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FORMAL NONASSOCIATIVE NUMBER THEORY

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1. Introduction. The *logarithmetic*, first studied by I.M.H. Etherington (see, for example, [2]), of a nonassociative algebra has been found to bear some resemblance to the arithmetic of natural numbers. In [1], Evans has characterized this logarithmetic (i.e. the arithmetic of the indices of powers of the general element in a nonassociative algebra) by a set of axioms analogous to Peano's axioms and calls the resulting system "non-associative number theory."

In this paper, we shall formalize these "Peano-like" axioms and develop some of the properties of nonassociative number theory as theorems of the formal theory. In the last section it will be shown that formal nonassociative number theory, **N**, is both essentially undecidable and incomplete. This is accomplished by showing that **N** contains an essentially undecidable subtheory.

Few of the proofs of the theorems of **N** have all of the steps given. However, with the metamathematical remarks given, it should be an easy matter for the interested reader to supply complete proofs.

2. An axiom system for nonassociative number theory. We define **N** (formal nonassociative number theory) to be the first-order theory whose only individual constant is a_1, whose only predicate letter is A_1^2, and whose only function letters are f_1^2, f_2^2, and f_3^2. We write 1 for a_1, $x_1 = x_2$ for $A_1^2(x_1,x_2)$, $x_1 + x_2$ for $f_1^2(x_1,x_2)$, $x_1 \cdot x_2$ for $f_2^2(x_1,x_2)$, *and* $x_1^{x_2}$ for $f_3^2(x_1,x_2)$. The proper axioms of **N** are the following:

(N1) $x_1 = x_2 \supset (x_1 = x_3 \supset x_2 = x_3)$
(N2) $x_1 = x_2 \supset (x_1 + x_3 = x_2 + x_3)$
(N3) $x_1 = x_2 \supset (x_3 + x_1 = x_3 + x_2)$
(N4) $x_1 + x_2 \neq 1$
(N5) $x_1 + x_2 = x_3 + x_4 \supset (x_1 = x_3 \wedge x_2 = x_4)$
(N6) $x_1 \cdot 1 = x_1$
(N7) $x_1 \cdot (x_2 + x_3) = x_1 \cdot x_2 + x_1 \cdot x_3$
(N8) $x_1^1 = x_1$
(N9) $x_1^{x_2+x_3} = x_1^{x_2} \cdot x_1^{x_3}$
(N10) (Nonassociative Induction):

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For any wf $\mathcal{A}$ *of* **N**,

$$\vdash \mathcal{A}(1) \supset ((x_1)(x_2)(\mathcal{A}(x_1) \wedge \mathcal{A}(x_2) \supset \mathcal{A}(x_1 + x_2)) \supset (x_1)\mathcal{A}(x_1))$$

By using generalization and the "particularization rule," $(x)\mathcal{A}(x) \vdash \mathcal{A}(t)$, where t is free for x in $\mathcal{A}(x)$, one can easily prove the

Lemma. For any terms t, s, r and u of **N**, *the following wfs are theorems of* **N**.

(N1') $t = r \supset (t = s \supset r = s)$
(N2') $t = r \supset (t + s = r + s)$
(N3') $t = r \supset (s + t = s + r)$
(N4') $t + r \neq 1$
(N5') $t + r = s + u \supset (t = s \wedge r = u)$
(N6') $t \cdot 1 = t$
(N7') $t \cdot (r + s) = t \cdot r + t \cdot s$
(N8') $t^1 = t$
(N9') $t^{r+s} = t^r \cdot t^s$

3. **N** as a first-order theory with equality.

Proposition 1. If t, r, and s are terms of **N** *then the following wfs are theorems of* **N**.

(a) $t = t$
(b) $t = r \supset r = t$
(c) $t = r \supset (r = s \supset t = s)$
(d) $r = t \supset (s = t \supset r = s)$
(e) $1 \cdot t = t$
(f) $t = r \supset t \cdot s = r \cdot s$
(g) $t = r \supset s \cdot t = s \cdot r$
(h) $t = r \supset t^s = r^s$
(i) $t = r \supset s^t = s^r$
(j) $t \neq 1 \supset (E\, x_1)(E\, x_2)(t = x_1 + x_2)$

Proof:
(a) Use (N1') and (N6').
(b) Use (N1') and part (a).
(c) Use (N1') and part (b).
(d) Use parts (c) and (b).
(e) Apply (N10) to the wf $\mathcal{A}(x_1)$: $1 \cdot x_1 = x_1$.
(f) Apply (N10) to the wf $\mathcal{A}(x_3)$: $x_1 = x_2 \supset x_1 \cdot x_3 = x_2 \cdot x_3$.
(g) This can be proved by several applications of (N10).

An outline of the proof is as follows. Apply (N10) to the wf $\mathcal{A}(x_2)$: $1 = x_2 \supset x_3 \cdot 1 = x_3 \cdot x_2$ and thus prove

$$\vdash (x_2)(1 = x_2 \supset x_3 \cdot 1 = x_3 \cdot x_2)$$

Denote by $\mathcal{B}(x_1)$ the wf

$$(x_2)(x_1 = x_2 \supset x_3 \cdot x_1 = x_3 \cdot x_2)$$

Then prove that

$$\mathcal{B}(x_1),\ \mathcal{B}(x_4),\ x_1 + x_4 = x_5 + x_6 \vdash x_3(x_1 + x_4) = x_3(x_5 + x_6)$$

Then

$$\mathcal{B}(x_1),\ \mathcal{B}(x_4) \vdash \mathcal{C}(x_5 + x_6)$$

where $\mathcal{C}(x_2)$ is the wf

$$x_1 + x_4 = x_2 \supset x_3 \cdot (x_1 + x_4) = x_3 \cdot x_2$$

By (N4) and a tautology,

$$\vdash \mathcal{C}(1)$$

Hence by tautologies and (N10),

$$\mathcal{B}(x_1),\ \mathcal{B}(x_4) \vdash (x_2)\ \mathcal{C}(x_2)$$

Hence

$$\mathcal{B}(x_1),\ \mathcal{B}(x_4) \vdash \mathcal{B}(x_1 + x_4)$$

and by (N10),

$$\vdash (x_1)\ \mathcal{B}(x_1)$$

(h) Apply (N10) to the wf $\mathcal{A}(x_3)$: $x_1 = x_2 \supset x_1^{x_3} = x_2^{x_3}$

(i) The proof is similar to that of part (h). This time denote by $\mathcal{B}(x_1)$ the wf $(x_2)(x_1 = x_2 \supset x_3^{x_1} = x_3^{x_2})$ and denote by $\mathcal{C}(x_2)$ the wf $x_1 + x_4 = x_2 \supset x_3^{x_1+x_4} = x_3^{x_2}$.

(j) Apply (N10) to the wf $a(x_3)$: $x_3 \neq 1 \supset (\mathrm{E}\,x_1)(\mathrm{E}\,x_2)(x_3 = x_1 + x_2)$

Proposition 2. **N** *is a first-order theory with equality, i.e.*

$$\vdash x_1 = x_1$$

and

$$\vdash x_1 = x_2 \supset (\mathcal{A}(x_1, x_1) \supset \mathcal{A}(x_1, x_2)),$$

where $\mathcal{A}(x_1, x_1)$ is any wf and $\mathcal{A}(x_1, x_2)$ is the result of replacing some, but not necessarily all, free occurrences of x_1 by x_2, with the proviso that x_2 is free for the occurrences of x_1 which it replaces.

Proof: This follows from Proposition 1, parts (a), (b), (c), (d), (f), (g), (h), and (i), (N2), (N3), and Proposition 2.26 of [3].

4. Some theorems of **N**.

Proposition 3. For any terms r, s, and t of **N** *the following wfs are theorems of* **N**.

(a) $t \neq r \supset t + r \neq r + t$

(b) $t \neq t + r$

(c) $t + (r + s) \neq (t + r) + s$

(d) $(\mathrm{E}\,x_1)(\mathrm{E}\,x_2)(x_1 \cdot x_2 \neq x_2 \cdot x_1)$

(e) $t \cdot (r \cdot s) = (t \cdot r) \cdot s$

Proof:
(a) By (N5)

$$\vdash x_1 + x_2 = x_2 + x_1 \supset x_1 = x_2$$

Hence by a tautology,

$$\vdash x_1 \neq x_2 \supset x_1 + x_2 \neq x_2 + x_1$$

(b) Apply (N10) to the wf $\mathcal{A}(x_1)$: $(x_2)(x_1 \neq x_1 + x_2)$
(c) By (N5)

$$\vdash x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3 \supset x_1 = x_1 + x_2$$

Hence

$$\vdash x_1 \neq x_1 + x_2 \supset x_1 + (x_2 + x_3) \neq (x_1 + x_2) + x_3$$

and by part (b) and Modus Ponens,

$$\vdash x_1 + (x_2 + x_3) \neq (x_1 + x_2) + x_3$$

(d) Prove that

$$\vdash ((1 + 1) + 1)(1 + 1) = (1 + 1)(1 + 1) \supset 1 + 1 = 1$$

Hence

$$\vdash 1 + 1 \neq 1 \supset ((1 + 1) + 1)(1 + 1) \neq (1 + 1)((1 + 1) + 1)$$

Then by (N4) and Modus Ponens,

$$\vdash ((1 + 1) + 1)(1 + 1) \neq (1 + 1)((1 + 1) + 1)$$

and hence

$$\vdash (\mathrm{E}\ x_1)(\mathrm{E}\ x_2)(x_1 \cdot x_2 \neq x_2 \cdot x_1)$$

(e) Apply (N10) to the wf $\mathcal{A}(x_3)$: $x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3$

Hence addition in **N** is noncommutative and anti-associative and multiplication in **N** is noncommutative and associative. The following proposition shows that both the right and left cancellation laws hold.

Proposition 4. For any terms t, r, and s, the following wfs are theorems of **N**.

(a) $t \cdot r = s \cdot r \supset t = s$
(b) $t \cdot r = 1 \supset (t = 1 \wedge r = 1)$
(c) $t \neq 1 \supset t^r \neq 1$
(d) $(t^r = t \wedge t \neq 1) \supset r \equiv 1$
(e) $t \cdot r = t \cdot s \supset r = s$

Proof:
(a) Apply (N10) to the wf $\mathcal{A}(x_2)$: $x_1 \cdot x_2 = x_3 \cdot x_2 \supset x_1 = x_3$.
(b) Apply (N10) to the wf $\mathcal{A}(x_2)$: $x_1 \cdot x_2 = 1 \supset (x_1 = 1 \wedge x_2 = 1)$.
(c) Using part (b) along with (N8) and (N9), apply (N10) to the wf $\mathcal{A}(x_2)$: $x_1 \neq 1 \supset x_1^{x_2} \neq 1$.

(d) Apply (N10) to the wf $\mathscr{A}(x_1)$: $x_1 \neq 1 \supset (\mathrm{E}\, x_3)(x_2^{x_1} = x_2^{x_3+1})$ to prove

$$\vdash (x_1)(x_1 \neq 1 \supset (\mathrm{E}\, x_3)(x_2^{x_1} = x_2^{x_3+1})$$

Then use parts (a) and (c) to prove

$$\vdash (x_1 \neq 1 \wedge x_2^{x_1} = x_2) \supset x_2 = 1.$$

Then by a tautology,

$$\vdash (x_2^{x_1} = x_2 \wedge x_2 = 1) \supset x_1 = 1.$$

(e) Apply (N10) to the wf $\mathscr{A}(x_3)$: $(x_1^{x_2})^{x_3} = x_1^{x_2 \cdot x_3}$ to prove

$$\vdash (x_3)((x_1^{x_2})^{x_3} = x_1^{x_2 \cdot x_3})$$

Then

$$x_1 = x_1 \cdot x_3 \vdash (x_2)(x_2^{x_1} = (x_2^{x_1})^{x_3})$$

and hence

$$x_1 = x_1 \cdot x_3 \vdash (1 + 1)^{x_1} = ((1 + 1)^{x_1})^{x_3}$$

By (N4) and part (c),

$$\vdash (1 + 1)^{x_1} \neq 1$$

Hence by part (d)

$$x_1 = x_1 \circ x_3 \vdash x_3 = 1$$

and thus

$$\vdash \mathscr{B}(1),$$

where $\mathscr{B}(x_2)$ is the wf $(x_3)(x_1 \cdot x_2 = x_1 \cdot x_3 \supset x_2 = x_3)$. Then

$$x_1 \cdot (x_2 + x_4) = x_1 \cdot x_3,\ x_3 = 1 \vdash x_2 + x_4 = 1.$$

Hence by (N4) and a tautology,

$$x_1 \cdot (x_2 + x_4) = x_1 \cdot x_3 \vdash x_3 \neq 1$$

Then using proposition 1, part (j), prove that

$$\mathscr{B}(x_2),\ \mathscr{B}(x_4),\ \vdash \mathscr{B}(x_2 + x_4)$$

Hence by (N10),

$$\vdash (x_2)\, \mathscr{B}(x_2)$$

5. Essential undecidability and incompleteness of **N**. Following the methods of [4], we shall establish that **N** is essentially undecidable and incomplete by showing that **N** contains an essentially undecidable subtheory.

Let **Q** be the first-order theory whose only individual constant is a_1, whose only predicate letter is A_1^2, and whose only function letters are f_1^2 and f_2^2. As usual, we write 1 for a_1, $x_1 = x_2$ for $A_1^2(x_1, x_2)$, $x_1 + x_2$ for $f_1^2(x_1, x_2)$, and $x_1 \cdot x_2$ for $f_2^2(x_1, x_2)$. The nonlogical axioms of **Q** are the following.

(Q1) $x_1 = x_1$
(Q2) $x_1 = x_2 \supset x_2 = x_1$
(Q3) $x_1 = x_2 \supset (x_2 = x_3 \supset x_1 = x_3)$
(Q4) $x_1 = x_2 \supset (x_1 + x_3 = x_2 + x_3 \wedge x_3 + x_1 = x_3 + x_2)$
(Q5) $x_1 = x_2 \supset (x_1 \cdot x_3 = x_2 \cdot x_3 \wedge x_3 \cdot x_1 = x_3 \cdot x_2)$
(Q6) $x_1 + 1 = x_2 + 1 \supset x_1 = x_2$
(Q7) $1 \neq x_1 + 1$
(Q8) $x_1 \neq 1 \supset (\mathrm{E}\, x_2)(x_1 = x_2 + 1)$
(Q9) $x_1 + (x_2 + 1) = (x_1 + x_2) + 1$
(Q10) $x_1 \cdot 1 = x_1$
(Q11) $x_1 \cdot (x_2 + 1) = x_1 \cdot x_2 + x_1$

It can be shown (see [4], p. 67) that **Q** ("Robinson's system") is essentially undecidable.

Denote by $x_1 \cong x_2$ the following wf of **N**: $(x)(x^{x_1} = x^{x_2})$ and denote by $x_1 \not\cong x_2$ the wf $\sim (x)(x^{x_1} = x^{x_2})$.

For each wf $\mathscr{A}$ of **Q**, let $\mathscr{A}'$ be the wf of **N** obtained from $\mathscr{A}$ by replacing each occurrence of = by $\cong$. Let **N'** be the first-order theory whose "nonlogical" symbols are those of **N**, and whose axioms are the set of wfs $\mathscr{A}'$, where $\mathscr{A}$ is an axiom of **Q**.

Lemma. **N'** *is a subtheory of* **N**.

Proof: Each axiom of **N'** is a theorem of **N**:

(i) $\vdash x_1 \cong x_1$

This follows from Proposition 1, part (a).

(ii) $\vdash x_1 \cong x_2 \supset x_2 \cong x_1$

This follows from Proposition 1, part (b).

(iii) $\vdash x_1 \cong x_2 \supset (x_2 \cong x_3 \supset x_1 \cong x_3)$

This follows from Proposition 1, part (c).

(iv) $\vdash x_1 \cong x_2 \supset (x_1 + x_3 \cong x_2 + x_3 \wedge x_3 + x_1 \cong x_3 + x_2)$

This follows from Proposition 1, parts (f) and (g) and (N9).

(v) $\vdash x_1 \cong x_2 \supset (x_1 \cdot x_3 \cong x_2 \cdot x_3 \wedge x_3 \cdot x_1 \cong x_3 \cdot x_2)$

From the proof of Proposition 4, part (e),

$$\vdash (x_1)(x_2)((x^{x_1})^{x_2} = x^{x_1 \cdot x_2}).$$

Then $\vdash x_1 \cong x_2 \supset (x_1 \cdot x_3 \cong x_2 \cdot x_3)$ follows from Proposition 1, part (h). Furthermore,

$$x_1 \cong x_2 \vdash (x^{x_3})^{x_1} = (x^{x_3})^{x_2}$$

and hence

$$x_1 \cong x_2 \vdash x_3 \cdot x_1 \cong x_3 \cdot x_2$$

The desired result then follows by a tautology.

(vi) $\vdash x_1 + 1 \cong x_2 + 1 \supset x_1 \cong x_2$

This follows from (N9) and Proposition 4, part (a).

(vii) $\vdash 1 \ncong x + 1$

By (N4), Proposition 4, part (d), and a tautology,

$$\vdash (1+1)^1 \neq (1+1)^{x_1+1}$$

Hence

$$\vdash \sim(x)(x^1 = x^{x_1+1})$$

(viii) $\vdash x_1 \ncong 1 \supset (\mathrm{E}\, x_2)(x_1 \cong x_2 + 1)$

Since $\vdash 1 \cong 1$,

$\vdash 1 \ncong 1 \supset (\mathrm{E}\, x_2)(1 \cong x_2 + 1)$ by a tautology

and Modus Ponens. Let $\mathscr{B}(x_1)$ denote $x_1 \ncong 1 \supset (\mathrm{E}\, x_2)(x_1 \cong x_2 + 1)$. Then prove

$$x_3 \cong 1 \vdash \mathscr{B}(x_1 + x_3)$$

and

$$x_3 \ncong 1,\ \mathscr{B}(x_3) \vdash \mathscr{B}(x_1 + x_3)$$

Then by a tautology,

$$\mathscr{B}(x_3) \vdash \mathscr{B}(x_1 + x_3)$$

and the desired result follows from (N10).

(ix) $\vdash x_1 + (x_2 + 1) \cong (x_1 + x_2) + 1$

This follows from (N9) and Proposition 3, part (e).

(x) $\vdash x_1 \cdot 1 \cong x_1$

This follows from (N6) and Proposition 1, part (i).

(xi) $\vdash x_1 \cdot (x_2 + 1) \cong x_1 \cdot x_2 + x_1$

This follows from (N7), (N6), and Proposition 1, part (i).

Proposition 4. If **N** *is consistent then* **N** *is (a) essentially undecidable (b) incomplete.*

Proof:

(a) We first note that **N'** is undecidable since a decision procedure for **N'** would yield a decision procedure for **Q**. Furthermore, **N'** is essentially undecidable. For suppose that **T** is a consistent decidable extension of **N'**. Let **Q'** be the first-order theory whose symbols are those of **Q** and such that $\vdash_{\mathbf{Q}'} \mathscr{A}$ *if and only if* $\vdash_{\mathbf{T}} \mathscr{A}'$. Clearly, **Q'** is consistent and decidable. Furthermore, $\vdash_{\mathbf{Q}} \mathscr{A} \Rightarrow \vdash_{\mathbf{N}'} \mathscr{A}' \Rightarrow \vdash_{\mathbf{T}} \mathscr{A}' \Rightarrow \vdash_{\mathbf{Q}'} \mathscr{A}$ and hence **Q'** is a consistent decidable extension of **Q**, which contradicts the essential undecidability of **Q**.

By the lemma, **N'** is a subtheory of **N** and so by Theorem 3, p. 16 of [4], **N** is essentially undecidable.

(b) Clearly, **N** is recursively axiomatizable and since **N** is also essentially undecidable, **N** is incomplete by Theorem 1, p. 14 of [4].

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