322
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THE STRONG DECIDABILITY OF CUT-LOGICS I: PARTIAL PROPOSITIONAL CALCULI

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1. *Introduction*. Recent study of partial propositional calculi in which any number of applications of the deduction rules is allowed has shown that such calculi tend to be highly undecidable; i.e., problems may be constructed concerning such calculi which are of any given degree of unsolvability (cf. [3], [4], [5]). However, in the case of calculi in which the number of applications of the deduction rules is limited, the situation is rather the reverse. It is proved below that all such calculi with finite numbers of axiom schemata and the rule modus ponens (or equivalently with finite numbers of axioms and the rules modus ponens and simultaneous substitution or substitution) are decidable and decidable in a rather strong way. The second part of this paper will concern the generalization of the decidability results to other classes of calculi (modal logics, higher order calculi, etc.)

2. *Definitions*. For the purposes of this first paper, a *partial propositional calculus* shall mean a triple $\langle M,R,N\rangle$ where M is a finite set of well-formed formulae which are theorems of the classical propositional calculus, R is either the rule modus ponens (MP) or the rule MP together with the rule simultaneous substitution (SS), or the rule MP together with the rule substitution (S) and N is a non-negative integer or infinity (∞). If N is infinity, $\langle M, \mathsf{MP},N\rangle$ is to be thought of as the calculus with axiom schemata corresponding to the elements of M, and sole rule MP; $\langle M, (\mathsf{MP},\mathsf{SS}),N\rangle$ is to be thought of as the corresponding calculus with the elements of M as axioms and MP and SS as rules, and similarly for $\langle M, (\mathsf{MP},\mathsf{S}),N\rangle$. The corresponding calculi when N is finite are the same calculi with the restriction that the rule MP may be applied N or fewer times only. Thus each $\langle M,R,n\rangle$ represents a subset of the calculus represented by $\langle M,R,m\rangle$ for $n < m$, and also a subset of the calculus represented by $\langle M,R,\infty\rangle$. Usually we shall identify a triple and the calculus it represents whenever this causes no confusion. The calculi with N finite will be called cut-propositional calculi.

In general, the proofs below will be carried out for the case that the connectives present are implication ($\supset$) and a constant false (f), but the modifications for other sets of connectives should be reasonably easy for

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the reader to see and will be covered in the second part of the paper. Any calculus of any of the above types is called decidable if there exists an effective method, given any well-formed formula of the propositional calculus, for determining whether that formula is a theorem of the given calculus. Such a calculus will be called strongly decidable if one can effectively find a finite set of well-formed formulae such that the set of theorems of the calculus coincides with the set of simultaneous substitution instances of the given set of formulae. For example, it was proved in [1] and [2] that, while both of the calculi $\langle (P \supset (Q \supset P)), \mathsf{MP}, \infty\rangle$ and $\langle ((P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))), \mathsf{MP}, \infty\rangle$ are decidable, neither is strongly decidable. On the other hand, it is proved in [1] that $\langle (((P \supset f) \supset f) \supset P), \mathsf{MP}, \infty\rangle$ is strongly decidable.

3. *Statement of theorems and proofs of lemmas.*

Theorem A. *For all M and for all finite N, $\langle M,R,N\rangle$ is strongly decidable; i.e. all cut-propositional calculi are strongly decidable.*

One would expect that the proof of such a theorem would involve complications with MP; i.e., there should be some sort of cut-elimination involved, if perhaps carefully disguised. In fact, this is so, but the reduction to questions about SS is of such a simple nature that the main burden falls on the theorem below which seemingly concerns SS alone. Given a well-formed formula W, let $W!$ be the set of all simultaneous substitution instances of W. We call $W!$ the *closure under* SS *of* W.

Theorem B. *For any two well-formed formulae W and X, $W! \cap X!$ is representable as a finite union of $A! \cup B! \cup \ldots \cup N!$ and given W and X, we can effectively find $A, B, \ldots, N$.*

Since Theorem B is needed in the proof of Theorem A, we prove Theorem B first. For any well-formed formula W, we shall write AW for the antecedent of W and CW for the consequent of W, provided W is of the form $X \supset Y$, i.e., $AW \supset CW$. In more complicated circumstances we shall write AAW, ACW, etc., with the obvious meanings. Now Theorem B requires us, given two formulae W and X, to characterize the set of all formulae which are at the same time simultaneous substitution instances of W and of X. This procedure can be carried out in two steps, roughly first finding the common SS instances ignoring the fact that some variables occur more than once in W or in X and then taking into account these coincidences of variables. The following lemma may appear obvious, and follows easily by induction on the length of formulae, but it is essential for the effectiveness of the decision problem involved.

Lemma 1. *Given two well-formed formulae W and X, one may effectively tell if X is an* SS *instance of W; if X is such an instance, one can effectively assign to each variable in W the subformula of X which was substituted for it in W to get X.*

Proof. We do induction on the length of W. If W is of length one, it is either a variable or a constant. If W is a constant, X is not an SS instance unless it is that same constant. In that case, the constant corresponds to itself for the effective assignment. If W is a variable, then X is an SS instance of W and the whole formula X corresponds to the single variable of W. Now assume the lemma true for W of length n-1 or less, and suppose that W is of length n. Then if X is of length n-1 or less, it is not an SS instance, since SS always maintains or increases length. If X is of length n or greater, it is an SS instance of W if and only if AX is an SS instance of AW and CX is an SS instance of CW and the formulae assigned to variables occurring in both AW and CW by the two assignments (AX to AW and CX to CW) are the same. The necessity of the condition is immediate. Further, if the condition is satisfied, we can list, for each variable in W, the formula to be substituted for it in W to attain X by SS, so that X is an SS instance of W. But the lengths of AW and CW are both of necessity shorter than the length of W, so that the lemma follows by induction, the effective assignment desired for the variables in W being the union of the effective assignments for AW and CW which union is consistent in the sense of assigning only one formula to each variable by the above discussion. Q.E.D.

Given a well-formed formula W, define its length to be the total number of signs appearing in it, counting repetitions but not parentheses (so that the length of $(P \supset (Q \supset P))$ is 5), and its variable length to be the total number of variables appearing in the formula, counting repetitions (so that the variable length of $(P \supset (Q \supset P))$ is 3). If W has variable length n, define the generalized version of W to be the formula that results from W by replacing its variables from left to right by the first n variables in a fixed ordering of the variables. (For purposes of examples, we shall assume that this list begins with the 26 letters of the English alphabet in the usual order, so that the generalized version of $(P \supset (Q \supset P))$ is $(A \supset (B \supset C))$.) For the generalized version of a formula W we shall write $W^{\#}$, then W is an SS instance of $W^{\#}$ so that we have

Lemma 2. *To every formula W, there corresponds a unique generalized version $W^{\#}$ of which W is an* SS *instance. Given W we can effectively find $W^{\#}$ and can effectively establish the correspondence between the variables of $W^{\#}$ and W through which W arises as an* SS *instance of $W^{\#}$.* Q.E.D.

Lemma 3. *Given two formulae W and X, each of which has the property that no variable occurs in it more than once, $W! \cap X!$ is either empty or representable as $A!$ for some formula A, and given W and X, we can effectively find A or show that the intersection is empty.*

Proof: We do induction on the length of the shorter of W and X, which without loss of generality, we may assume to be W. If the length of W is one, either it is a constant or a variable. If it is a constant, either X is the same constant, a different constant, a variable standing alone, or a formula of length greater than one. If X is the same constant, say c, then take $A = c$. If X is a different constant, the intersection is empty. If X is a variable, take $A = c$. If the length of X is greater than one, all its substitution instances must contain a connective, so we take the intersection empty. The other possibility is that W is a variable. In that case, we take $A = X$.

Now assume that the lemma is true for all cases in which the shorter of the two formulae has length less than n, and assume that the length of W is n. Since we are in the case $n > 1$, W must be of the form $AW \supset CW$, and since the length of X is greater than or equal to the length of W, X must be of the form $AX \supset CX$. Hence, by the induction assumption, we can find a formula A such that $A! = AW! \cap AX!$ and formula B such that $B! = CW \cap CX$. Now $A \supset B$ is a formula such that $(A \supset B)!$ is $W! \cap X!$. First, if U is an SS instance of W and of X, its antecedent and consequent are SS instances of the antecedents and consequents of W and X, respectively. Second, since the antecedent and consequent of W and X have no variables in common, if U' is an SS instance of both the antecedents and U'' such an instance of both the consequents, then $U' \supset U''$ is an SS instance of both W and X as desired. We are now done, since the above process is effective at every step, either telling us that there is no instance in common or giving us a formula A such that the desired intersection is $A!$. Q.E.D.

Corollary. *Given two formulae W and X, one can effectively find a formula, call it $(W,X)^{\#}$ such that $W! \cap X! = (W,X)^{\#}!$.*

Proof: $W!$ and $X!$ both have the property that no variable occurs more than once. Q.E.D.

Lemma 4. *In Lemma 3, to each variable of W (of X), there corresponds a unique effectively determinable subformula of A such that A results from W (from X) by* SS *of the subformulae for the variables of W (of X).*

Proof: By induction, A is an SS instance of W and of X. Hence apply lemma 1. Q.E.D.

Corollary. *Given two formulae W and X, to each variable v of W and to each variable v' of X, one may effectively associate a finite collection C_v $(C_{v'})$ of subformulae of $(W,X)^{\#}$ such that C is the collection of those subformulae that are associated with the variables of $W^{\#}(X^{\#})$ which are associated with v.*

Proof: Lemmas 2 and 4. Q.E.D.

Note that, in the above process, since $W!$ and $X!$ have the property that no variable occurs more than once in them singly, by a change of variables (say, instead of using the first m variables for X, we use the variables from the first one not used in $W^{\#}$ on to form $X^{\#}$) we can assure that $(W,X)^{\#}$ also has the property that no variable occurs more than once in it. So, by another effective change of variables we may assume that if the variable length of $(W,X)^{\#}$ is p, then the variables of $(W,X)^{\#}$ from left to right are the first p variables.

We now form a collection of formulae, the *variable instances of* $(W,X)^{\#}$ as follows. If the variable length of $(W,X)^{\#}$ is p, there are p^p distinct sequences of length p with elements in the sequence chosen from the first p variables. Call the n-th such sequence, (listing the sequences in alphabetical order) $s_n = n_1, n_2, \dots, n_p$. Then the variable instances of $(W,X)^{\#}$ are the formulae resulting from it by the simultaneous substitution of $n_1, n_2, \dots, n_p$ for the first p variables in alphabetical order, respectively. Since, given

$(W,X)^\#$ and a particular variable instance, say S_n corresponding to the substitution of s_n, we have an effective correspondence between the variables of S_n and $(W,X)^\#$, by the above lemmas we have

Lemma 5. *Given two formulae W and X, to each variable v of W (of X) one may effectively associate a finite collection C' of subformulae of any variable instance S_n of $(W,X)^\#$ such that C' is the collection of those subformulae that result from the subformulae of C (cf. Corollary to Lemma 4) by the substitution associated with s_n.* Q.E.D.

Now, given W and X, call any of the corresponding S_n *W-good* (*X-good*) if for each variable in $W(X)$, the corresponding collection of subformulae given by Lemma 5 consists of exactly one formula, possibly repeated more than once. A formula which is both X-good and W-good will be called a *mutual* SS *instance of X and W.*

4. *Proofs of theorems.*

Proof of Theorem B: Let $A, B, \ldots, N$ be the collection of mutual SS instances of W and X, a collection effectively available given W and X. Suppose that I is an SS instance of one of the mutual SS instances, say of K. Then K is an SS instance of $W^\#$ and moreover an SS instance of W: namely for each variable v in W substitute the unique formula in the collection C' given in Lemma 5. Thus since I is an SS instance of an SS instance of W, it is an SS instance of W and so belongs to $W!$. Similarly, I belongs to $X!$, so that I belongs to $W! \cap X!$. Thus $A! \cup B! \cup \ldots \cup N! \subseteq W! \cap X!$. On the other hand, let I belong to $W! \cap X!$. Then, since I is an SS instance of $W!$ and of W and of X and W and X are SS instances of $(W,X)^\#$, I is an SS instance of $(W,X)^\#$. Hence, it is an instance of some of the variable instances of $(W,X)^\#$ (since this collection contains $(W,X)^\#$ itself). We need only show that it is an instance of one at the $A, B, \ldots, N$ chosen. Now let us start with $(W,X)^\#$ and identify its variables in such a way that we get a mutual SS instance of W and X of which I is also an SS instance. First, we may divide $(W,X)^\#$ into segments corresponding to the variables of W. Call these segments the first segment, the second segment, etc., going from left to right. If the first segment corresponds to a variable v_i in W which occurs elsewhere in W, we take the segments in $(W,X)^\#$ corresponding to the other occurrences of v_i in W and replace each one by the first segment. By the construction, this amounts to an SS in $(W,X)^\#$. Now we underline the first segment and all these changed segments. We now consider the first non-underlined segment, counting from the left and go through the same process, comparing it with all non-underlined segments to the right of it—there are none available to the left. We repeat this process until all segments are underlined. This is possible since there are only a finite number of segments. Call the resulting formula I'. This formula results from I by making the minimum number of identifications so that the result is an SS instance of W. Since I is an SS instance of $(W,X)^\#$ which is also an SS instance of W, it must have at least these identifications made and hence it is also an SS instances of I'. Similarly, starting with X and $(W,X)^\#$ we can get a formula I'', one of the

X-good S_n. Now begin again with $(W,X)^{\#}$ and its first variable. To get I', this variable was identified with a collection of variables occurring to its right. Replace each of these variables by this first variable. Do the same to those variables which were identified with this first variable in I'' (call the resulting formula I_1). Now consider the variable occurring in the second position for variables in the resulting formula. This will be either the second variable of $(W,X)^{\#}$ or else the first variable appearing again because of one of the substitutions made previously. At any rate, call this variable V and the variable to which it corresponds in $(W,X)^{\#}$, V'. Replace all the variables in I_1 which correspond to variables in $(W,X)^{\#}$ which were identified with V' to form I' or I'' by V, to get a formula I_2. Note that this new substitution does not change any of the variables previously changed. Now continue with this procedure until the variable corresponding to the rightmost variable of $(W,X)^{\#}$ has been reached. At this point, we have a formula I_n which is such that the minimal number of identifications have been made so that the given formula is an SS instance both of W and of X, i.e., so that it is a mutual SS instance of W and X. Also note that I is an SS instance of I_n since it is an SS instance of $(W,X)^{\#}$ and must have at least the identifications made that were made in $(W,X)^{\#}$ to make this formula an SS instance of W and of X (since it is an SS instance of both these formulae). Hence I is an SS instance of one of our $A, B, \ldots, N$ as desired. Q.E.D.

Given this theorem, the proof of Theorem A is not very difficult.

Proof of Theorem A: The proof will be by induction on the number N. The two cases, axioms and axiom schemata do not significantly differ since we are allowed as many applications as we desire of SS, no matter what N is. Also note that the use at S and SS are equivalent for our purposes, since any SS may be accomplished by a sufficient number of applications of S. Hence, for each $N\langle M,R,N\rangle$ is closed under SS and we need show only that we can find a finite set of wffs, the set of all of whose SS instances is $\langle M,R,N\rangle$. Call $\langle M,R,0\rangle$ the collection of SS instances of the axioms. Using the axioms as the finite set, we have that $\langle M,R,0\rangle$ is strongly decidable. Now the elements of $\langle M,R,1\rangle$ arise from those of $\langle M,R,0\rangle$ by the use of MP. Again, since unlimited use of SS is available, we know that $\langle M,R,1\rangle$ is closed under SS and need show only that we can find the appropriate finite set of formulae. This set includes the finite set found for $\langle M,R,0\rangle$ and certain other axioms arising from the axioms in $\langle M,R,0\rangle$ by the use of MP. Namely, for each pair (W,X) from the finite set for $\langle M,R,0\rangle$, we apply Theorem B to AW and X to get the finite set of formulae $A, B, \ldots, N$. In applying MP for the pair (W,X), we are covering the cases in which we take an SS instance W' of W as major premises and an SS instance X' of X as minor premise such that $AW' = X'$. The new theorem will be CW', but every such $X' = AW'$ is an SS instance of one of $A, B, \ldots, N$ and conversely, so that the new theorems are precisely the SS instances that result from the formulae obtained from CW by making the SS by which A resulted from AW, or from the formulae obtained from CW by making the SS by which B resulted from AW, etc. Call this finite set of formulae $(W,X)^*$. Then all of the new consequences are SS instances of some formulae in some $(W,X)^*$

and conversely. Hence, our new finite set of formulae is the union of the $(W,X)^*$ and the finite set of formulae available for $\langle M,R,0\rangle$. Given any $\langle M,R,N\rangle$, assume that it has the given structure, i.e., that there exists an appropriate finite set of formulae. Then applying the above process, we get another finite collection of formulae which will do for $\langle M,R,N+1\rangle$, so that the theorem follows by induction. Q.E.D.

The proofs in this case may seem quite specific, in that we limited our calculi to those with $\supset$ as the only available connective. In fact, this is no limitation, as the second part of this paper will demonstrate.

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