpq\alpha$ are provable.

Proof. We have $\mathbf{S}(\alpha,\alpha,p,p,x)$ for some place index x, and so by Lemma 3, $C\alpha CCpp\alpha$ is provable. Furthermore, we have $\mathbf{S}(CCpp\alpha,\ CCqp\alpha,p,q,\ 11)$, and so again by Lemma 3 we have $CCCpp\alpha CCpqCCqp\alpha$. From the provability of these two sentences and Lemma 1, we have $C\alpha CCpqCCqp\alpha$. A similar proof shows that $C\alpha CCqpCCpq\alpha$ is provable.

Lemma 9. If α contains the variable p as a proper part, then the sentences $C\alpha C^kCpqC^kCqp\alpha$ and $C\alpha C^kqpC^kpq\alpha$ are provable. ($k = 1, 2, \ldots$)

Proof. By induction on k. For $k = 1$, Lemma 8 is sufficient. Let the Lemma be assumed true for k-1. Then

(1) $C\alpha C^{k-1}CpqC^{k-1}Cqp\alpha$

and

(2) $C\alpha C^{k-1}CqpC^{k-1}Cpq\alpha$

are provable. Substitution in Lemma 8 (**S**: $\alpha/C^{k-1}Cqp\alpha$) yields

(3) $CC^{k-1}Cqp\alpha CCpqCCqpC^{k-1}Cqp\alpha$,

and another substitution in Lemma 8 (**S**: $\alpha/C^{k-1}Cpq\alpha$) yields

(4) $CC^{k-1}Cpq\,\alpha CCqpCCpqC^{k-1}Cpq\alpha$.

By the definition of the superscript, we have

(5) $CC^{k-1}Cqp\alpha CCpqC^kCqp\alpha$,

and

(6) $CC^{k-1}Cpq\alpha CCqpC^kCpq\alpha$

from (3) and (4). By substitution in Lemma 4 (**S**: α_1/α, α_i/Cpq ($i = 2, 3, \ldots, k$), $\gamma/C^{k-1}Cqp\alpha$, $\delta/CCpqC^kCqp\alpha$) we obtain

(7) $C\alpha C^{k-1}CpqCCpqC^kCqp\alpha$

from (1) and (5). To complete the proof, we need the following property of the superscript notation:

$C^{k-1}\alpha C\alpha\beta = C^k\alpha\beta$.

This is easily proved by induction. For $k = 1$, the equivalence follows from the definition. Assuming the property for k-1, we have $C^{k-2}\alpha C\alpha\beta = C^{k-1}\alpha\beta$. Then $C^{k-1}\alpha C\alpha\beta = C\alpha C^{k-2}\alpha C\alpha\beta = C\alpha C^{k-1}\alpha\beta = C^k\alpha\beta$ by the definition. We use this property of the superscript notation to find (7) equivalent to $C\alpha C^kCpqC^kCqp\alpha$, proving one half of the Lemma. The other half follows from (2) and (6) by Lemma 4 and a similar use of properties of the superscript notation.

(In Lemmas 1-9, various meta-theorems concerning deducibility in all of the propositional calculi defined by finite normal matrices were established. To this point, we have used the sentences (Syl_1), (Syl_2), (Transp_1), and (Transp_2) from the antecedent of the Hauptsatz. In the remaining Lemmas, we will use (Id*) and we will assume that $\mathfrak{M}$ refers to a fixed finite normal matrix. (Syl_1), (Syl_2), (Transp_1), (Transp_2), and (Id*), all of which satisfy $\mathfrak{M}$ by hypothesis, will be taken as axioms of a deductive system using $\vdash$ as its deducibility relation. This set of five axioms will then be constructively enlarged until it can be shown that any sentence satisfying $\mathfrak{M}$ can be deduced from the enlarged, but finite, axiom set. The existence of this axiom set is sufficient to establish the Hauptsatz.)

We let $\mathbf{E}_{\mathfrak{M}}(\alpha, \beta)$ express the equivalence of α and β with respect to $\mathfrak{M}$, that is, the fact that α and β are assigned the same value by $\mathfrak{M}$ whenever those variables common to α and β are assigned the same value from $\mathfrak{M}$. $\mathbf{E}_{\mathfrak{M}}$ is obviously reflexive, commutative, and transitive. We let $\mathbf{E}^{\mathfrak{M}}(\alpha, \beta)$ express the fact that $C\alpha\beta$ and $C\beta\alpha$ are both assigned the value 1 (this for L_n, otherwise any designated value) on every assignment of values to their constituent variables. Clearly, if $\mathbf{E}_{\mathfrak{M}}(\gamma, \delta)$, then if α satisfies $\mathfrak{M}$, so will β, where β is obtained from α by substituting γ for δ at one or more occurrences of γ in α. We can thus proceed as though $\mathbf{E}_{\mathfrak{M}}(\gamma,\delta)$ provided us with a replacement rule.

Lemma 10. If $\mathbf{E}_{\mathfrak{M}}(\alpha,\beta)$, *then* $\mathbf{E}^{\mathfrak{M}}(\alpha,\beta)$.

Proof. We can establish *Cpp* (Id) from (Id*). *CCqqCqq* is first obtained from Lemma 1, using (Id*) and *CCppCqq*, an obvious substitution instance of (Id*). Substitution in (Id*) (**S**: *q/Cqq*) yields *CCCqqCqqCpp*. By modus ponens, the two sentences just obtained give (Id). We therefore have $C\alpha\alpha$, by consideration of the role of small greek letters. By the replacement rule equivalent to the use of $\mathbf{E}_{\mathfrak{M}}(\alpha,\beta)$, $C\alpha\beta$ and $C\beta\alpha$ follow from $C\alpha\alpha$, and the Lemma is proved.

For convenience in establishing the next Lemmas, we will assume that the variables of the propositional calculi are ordered in this fashion: p_1, p_2, p_3, and so on. To preserve continuity with the earlier part of the paper, we may assume the well known convention that p_1 is p, p_2 is q, p_3 is r, and so forth. We then define $\mathbf{V}(n)$ as the set of sentences in which only propositional variables identical with one of p_1 to p_n occur.

Lemma 11. There is a finite set $N(n) \subset \mathbf{V}(n)$ *such that for every sentence* $\alpha \in \mathbf{V}(n)$, *we have an element* $\beta \in N(n)$ *such that* $\mathbf{E}_{\mathfrak{M}}(\alpha,\beta)$.

Proof. It is easy to see that such a finite set exists. Let m be the number of values in the matrix $\mathfrak{M}$. Then the possible value assignments to the n propositional variables $p_1,\ldots,p_n$ are m^n in number. Any particular sentence could have any of the m values assigned to each of its constituent variables. There are thus at most m^{m^n} distinct functions given $\mathfrak{M}$ from sentences with n variables to values in $\mathfrak{M}$. One could construct a set $N(n)$ by finding representative sentences for $n = 1, n = 2$, and so on, on the basis of the given matrix $\mathfrak{M}$. To fix ideas, we will adopt the following procedure. We start with the set T, consisting of the propositional variables $p_1, p_2, \ldots,$ p_n. Then a series of sets T_2, $T_3, \ldots$ is formed by the following recursive strategy. If N_i is the set of all sentences of the form $C\alpha\beta$, $C\beta\alpha$, and $N\alpha$, where α is in T_{i-1}, and β is in any of the sets T_1, $T_2, \ldots,$ T_{i-1}, then T_i is any subset of N_i which contains a single sentence β for every sentence α of N_i which is not equivalent with respect to $\mathfrak{M}$ to some sentence in one of the sets T_1, $T_2, \ldots, T_{i-1}$. By the observation made above, some set T_i will be the first empty set of the sequence. (It is easy to find an upper bound on the value of i within which the first empty set will appear.) The union of the sets $T_1 \cup T_2 \cup \ldots \cup T_{i-1}$ will have the properties attributed to $N(n)$ by Lemma 11.

Lemma 12. Every infinite set of sentences which is a subset of $\mathbf{V}(n)$ *contains as a proper part an infinite subset of sentences which are all equivalent with respect to* $\mathfrak{M}$.

Proof. Obvious as a corollary to Lemma 11.

Lemma 13. If $\alpha \in \mathbf{V}(m)$ (*where* m *is the number of values in* $\mathfrak{M}$), *then there exists a certain* $\alpha' \in N(m)$ *such that* $\vdash C\alpha\alpha'$ *and* $\vdash C\alpha'\alpha$.

Proof. If α is a variable, then α is identical with one of the variables $p_1, \ldots, p_m$. By construction of $N(m)$, all of the variables are elements of $N(m)$. Hence the Lemma is trivial in view of the provability of (Id). The remaining cases are where α is of the form $N\beta$ or $C\beta\gamma$. To prove Lemma 13 in these cases, we will start our definition of the set of sentences Ax which will eventually establish the axiomatization of the sentences satisfying $\mathfrak{M}$. The first step is to define the set A_1 which will be a subset of Ax. A_1 contains (Syl_1), (Syl_2), (Transp_1), (Transp_2), (Id*) as well as all sentences satisfying $\mathfrak{M}$ which are in $N(m)$ *and* all sentences of the form $CC\alpha\beta\gamma$ and $C\gamma C\alpha\beta$, where α, β, γ are sentences in $N(m)$ and γ is equivalent to $C\alpha\beta$ with respect to $\mathfrak{M}$ *and* all sentences of the form $CN\alpha\gamma$ and $C\gamma N\alpha$ where α, γ are sentences in $N(m)$ and γ is equivalent to $N\alpha$ with respect to $\mathfrak{M}$. Since $N(m)$ is finite, A_1 is obviously finite. We prove just one of the cases. Let α be of the form $C\beta\gamma$. Then there are sentences β', γ' in $N(m)$ such that the statements $C\beta\beta'$, $C\beta'\beta$, $C\gamma\gamma'$, and $C\gamma'\gamma$ are provable by Lemmas 10 and 11. A_1 contains the sentences (1) $CC\beta'\gamma'\alpha'$ and (2) $C\alpha'C\beta'\gamma'$ in view of the construction of A_1. By replacement (which can in this case be rigorously justified by means of Lemma 3), we can also prove $CC\beta\gamma\alpha'$ and $C\alpha'C\beta\gamma$. Thus we have $\mathsf{E}_{\mathfrak{M}}(\alpha, C\beta'\gamma')$ and $\mathsf{E}_{\mathfrak{M}}(C\beta'\gamma', \alpha')$, therefore $\vdash C\alpha\alpha'$ and $\vdash C\alpha'\alpha$ by Lemma 10. The case where α is of the form $N\beta$ can be proved in an analogous fashion.

A sentence containing *exactly* k distinct variables will be called k-dimensional. The set of all k-dimensional sentences will be designated by V_k. V_k should be distinguished from $\mathsf{V}(k)$, the set of sentences containing only propositional variables from $p_1, p_2, \ldots, p_k$ which was introduced in Lemma 11.

Lemma 14. If α satisfies $\mathfrak{M}$ and is at most m-dimensional, then α is provable.

Proof. We assume that $\alpha \in \mathsf{V}(m)$, that is, uses at most the variables $p_1, p_2, \ldots, p_m$. Otherwise α is equivalent to some β which has this property, and β is taken as the α of the Lemma. By Lemma 13, there is an element α' of $N(m)$ such that $\vdash C\alpha\alpha'$ and $\vdash C\alpha'\alpha$. $C\alpha\alpha'$ clearly satisfies $\mathfrak{M}$, and since α satisfies $\mathfrak{M}$ by the assumption of the Lemma, α' also satisfies $\mathfrak{M}$. Since all sentences satisfying $\mathfrak{M}$ which are elements of $N(m)$ are elements of A_1, α' is an element of A_1. But from this fact and the provability of $C\alpha'\alpha$, it follows that α is provable (given A_1).

We now consider the set of all sentences ϕ of the form $C^iCp_1p_2p_3$ ($i = 0, 1, 2, \ldots$). This set is an infinite subset of $\mathsf{V}(3)$ and must therefore (in view of Lemma 12) contain two sentences which are equivalent to each other with respect to $\mathfrak{M}$. Let ρ, $\rho + \sigma$ ($\sigma \neq 0$) be the pair of smallest indices such that $\phi_\rho = C^\rho Cp_1p_2p_3$ and $\phi_{\rho+\sigma} = C^{\rho+\sigma}Cp_1p_2p_3$ are equivalent to each other with respect to $\mathfrak{M}$. By Lemma 10 we have $\vdash CC^{\rho+\sigma}Cp_1p_2p_3C^\rho Cp_1p_2p_3$. By our informal convention interchanging subscripted variables with the usual notation, this means that we can prove (Red) $CC^{\rho+\sigma}CpqrC^\rho Cpqr$. We add (Red) to the sentences of A_1 to form the set A_2.

Lemma 15. The following sentences are provable given A_2:

(a) $C^{\sigma}CqpCC^{\sigma}CppC^{\rho}CpqrC^{\rho}Cpqr$,
(b) $CCpqCCppCpq$,
(c) $CCpqCCrrCpq$,
(d) $CCpqC^{\sigma}CrrCpq$,
(e) $CC^{\rho}CpqrC^{\sigma}CppC^{\rho}Cpqr$,
(f) $C^{\sigma}CqpCC^{\sigma}CppC^{\rho}CpqrC^{\sigma}CppC^{\rho}Cpqr$,
(g) $C^{\sigma}CqpCrr$,
(h) $CCrsC^{\sigma}CqpCrs$,
(i) $CCC^{\sigma}CqpCrstCCrst$,
(j) $CC^{\rho+k\sigma}CpqrC^{\rho}Cpqr$ ($k = 1, 2, \ldots$).

Proofs:

(a) Proof from Lemma 7(a) by substitution (**S**: α/(Red), $\beta/CC^{\sigma}CppC^{\rho}CpqrC^{\rho}Cpqr$, k/σ, $1/0$, γ/q, δ/p).
(b) Proof from (Syl_2) by substitution (**S**: r/p).
(c) Use (b), Lemma 3, and substitution in (Id*) or (Id).
(d) Substitution in (c) (**S**: p/Crr, q/Cpq) yields (1) $CCCrrCpqCCrrCCrrCpq$. (1) and (c), in view of Lemma 1, yield $CCpqCCrrCCrrCpq$, or $CCpqC^2CrrCpq$. Repeated use of this strategy yields (d) for the particular value of σ required.
(e) Substitution in (d) (**S**: p/Cpq, $q/C^{\rho-1}Cpqr$, r/p).
(f) In Lemma 4, let $C^{\rho}Cpqr$ be substituted for γ. Then (a) and (e) are obvious substitutions into the sentences assumed in Lemma 4, and (f) follows from an application of the Lemma.
(g) In Lemma 4, let $k = \sigma$. Then substitution (**S**: α_i/Cqp, $\gamma/CC^{\sigma}CppC^{\rho}CpqrC^{\sigma}CppC^{\rho}Cpqr$, δ/Crr) and the Lemma yield (g). The assumptions of the Lemma after substitution are (f) and a substitution instance of (Id*) (**S**: $q/C^{\sigma}CppC^{\rho}Cpqr$).
(h) Substitution in Lemma 3 (**S**: $\alpha/C^{\sigma}CqpCrr$, $\beta/C^{\sigma}CqpCrs$, γ/r, δ/s), and modus ponens on the result with (g).
(i) Substitution in (Syl_1)(**S**: p/Crs, $q/C^{\sigma}CqpCrs$, r/t) yields a sentence from which a use of modus ponens with (h) yields (i).
(j) Induction on k using (Red) and Lemma 1.

In order to state and prove the remaining Lemmas, we introduce a number of definitions:

(a) $\mathrm{gr}(\alpha, \beta)$ = the number of the even place indices x such that $\beta = \alpha x$. If α is a variable, x is defined as 1 or 0 depending on whether $\beta = \alpha$ or $\beta \neq \alpha$.
(b) $\mathrm{ngr}(\alpha, \beta)$ = the number of the odd place indices x such that $\beta = \alpha x$. If α is a variable, then $\mathrm{ngr}(\alpha, \beta) = 0$.
(c) $\mathrm{df}(\alpha, \beta) = \mathrm{gr}(\alpha, \beta) - \mathrm{ngr}(\alpha, \beta)$.
(d) $\mathrm{div}(\alpha)$ = the greatest common divisor of the numbers $\mathrm{df}(\alpha, \beta)$ different from 0, where β is a variable occurring in α; 0 if no such divisor otherwise exists.

(e) $\mathrm{div}(X)$ (X is a *set* of sentences) = the greatest common divisor of all numbers $\mathrm{div}(\alpha)$, where $\alpha \in X$; $\mathrm{div}(X) = 0$ if no such divisor otherwise exists.

(f) $\mathrm{div}(\mathfrak{M}) = \mathrm{div}(\mathbf{E}(\mathfrak{M}))$, where $\mathbf{E}(\mathfrak{M})$ is the set of sentences satisfying $\mathfrak{M}$.

Examples. $\mathrm{gr}(C^kpq,p) = 0$, $\mathrm{ngr}(C^kpq,p) = k$, $\mathrm{df}(C^kpq,p) = -k$, $\mathrm{gr}(p,p) = \mathrm{df}(p,p) = 1$. The divisors (div) of (Id), (Id*), (Syl_1), (Syl_2), (Transp_1), and (Transp_2) are equal to 0, on the other hand the divisor of (Red) is equal to σ.

Lemma 16. Suppose β and α are not variables, and substitution of q for p turns β into α. If $\mathrm{gr}(\alpha,p) = k$ and $\mathrm{ngr}(\alpha,p) = l$, then $C\beta C^{k+m}CqpC^{l+m}Cpq\alpha$ and $C\beta C^{l+m}CpqC^{k+m}Cqp\alpha$ are both provable, where m is any natural number.

Proof. Using Lemma 6, and the conditions on this Lemma, we can prove $C\beta C^kCqpC^lCpq\alpha$ and $C\beta C^lCpqC^kCqp\alpha$. Using Lemma 9 ($\mathbf{S}$: k/m ($m = 1, 2, \ldots$), $\alpha/C^lCpq\alpha$ or $C^kCqp\alpha$) we can then prove $CC^lCpq\alpha C^mCqpC^mCpqC^lCpq\alpha$ and $CC^kCqp\alpha C^mCpqC^mCqpC^kCqp\alpha$. The Lemma is then provable from these sentences by means of Lemma 4.

Lemma 17. If α satisfies $\mathfrak{M}$, and $\mathrm{df}(\alpha,p) = +m$ ($m = 1, 2, \ldots$), then $C^mCqpCrr$ satisfies $\mathfrak{M}$.

Proof. If $\alpha = \beta$ ($\mathbf{S}$: p/q) and α satisfies $\mathfrak{M}$, so will β. From Lemma 16, (1) $C^kCqpC^lCpq\alpha$ satisfies $\mathfrak{M}$. Assuming $k \geqslant l$, and setting $k - l = m$, we write (1) as (2) $C^mCqpC^lCqpC^lCpq\alpha$. α contains p as a proper part, since not both $\mathrm{gr}(\alpha,p)$ and $\mathrm{ngr}(\alpha,p)$ are equal to 0. By substitution in Lemma 9 and use of modus ponens with α, it follows that $C^lCqpC^lCpq\alpha$ satisfies $\mathfrak{M}$. Since Crr satisfies $\mathfrak{M}$, we have (3) $CC^lCqpC^lCpq\alpha Crr$, by Lemma 10. Using Lemma 4 on (2) and (3), we have $C^mCqpCrr$. If $k < l$, then letting $l - k = m$, and exchanging p and q, we obtain the same result.

Lemma 18. If $C^kCqpCrr$ and $C^lCqpCrr$ (k, $l = 1, 2, \ldots$) satisfy $\mathfrak{M}$, and m is the greatest common divisor of k and l, then $C^mCqpCrr$ satisfies $\mathfrak{M}$.

Proof. To establish Lemma 18, we first show that if (1) $C^{t+u}CqpCrr$ and (2) $C^uCqpCrr$ satisfy $\mathfrak{M}$, then (3) $C^tCqpCrr$ satisfies $\mathfrak{M}$. (1) is equivalent to (4) $C^tCqpC^uCqpCrr$. From (2), and the fact that Crr satisfies $\mathfrak{M}$, we have (5) $CC^uCqpCrrCrr$ by Lemma 10. Lemma 4 with (4) and (5) yields (3), proving this new rule of inference. This rule can now be used with (6) $C^kCqpCrr$ and (7) $C^lCqpCrr$ from the antecedent of Lemma 18 to obtain $C^mCqpCrr$ by means of number theoretic considerations. Without loss of generality, we assume $k > l$, If $m = l$, the theorem is proved. We therefore assume $m < l$. We can set $k = ma$ and $l = mb$, with $a > b$. Further, by virtue of the fact that m is the greatest common divisor of k and l, a and b must be relatively prime. Now consider the quantity $(k-l)$. Either $(k-l) < l$ or $l < (k-l)$. We establish the remainder of the proof for the case $(k-l) < 1$. Similar remarks could be constructed if $1 < (k-l)$. By the rule of inference introduced, we obtain $C^{(k-l)}CqpCrr$ from (6) and (7). Continuing the use of the rule, we can obtain $C^{l-(k-l)}CqpCrr$, $C^{l-2(k-l)}CqpCrr$, and so on, until we obtain $C^{l-\lambda(k-l)}CqpCrr$ with $l - \lambda(k-l)$ the smallest such number larger than

0. If $l-\lambda(k-l) = m$, the theorem is proved. Otherwise, since a and b are relatively prime, $l-\lambda(k-l) < k-l$, and $(k-l) \neq c(l-(k-l))$ for $c = 1, 2, \ldots$ If this were not so, since m divides $(k-l)$ and $(l-(k-l))$, both k and l would be multiples of $(k-l)$, and hence of md ($d = 2, 3, \ldots$), so that k and l would not be relatively prime after division by m, contrary to hypothesis. We use $(l-(k-l))$ to obtain $C^{(k-l)-(l-(k-l))}CqpCrr$, and so on, until either we reach $C^{m}CqpCrr$ or some sentence $C^{mn}CqpCrr$ which can be reduced by repeating the same procedure until $C^{m}CqpCrr$ is finally obtained.

Lemma 19. $C^{\mathrm{div}(\mathfrak{M})}CqpCrr$ *satisfies* $\mathfrak{M}$.

Proof. This follows immediately from Lemma 17 and 18 and the definition of $\mathrm{div}(\mathfrak{M})$.

Lemma 20. $\mathrm{div}(\mathfrak{M}) = \sigma$.

Proof. σ is given a value in (Red) such that $\sigma = \mathrm{df}((\mathrm{Red}),p)$ and (Red) satisfies $\mathfrak{M}$. σ is therefore some multiple of $\mathrm{div}(\mathfrak{M})$, say $k\ \mathrm{div}(\mathfrak{M})$. Since both σ and $\mathrm{div}(\mathfrak{M})$ are greater than 0, we need only show that $k = 1$. By the definition of (Red), $\rho + \sigma$ is the smallest number greater than ρ for which (Red) $CC^{\rho+\sigma}CpqrC^{\rho}Cpqr$ satisfies $\mathfrak{M}$. By Lemma 19, we have (1) $C^{\mathrm{div}(\mathfrak{M})}CqpCrr$ satisfies $\mathfrak{M}$. Repeating the derivation of (i) from (g) in Lemma 15, using Lemma 3 and (Syl_1), except that $\mathrm{div}(\mathfrak{M})$ replaces σ everywhere in the derivation, we can obtain (2) $CCC^{\mathrm{div}(\mathfrak{M})}CqpCrstCCrst$ from (1). Substituting in (2) ($\mathbf{S}$: q/p, p/q, r/Cpq, $s/C^{\rho+\sigma-\mathrm{div}(\mathfrak{M})-1}Cpqr$, $t/C^{\rho}Cpqr$), and using the identity $\sigma = k\ \mathrm{div}(\mathfrak{M})$, we obtain (3) $C(\mathrm{Red})CC^{(k-1)\mathrm{div}(\mathfrak{M})+\rho}CpqrC^{\rho}Cpqr$. Using modus ponens with (Red) on (3), we obtain that (4) $CC^{(k-1)\mathrm{div}(\mathfrak{M})+\rho}CpqrC^{\rho}Cpqr$ satisfies $\mathfrak{M}$. Now, unless $k-1 = 0$, the fact that (4) satisfies $\mathfrak{M}$ is incompatible with the stipulation used to define (Red) that ρ,σ are the smallest indices such that $CC^{\rho+\sigma}CCpqrC^{\rho}Cpqr$ satisfies $\mathfrak{M}$. (An easily proved property of exponents is required.) Therefore $k = 1$.

Lemma 21. Let α *satisfy* $\mathfrak{M}$ *and contain the variable* p *as a proper part. Let* β *be provable from* A_2 *where* β *is obtained from* α *if* p *is everywhere replaced by* q. *Then the sentences* $C^{\rho}CqpC^{\rho}Cpq\alpha$ *and* $C^{\rho}CpqC^{\rho}Cqp\alpha$ *are provable.*

Proof. If the assumptions of the Lemma hold, and $\mathrm{gr}(\alpha, p) = k$ and $\mathrm{ngr}(\alpha,p) = l$, then by Lemma 16 the statements (1) $C^{k+m}CqpC^{l+m}Cqp\alpha$ and (2) $C^{l+m}CpqC^{k+m}Cqp\alpha$ are provable, for any m. With a suitable choice of m, we have $k+m \equiv \rho \pmod{\sigma}$ and $l+m \equiv \rho \pmod{\sigma}$. This follows from the fact that since α satisfies $\mathfrak{M}$, and $k-l \equiv \mathrm{df}(\alpha,p)$, σ (which is by Lemma 20 identical with $\mathrm{div}(\mathfrak{M})$) must divide $k-l$. We therefore have $k \equiv l \pmod{\sigma}$. Using (Red) and Lemma 4, we can always obtain the sentences desired from (1) and (2) with suitable choice of m.

We now introduce a sentence-type which satisfies $\bar{\mathfrak{M}}$, and when added to A_2 to yield A_3, results in an axiomatization of the sentences satisfying $\bar{\mathfrak{M}}$, with A_3 as the axiom set. This sentence, to be referred to as (Fin_m), depends on the degree of $\mathfrak{M}$, and on ρ and σ. Consider sentences of the form $C^{\rho}Cp_kp_1C^{\rho}p_1p_kCqr$, where $k < l$, and $k, l = 1, 2, \ldots, m+1$. There will be

$m(m+1)/2$ sentences of this form, which we can designate $\phi_1, \phi_2, \ldots, \phi_{m(m+1)/2}$. ($\mathsf{Fin}_m$) is then the following:

$C^\sigma \phi_1 C^\sigma \phi_2, \ldots, C^\sigma \phi_{m(m+1)/2} C^{\sigma+1} \phi_{m(m+1)/2} C^{2\rho} CqqCqr$. (For example, with $m = 2$, we have (Fin_2):

$C^\sigma C^\rho Cp_1p_2 C^\rho Cp_2p_1 CqrC^\sigma C^\rho Cp_1p_3 C^\rho Cp_3p_1 CqrC^\sigma C^\rho Cp_2p_3 C^\rho Cp_3p_2 CqrC^{\sigma+1} C^\rho Cp_2p_3 C^\rho Cp_3p_2CqrC^{2\rho} CqqCqr$. Notice that we do not strictly adhere to our convention in which $p_1 = p$, $p = q$, and so on, because q and r are used in (Fin_m) as variables distinct from any of the p_i ($i = 1, 2, \ldots, m+1$).)

Lemma 22. (Fin_m) *satisfies* $\mathfrak{M}$.

Proof. First we derive some sentences from A_3:

(a) $CCpqC^\sigma C^\rho CrsC^\rho CsrC^\rho CtuC^\rho CutCvwCpq$. (Substitution in Lemma 15(h) (**S**: r/p, s/q, q/Crs, $p/C^{\rho-1}CrsC^\rho CsrC^\rho CtuC^\rho CutCvw$).)

(b) $C^\rho CpqC^\rho CqpCpp$. (Substitution in Lemma 9 (**S**: α/Cpp, k/ρ) yields $CCppC^\rho CpqC^\rho CqpCpp$, which in turn yields (b) after modus ponens with (Id).)

(c) $C^\rho CpqC^\rho CqpCrr$. (From (b) and $CCppCrr$ (Substitution in (Id*)) using Lemma 4.)

(d) $CCrsC^\rho CpqC^\rho CqpCrs$. (Substitution in Lemma 3 (**S**: $\alpha/$(c), $\beta/C^\rho CpqC^\rho CqpCrs$, γ/r, δ/s) yields C(c)(d). Modus ponens with (c) obviously then yields (d).

(e) $CCvwC^\rho CtuC^\rho CutCvw$. (Substitution in (d) (**S**: r/v, s/w, p/t, q/u).)

(f) $CC^\rho CtuC^\rho CutCvwC^\rho CrsC^\rho CsrC^\rho CtuC^\rho CutCvw$. (Substitution in (e) (**S**: v/Ctu, $w/C^{\rho-1}CtuC^\rho CutCvw$, t/r, u/s).)

(g) $CCpqC^{\sigma-1}C^\rho CrsC^\rho CsrCvwCC^\rho CtuC^\rho CutCvwCpq$. (Using Lemma 5, with $\alpha/$(a), we let (e) be $C\delta_1\gamma_1$. Replacing γ_1 by δ_1 at σ-1 odd places in (a), we obtain $CCpqC^{\sigma-1} C^\rho rsC^\rho srCvwCC^\rho CrsC^\rho CsrC^\rho CtuC^\rho CutCvwCpw$, as α_k of Lemma 5. Now, using (f), we obtain (g) as α_{k+1} of Lemma 5.)

Returning to (Fin_m), some pair of variables p_i, p_k ($i \neq k, i, k = 1, 2, \ldots, m+1$) must have equal values on any given valuation. As a result, (Fin_m) satisfies $\mathfrak{M}$ on any particular valuation, and hence on all valuations, that is, *satisfies* $\mathfrak{M}$ in the sense of the Lemma. This is shown only for the case where $m = 2$, and p_1 and p_2 are assigned the same value. (Fin_2) is then of the following form:

(1) $C^\sigma C^{2\rho} Cp_2p_2 CqrC^{2\sigma+1} C^\rho Cp_2p_3 C^\rho Cp_3p_2 CqrC^{2\rho} CqqCqr$.

If, in this sentence, we replace Cp_2p_2 with Cqq everywhere, we obtain a sentence (2) such that C(1)(2) and C(2)(1) are provable by (Id*) and Lemma 7. (2) satisfies $\mathfrak{M}$ since it is a substitution instance of

(3) $CCpqC^{\sigma-1} C^\rho CrsC^\rho CsrCvwCC^\rho tuC^\rho CutCvwC^{2\sigma} CxyCpq$.

(3) may be proved from Lemma 22(g) and Lemma 15(h) using Lemma 4. It is easily seen that other cases can be proved in a similar manner, and hence that (1), or (Fin_m) in the general case, satisfies $\mathfrak{M}$. We now add (Fin_m) to A_2 to obtain A_3 and state Lemma 23.

Lemma 23. All sentences which satisfy $\mathfrak{M}$ are provable from A_3.

Proof. The proof proceeds by establishing this inductive statement: If, for any $s \geqslant m$, all the s-dimensional (or less) sentences satisfying $\mathfrak{M}$ are provable, then the s+1-dimensional (or less) sentences satisfying $\mathfrak{M}$ are also provable. Let α be any s+1-dimensional sentence in which all of the variables $p_1, p_2, \ldots, p_{m+1}$ occur. Consider the sentences obtained from α by replacing occurrences of a variable p_k with another variable p_i where $p_i < p_k$. By hypothesis, all such sentences satisfy $\mathfrak{M}$ and are provable. By Lemma 21, therefore, all sentences of the form $C^{\rho}Cp_kp_iC^{\rho}Cp_ip_k\alpha$ are provable. If α is of the form $C\beta\gamma$, then by means of (Fin_m), it is possible to prove $C^{2\rho}C\beta\beta C\beta\gamma$, and therefore also $C\beta\gamma$. If α is of the form $N\delta$, then the sentence (1) $CC\delta\delta N\delta$ satisfies $\mathfrak{M}$ by substitution in (Transp_2). (1) is of the form $C\beta\gamma$ and is also s+1-dimensional. Therefore (1) and hence α is provable by the strategy just suggested. As α is provable no matter what its form, Lemma 23 is proved. This also suffices to establish the Hauptsatz, since the construction of A_3 completes the development of an axiom set Ax for the sentences satisfying the arbitrary matrix $\mathfrak{M}$.

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