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SOME COMPLETENESS RESULTS FOR INTERMEDIATE PROPOSITIONAL LOGICS

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1. In my paper "Implicationless wffs of **IC**" [4] I made use of one of the results of V. A. Jankov, which were stated by him without proof in [3]. R. Harrop in a recent article [2] has commented on their importance, so it is perhaps worthwhile to supply a proof. In addition a new and much simpler proof of an older result of Dummett [1] is presented.

2. Intermediate Propositional Logics (**IPL**'s) can be characterized in either of two ways. Firstly *syntactically*: let S be the set of **IPL**'s, K the set of classically valid wffs and I the set of intuitionistically valid wffs. Then $S = \{L : I \subseteq L \subseteq K\}$. If **IC** is Heyting's axiomatization of I then we can obtain axiomatizations for each $L \varepsilon S$ by augmenting **IC** with a set of new axioms A, $A \subset K$. We write such axiomatizations as $<$**IC**,$A>$. It is clear that one logic may have a number of different axiomatizations, and for this reason we distinguish between the axiomatizations and the logic.

Secondly we can characterize S *semantically*. By an *algegra* I mean a pseudo-complemented lattice. Let **J** be the direct product of all the algebras in the Jaśkowski sequence, and B(**J**) the set of all subalgebras of **J**. If $\mathcal{L} \varepsilon B(\mathbf{J})$ then $\mathcal{L}$ will be said to admit an interpretation of an **IPL**,L, if under the normal mapping (for details [5]) $\phi : L \rightarrow \mathcal{L}$ for each $P \varepsilon L$ $\phi(P)$ vanishes identically in $\mathcal{L}$. We can say that $\mathcal{L}$ is a *model* for L. If $\phi(P)$ vanishes identically in $\mathcal{L}$ iff $P \varepsilon L$, then we say that $\mathcal{L}$ is a *characteristic model* for L. L will be said to be *complete with respect* to $\mathcal{L}$.

It is well known that there is an infinite sequence of (Boolean) algebras of 2^k elements, $k = 1, 2, \ldots$ each of which is characteristic for K. From the viewpoint of logic the difference between these algebras is inessential.[1] To deal with such cases we define an equivalence relation on B(**J**). We put, for $\mathcal{L}_1, \mathcal{L}_2 \varepsilon B(\mathbf{J})$, $\mathcal{L}_1 \equiv \mathcal{L}_2$ if $\mathcal{L}_1$ and $\mathcal{L}_2$ are characteristic for the same logic L. Let $\overline{B(\mathbf{J})}$ be the resulting set of equivalence classes. If $\mathcal{L}$ is a model for a logic L, then we say that $\overline{\mathcal{L}}$ is a *model-set* for L. Similarly if $\mathcal{L}$ is a characteristic model for L, we say that $\overline{\mathcal{L}}$ is the *characteristic model-set* for L. S is partially ordered under set inclusion. We can partially order

1. A. S. Troelstra pointed out the need for taking this into account.

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$\overline{B(\mathbf{J})}$ thus: if $\overline{\mathscr{L}}_1, \overline{\mathscr{L}}_2 \varepsilon \overline{B(\mathbf{J})}$ then we put $\mathscr{L}_1 \subseteq' \mathscr{L}_2$ iff every expression $\phi(P)$ which fails in $\mathscr{L}_2$ fails also in $\mathscr{L}_1$. $<\overline{B(\mathbf{J})}, \subseteq'>$ is order-isomorphic to $<S, \subseteq>$. After Troelstra [6] we say that a logic K *precedes* a logic L, if $K \subset L$. If L covers K, then K is said to be an *immediate predecessor* of L. The notion of *successor* and *immediate successor* can be defined analogously, and the terminology can be transferred to cover $\overline{B(\mathbf{J})}$ as well.

3. We now prove the theorem from which the results of this paper will follow.

Theorem 1. *Let L be an arbitrary* **IPL** *and let $<\mathbf{IC}, A>$ be an axiomatization of L. Then $\overline{\mathscr{L}} \varepsilon \overline{B(\mathbf{J})}$ is the characteristic model-set for L iff $\overline{\mathscr{L}}$ is a model-set for A, and for each immediate predecessor $\overline{\mathscr{K}}$ of $\overline{\mathscr{L}}$ there exists some $P \varepsilon A$ such that $\phi(P)$ fails in $\overline{\mathscr{K}}$.*

Proof. (Sufficiency) Suppose that $\overline{\mathscr{L}}$ is a model-set for A, and that for each immediate predecessor $\overline{\mathscr{K}}$ of $\overline{\mathscr{L}}$ there is some $P \varepsilon A$ such that $\phi(P)$ fails in $\overline{\mathscr{K}}$, and that contrary to the theorem $\overline{\mathscr{L}}$ is not characteristic for L. Then either 1) for some P, $P \varepsilon L$, $\phi(P)$ fails in $\overline{\mathscr{L}}$ or 2) for some $P \not\varepsilon L$, $\phi(P)$ vanishes identically in $\overline{\mathscr{L}}$. As to case 1) if $P \varepsilon L$ then P is derivable from $<\mathbf{IC}, A>$. But if $\phi(P)$ fails in $\overline{\mathscr{L}}$, then since $\overline{\mathscr{L}}$ is a model-set for A, P is not derivable from $<\mathbf{IC}, A>$. Contradiction. In case 2) if $P \not\varepsilon L$ then P is not derivable from $<\mathbf{IC}, A>$, in which case there is some element $\overline{\mathscr{J}} \varepsilon \overline{B(\mathbf{J})}$ such that $\overline{\mathscr{J}}$ is a model-set for $<\mathbf{IC}, A>$ and $\phi(P)$ fails in $\overline{\mathscr{J}}$. $\overline{\mathscr{J}}$ is either incomparable with (i.e. $\overline{\mathscr{J}} \not\subset \overline{\mathscr{L}}$ and $\overline{\mathscr{L}} \not\subset \overline{\mathscr{J}}$) or precedes $\overline{\mathscr{L}}$. In both cases this leads to a contradiction. In the former $\overline{\mathscr{L} \times \mathscr{J}}$ precedes $\overline{\mathscr{L}}$. Hence for some P, $P \varepsilon A$ $\phi(P)$ fails in $\overline{\mathscr{L} \times \mathscr{J}}$ but this contradicts the assumption that $\overline{\mathscr{L}}$ and $\overline{\mathscr{J}}$ are model-sets for A. If in the second case $\overline{\mathscr{J}} \subset \overline{\mathscr{L}}$ then $\overline{\mathscr{J}} \subseteq \overline{\mathscr{K}}$ and hence once again for some $P \varepsilon A$, $\phi(P)$ fails in $\overline{\mathscr{J}}$, which contradicts the assumption that $\overline{\mathscr{J}}$ is a model set for $<\mathbf{IC}, A>$.

(Necessity) Suppose $\overline{\mathscr{L}} \varepsilon \overline{B(\mathbf{J})}$ is characteristic for L, and that for at least one immediate predecessor $\overline{\mathscr{K}}$ of $\overline{\mathscr{L}}$ for each $P \varepsilon A$, $\phi(P)$ vanishes identically. Since $\overline{\mathscr{K}}$ immediately precedes $\overline{\mathscr{L}}$ there is some wff $P \varepsilon L$ such that $\phi(P)$ fails in $\overline{\mathscr{K}}$. But then P is not derivable from $<\mathbf{IC}, A>$. But $\phi(P)$ does vanish identically in $\overline{\mathscr{L}}$ and hence if $\overline{\mathscr{L}}$ is characteristic for L then P is derivable from $<\mathbf{IC}, A>$. Hence we obtain a contradiction.

4. We now apply theorem 1 in the proof of Jankov's results. Consider the following sets of algebras:

1) The Jaśkowski-sequence, J_1 = 2-element Boolean algebra. $i = 1, 2, \ldots$ $\mathbf{J}_{i+1} = \Gamma(J_i)^i$ where for an algebra L, $\Gamma(L)$ is the result of applying to L the ordinal addition from above of a one element algebra.
2) $\mathbf{J}_i^*$, $i = 1, 2, 3 \ldots$ where $\mathbf{J}_i^*$ is obtained from $\mathbf{J}_i$ by taking the least element of $\mathbf{J}_i$ and all those elements of $\mathbf{J}_i$ whose pseudo-complements are equal to the least element, preserving among these elements the ordering they had in $\mathbf{J}_i$.
3) $\mathbf{E}_i = \Gamma(\mathbf{J}_i)^i$ $i = 1, 2, 3, \ldots$

Let $\mathbf{J}$, $\mathbf{J}^*$, and $\mathbf{E}$ denote the direct product of all the algebras in the relevant sequences, and let $\overline{\mathbf{J}}$, $\overline{\mathbf{J}^*}$ and $\overline{\mathbf{E}}$ be the corresponding model-sets.

Jankov states as his first theorem:

Theorem A

1) $<\mathsf{IC},A>$ *is complete with respect to the model-set* $\overline{\mathsf{J}}_1$, *iff* $\overline{\mathsf{J}}_1$ *is a model-set for* A, *and for some* $P\varepsilon A$, $\phi(P)$ *fails in* $\overline{\mathsf{J}}_2$
2) $<\mathsf{IC},A>$ *is complete with respect to the model-set* $\overline{\mathsf{J}}^*$, *iff* $\overline{\mathsf{J}}^*$ *is a model-set for* A, *and for some* $P\varepsilon A$, $\phi(P)$ *fails in* $\overline{\mathsf{E}}_2$
3) $<\mathsf{IC},A>$ *is complete with respect to the model-set* $\overline{\mathsf{E}}$ *iff* $\overline{\mathsf{E}}$ *is a model-set for* A *and for some* $P\varepsilon A$, $\phi(P)$ *fails in* $\overline{\Gamma(\mathsf{J}_2)}$
4) $<\mathsf{IC},A>$ *is complete with respect to the model-set* $\overline{\mathsf{J}}_2$ *iff* $\overline{\mathsf{J}}_2$ *is a model-set for* A *and for some* P_1, $P_2\varepsilon A$ $\phi(P_1)$ *and* $\phi(P_2)$ *fails in* $\overline{\Gamma(\mathsf{J}_2)}$ *and* $\overline{\mathsf{E}}_2$ *respectively.* (P_1 *may coincide with* P_2).

Proof. By inspection of the usual diagrams we can check that $\overline{\mathsf{J}}_2$ is the only immediate predecessor of $\overline{\mathsf{J}}_1$, $\overline{\mathsf{E}_2 \times \mathsf{J}^*}$ is the only immediate predecessor of $\overline{\mathsf{J}}^*$, $\overline{\mathsf{E} \times \Gamma(\mathsf{J}_2)}$ is the only immediate predecessor of $\overline{\mathsf{E}}$, and $\overline{\Gamma(\mathsf{J}_2)}$ and $\overline{\mathsf{E}}_2$ are the only two immediate predecessors of $\overline{\mathsf{J}}_2$. The theorem then follows by theorem 1.

Consider the following axiomatizations of ILP's.

$K = <\mathsf{IC}, \neg\neg a \supset a>$
$L = <\mathsf{IC}, \neg\neg a \vee \neg a>$
$M = <\mathsf{IC}, (\neg\neg a \wedge (a \supset b) \wedge (b \supset a) \supset a)) \supset b>$
$N = <\mathsf{IC}, \neg\neg a \vee \neg a, (\neg\neg a \wedge (a \supset b) \wedge (b \supset a) \supset a)) \supset b>$

Jankov then states as his second theorem.

Theorem B.

1) $\overline{\mathsf{J}}_1$ *is the characteristic model-set for* K
2) $\overline{\mathsf{J}}^*$ *is the characteristic model-set for* L
3) $\overline{\mathsf{E}}$ *is the characteristic model-set for* M
4) $\overline{\mathsf{J}}_2$ *is the characteristic model-set for* N

Proof. It can be checked that the indicated axioms satisfy the conditions laid down in the previous theorem.

Theorem C.

1) *An axiomatization* $<\mathsf{IC},A>$ *is deductively equivalent to* K *iff* $\overline{\mathsf{J}}_1$ *is a model-set for* A *and for some* $P\varepsilon A$, $\phi(P)$ *fails in* $\overline{\mathsf{J}}_2$.
2) *An axiomatization* $<\mathsf{IC},A>$ *is deductively equivalent to* L *iff* $\overline{\mathsf{J}}^*$ *is a model-set for* A, *and for some* $P\varepsilon A$, $\phi(P)$ *fails in* $\overline{\mathsf{E}}_2$.
3) *An axiomatization* $<\mathsf{IC},A>$ *is deductively equivalent to* M *iff* $\overline{\mathsf{E}}$ *is a model-set for* A *and for some* $P\varepsilon A$, $\phi(P)$ *fails in* $\overline{\Gamma(\mathsf{J}_2)}$.
4) *An axiomatization* $<\mathsf{IC},A>$ *is deductively equivalent to* N *iff* $\overline{\mathsf{J}}_2$ *is a model-set for* A *and for some* P_1 *and* $P_2\varepsilon A, \phi(P_1)$ $\phi(P_2)$ *fail in* $\overline{\Gamma(\mathsf{J}_2)}$ *and* $\overline{\mathsf{E}}_2$ *respectively.*

Proof. Two axiomatizations $<\mathsf{IC},A>$ and $<\mathsf{IC},B>$ are deductively equivalent iff the axiom-set B is derivable from $<\mathsf{IC},A>$ by means of the usual rules of inference, and conversely the axiom-set A is derivable from $\langle \mathbf{IC}, B \rangle$. As we remarked initially to each **IPL** there may correspond several distinct axiomatizations. However, if we gather the distinct axiomatizations into equivalence classes under the relation of deductive equivalence then the axiomatizations and **IPL**'s are in one-to-one correspondence. Further, there will be a one-to-one correspondence between the equivalence classes of axiomatizations and characteristic model-sets, such that if $\bar{A}$ is an equivalence class of axiomatizations then A will belong to $\bar{A}$ iff it is complete with respect to a unique model-set $\mathcal{Z}$. Jankov's third theorem follows from this observation.

5. Now we give a new proof of Dummett's result that $\mathbf{LC} = \langle \mathbf{IC}, (a \supset b) \vee (b \supset a) \rangle$ is complete with respect to the model-set $\mathcal{Z}$ where $\mathcal{L}$ is the algebra defined on the set $\{0, 1, 2 \ldots \ldots \omega\}$ where $0 > 1 > 2 > \ldots n > n{+}1 > \ldots > \omega$.

By inspection we can check that the only immediate predecessors of $\mathcal{Z}$ are $\overline{\mathcal{L} \times \mathcal{K}_1}$ and $\overline{\mathcal{L} \times \mathcal{K}_2}$ where $\mathcal{K}_1$ is $\mathbf{E}_2$ and $\mathcal{K}_2$ is obtained by an ordinal addition of a one element algebra at the bottom of $\mathbf{E}_2$. To verify Dummett's result one need only note that $\phi((a \supset b) \vee (b \supset a))$ fails in $\overline{\mathcal{K}}_1$ and $\overline{\mathcal{K}}_2$ and vanishes identically in $\mathcal{Z}$.

The methods used in this paper are obviously very powerful in obtaining completeness results for whole classes of **IPL**'s. I hope to extend these methods in a forthcoming paper.

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