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OMITTING THE REPLACEMENT SCHEMA IN RECURSIVE ARITHMETIC

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We shall consider a formalisation of primitative recursive arithmetic similar to that in Goodstein [1] in which the replacement schema (Goodstein's **Sb**$_2$) is deduced from certain special cases using a double recursive uniqueness rule. In Lee [2] the replacement schema is deduced from an infinite number of instances, whereas here we need only a finite number of instances. In our formalisation we make use of only special cases of the singly recursive uniqueness rule.

§1. **Axioms and Rules of Inference.** As axioms we take all definitions by recursion and substitution. Among these we specify

$$\begin{aligned} a + 0 &= a \\ a + Sb &= S(a + b) \\ a.0 &= 0 \\ a.Sb &= a.b + a \\ a \dot{-} 0 &= a \\ 0 \dot{-} a &= 0 \\ Sa \dot{-} Sb &= a \dot{-} b \end{aligned}$$

We shall denote recursive functions by $f,g,h, \ldots$, and recursive terms by $A,B,C, \ldots$. We take the following rules of inference

$$fx = gx \vdash fA = gA$$

$$\begin{aligned} A = B &\vdash SA = SB \\ A = B &\vdash x + A = x + B \\ A = B &\vdash A + x = B + x \\ A = B &\vdash x \dot{-} A = x \dot{-} b \\ A = B &\vdash A \dot{-} x = B \dot{-} x \end{aligned}$$

$$A = B, A = C \vdash B = C \qquad \ldots\ldots \quad (\mathbf{T})$$

$$fSx = fx \vdash fx = f0 \qquad \ldots\ldots \quad (\mathbf{E}_1)$$

$$f0 = g0, fSx = Sfx, gSx = Sgx \vdash fx = gx \qquad \ldots\ldots \quad (\mathbf{S})$$

$$f0 = g0, fSx = gSx \vdash fx = gx \qquad \ldots\ldots \quad (\mathbf{F})$$

$$f(m,0) = g(m,0), f(0,n) = g(0,n), f(Sm,Sn) = f(m,n), g(Sm,Sn) = g(m,n) \vdash f(m,n) = g(m,n) \qquad \ldots\ldots \quad (\mathbf{E}_2)$$

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Abbreviations

$$
\begin{aligned}
1 &\equiv S0\\
PA &\equiv A \dot{-} 1\\
\alpha A &\equiv 1 \dot{-} (1 \dot{-} A)\\
|A,B| &\equiv (A \dot{-} B) + (B \dot{-} A)\\
A = B \to C = D &\equiv (1 \dot{-} |A,B|)\,|C,D| = 0
\end{aligned}
$$

§2. **Development.** We now develop the theory to the point in which we have proved the axioms and rules of inference of Goodstein's system $\Re_1$. This means we must prove the key equation

$$a + (b \dot{-} a) = b + (a \dot{-} b)$$

and the substitution theorem

$$a = b \to fa = fb$$

which will give us the replacement schema by substitution and modus ponens.

First we show equality is an equivalence relation. We obtain $A + 0 = A$ by substitution and hence $A = A$ by **T**. Then we get $A = B \vdash B = A$ by taking C as A in **T**. We shall henceforth make free use of these properties of equality, the substitution schema, and replacement in functions formed from S, $+$, and $\dot{-}$ by substitution. Whenever an equation follows by one of the uniqueness rules we shall name the rule on the right.

$$
\begin{aligned}
Sa &= S(a + 0) = a + 1\\
PSa &= Sa \dot{-} S0 = a \dot{-} 0 = a\\
a \dot{-} a &= 0 && \mathbf{E}_1\\
\alpha 0 &= 0\\
\alpha Sa &= 1\\
0 + a &= a && \mathbf{S}\\
Sb + a &= S(b + a) && \mathbf{S}\\
b + a &= a + b && \mathbf{S}\\
a \dot{-} Sb &= P(a \dot{-} b) && \mathbf{E}_2
\end{aligned}
$$

We can now prove the key equation.

Theorem $a + (b \dot{-} a) = b + (a \dot{-} b)$

Proof Let $f(a,b) \equiv a + (b \dot{-} a)$

then $f(a,b) = f(Pa,Pb) + \alpha(a + b)$ **F**

$f(a \dot{-} n, b \dot{-} n) + \phi(n,a,b) = f(a \dot{-} Sn,\ b \dot{-} Sn) + \phi(Sn,a,b)$

where $\phi(0,a,b) = 0$

and $\phi(Sn,a,b) = \alpha((a \dot{-} n) + (b \dot{-} n)) + \phi(n,a,b)$

since $(a + \alpha b) + c = a + (\alpha b + c)$ **F**

$\therefore$ $f(a,b) = (a \dot{-} b) + \phi(b,a,b)$

Similarly $b + (a \dot{-} b) = (a \dot{-} b) + \phi(b,a,b)$

and so $a + (b \dot{-} a) = b + (a \dot{-} b)$ Q.E.D.

$$(a + c) \dot{-} (b + c) = a \dot{-} b \qquad \mathbf{E}$$
$$(a + c) \dot{-} c = a$$
$$c \dot{-} (b + c) = 0$$
$$|a + c, b + c| = |a,b|$$

$A + B = 0 \vdash A = 0,$ since $A = (A + B) \dot{-} B$.
$|A,B| = 0 \vdash A = B,$ since $A = A + (B \dot{-} A) = B + (A \dot{-} B) = B$.
$A = B \vdash |A,B| = 0,$ since $A \dot{-} B = 0 = B \dot{-} A$.

$f0 = g0,\ fSx = fx + A,\ gSx = gx + A \vdash fx = gx$ **P**
follows by $\mathbf{E}_1$, since $|fSx, gSx| = |fx, gx|$.

$$A = B \vdash Ac = Bc \qquad \mathbf{P}$$
$$a + (b + c) = (a + b) + c \qquad \mathbf{P}$$
$$0.a = 0 \qquad \mathbf{P}$$
$$Sb.a = b.a + a \qquad \mathbf{P}$$
$$a.b = b.a \qquad \mathbf{P}$$
$$a.(b + c) = ab + ac \qquad \mathbf{P}$$

We can now apply replacement to any function formed by substitution from S, $+$, $.$, and $\dot{-}$.

$$c(a \dot{-} b) = ca \dot{-} cb \qquad \mathbf{E}_2$$
$$c|a,b| = |ca,cb|$$

Deduction Theorem *If* $A = B \vdash C = D$ *with no substitution or recursion in the variables of the hypothesis, then* $A = B \rightarrow C = D$.

Proof We multiply the proof throughout by $R \equiv 1 \dot{-} |A,B|$ in order to obtain a proof of $A = B \rightarrow C = D$.

$$RA = RB, \text{ since } |RA,RB| = R|A,B| = 0,$$

and the rules of inference remain valid.
E.g. $RA = RB \vdash R(A \dot{-} c) = R(B \dot{-} c)$

$$Rf0 = Rg0,\ RfSx = RSfx,\ RgSx = RSgx \ \vdash Rfx = Rgx \qquad \mathbf{P}$$
since $RfSx = Rfx + R$, and $RgSx = Rgx + R$. Q.E.D.

Induction Theorem $p0 = 0,\ (1 \dot{-} pn)pSn = 0 \vdash pn = 0$

Proof Let $q0 = 1$ and $qSn = qn(1 \dot{-} pn)$,
then $qSSn = qn(1 \dot{-} pn)\,(1 \dot{-} pSn) = qSn$
and so $qSn = 1$. $\mathbf{E}_1$
$\therefore$ $pn = qSn.pn = 0$ Q.E.D.

Double Induction Theorem $p(m,0) = 0 = p(0,n),\ (1 \dot{-} p(m,n))p(Sm,Sn) = 0 \vdash p(m,n) = 0$

Proof Let $q(m,0) = 1 = q(0,n)$ and $q(Sm,Sn) = q(m,n)(1 \dot{-} p(m,n))$,
then $q(SSm,SSn) = q(m,n)(1 \dot{-} p(m,n))(1 \dot{-} p(Sm,Sn)) = q(Sm,Sn)$
and so $q(Sm,Sn) = 1$. $\mathbf{E}_2$
$\therefore$ $p(m,n) = q(Sm,Sn)\,.\,p(m,n) = 0$ Q.E.D.

Robinson has shown in [3] that all primitive recursive functions are generated from 0, x, Sx, $x + y$, and $x \dot{-} y$ by substitution and the recursion without parameter

$$f0 = 0$$
$$fSx = g(x, fx).$$

The substitution theorem follows since it obviously persists under substitution and the following theorem shows it persists under the above recursion.

Theorem $a = b \rightarrow (c = d \rightarrow g(a,c) = g(b,d)) \vdash a = b \rightarrow fa = fb$

Proof by double induction

$$a = 0 \rightarrow fa = f0 \qquad \mathbf{F}$$
$$0 = b \rightarrow f0 = fb \qquad \mathbf{F}$$
$$(a = b \rightarrow fa = fb) \rightarrow (Sa = Sb \rightarrow fSa = fSb)$$

follows by the deduction theorem from

$Sa = Sb,\ a = b \rightarrow fa = fb \vdash g(a, fa) = g(b, fb) \vdash fSa = fSb$

which is true since $Sa = Sb \rightarrow a = b$ and $a = b \rightarrow (fa = fb \rightarrow g(a,fa) = f(b,fb))$ by hypothesis. Q.E.D.

§3. **Alternative Systems.** First we note that we deduced the key equation by applying **F** only to functions formed by substitution from S, $+$, and $\dot{-}$ and so we need assume **F** only in these cases and then derive the general case as Goodstein does in $\Re_1$.

If we assume $a + b = b + a$ then we may discard **S**, since we may prove **P** as before and deduce **S**.

If we assume $a \dot{-} Sb = P(a \dot{-} b)$ then we need only assume the homogeneous case of $\mathbf{E}_2$.

$$f(m,0) = 0 = f(0,n), f(Sm, Sn) = f(m,n) \vdash f(m,n) = 0,$$

since we may prove $|A, B| = 0 \Longleftrightarrow A = B$ as before and deduce $\mathbf{E}_2$.

If we assume the key equation then the only uniqueness rules we need assume are

$$f0 = 0, fSx = fx \vdash fx = 0,$$

and

$$f(m,0) = 0 = f(0,n), f(Sm,Sn) = f(m,n) \vdash f(m,n) = 0,$$

since we may prove $\mathbf{E}_1$ and $|A,B| = 0 \Longleftrightarrow A = B$ as Goodstein does in $\Re_1$ and deduce $\mathbf{E}_2$ and **S** by considering $|fx, gx|$. I suspect that any finite number of instances of replacement are inadequate to derive recursive arithmetic without a double recursive uniqueness rule.

REFERENCES

[1] Goodstein, R. L., Logic-free formalisation of recursive arithmetic, *Mathematica Scandinavia*, v. 2, 1954, pp. 247-61.

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[3] Robinson, R. M., Primitive recursive functions, *Bulletin of the American Mathematical Society*, v. 53, 1947, pp. 925-42.

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