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ALGEBRAIC INDEPENDENCE IN AN INFINITE STEINER TRIPLE SYSTEM

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In a recent note [1] W. J. Frascella has given an effective construction of a Steiner triple system on a set of the power of the continuum. With every Steiner triple system an idempotent binary operation is associated in a natural way. The triple system can be regarded as an algebra, and one can consider the algebraic independence in the sense of Marczewski [2] on it.

Frascella's triple system gives rise to an algebra in which every two elements are independent, while every three of them are dependent. Thus the numerical characteristics ι and ι^* introduced by Marczewski in [3] for finite algebras are both equal 2 here. The purpose of the present paper is to prove these facts.

1. A Steiner triple system on a set S is a class of three-element sets (called *Steiner triples*) such that every pair of elements of S belongs to exactly one Steiner triple. A given Steiner triple system on S determines a binary operation on S such that

(1) $$x \circ x = x ,$$

and for $x \neq y$, $x \circ y$ is the third element of the triple determined by x and y. Hence the binary operation "$\circ$" has the following additional properties

(2) $x \circ y = y \circ x$,
(3) $x \circ (x \circ y) = y$, $\quad y \circ (x \circ y) = x$,
(4) if $x \neq y$ then $x \neq x \circ y \neq y$.

2. Consider the algebra $\langle S,\circ\rangle$ with the single fundamental operation "$\circ$".

Proposition. *Every algebraic binary operation* $\mathsf{f}(x,y)$ *of the algebra* $\langle S,\circ\rangle$ *is either* $x \circ y$ *or one of the trivial operations* $\mathsf{e}_1^2(x,y) = x$, $\mathsf{e}_2^2(x,y) = y$.

Proof. The binary operation $\mathsf{f}(x,y)$ can be expressed as a word consisting of the letters x, y, the symbol $\circ$ and brackets. The number of letters in the word is called its length. An operation which can be

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expressed by a word of length one is one of the trivial operations. Suppose now that the proposition holds for operations expressible by words of length $< n$, and let $\mathsf{f}(x,y)$ be expressed by a word of length $n > 1$. Then we have $\mathsf{f}(x,y) = \mathsf{u}(x,y) \circ \mathsf{v}(x,y)$ where $\mathsf{u}(x,y)$ and $\mathsf{v}(x,y)$ are operations expressed by words of length $< n$. Hence there are the following possibilities:

(1) $\mathsf{u}(x,y) = x \circ y$, (2) $\mathsf{u}(x,y) = x$, (3) $\mathsf{u}(x,y) = y$,
(4) $\mathsf{v}(x,y) = x \circ y$, (5) $\mathsf{v}(x,y) = x$, (6) $\mathsf{v}(x,y) = y$.

Combining (1) and (3) or (2) and (6) or (3) and (4) and using formulae (1) - (3) we obtain $\mathsf{f}(x,y) = x \circ y$. In the remaining 6 cases we get one of the trivial operations.

Thus the proposition is proved by induction on the length of the word which expresses it. As a corollary we have, in view of (4), the following:

Theorem 1. *In the algebra associated with a Steiner triple system every set of two elements is independent.*

3. We now prove two lemmas which are valid in any algebra associated with a Steiner triple system, and which will be used in section 5.

Lemma 1. *Three distinct elements a, b, c, $\in S$ are dependent if and only if the elements a, $a \circ b$, $a \circ c$ are dependent.*

Proof. Note first that the elements a, $a \circ b$, $a \circ c$ are distinct: $a \circ b \neq a \neq a \circ c$ by (4); $a \circ b \neq a \circ c$, since otherwise the two elements a and $a \circ b$ would belong to two different Steiner triples.

Further, since $b = (a \circ b) \circ a$ and $c = (a \circ c) \circ a$, it suffices to prove only one implication. If a, b, c are dependent then there are two different binary algebraic operations $\mathsf{f}(x,y,z)$, $\mathsf{g}(x,y,z)$ such that

$$\mathsf{f}(a,b,c) = \mathsf{g}(a,b,c) .$$

Consider the operations

$$\mathsf{f}^*(x,y,z) = \mathsf{f}(x, x \circ y, x \circ z) ,$$
$$\mathsf{g}^*(x,y,z) = \mathsf{g}(x, x \circ y, x \circ z) .$$

We have

$$\mathsf{f}(x,y,z) = \mathsf{f}^*(x, x \circ y, x \circ z) ,$$
$$\mathsf{g}(x,y,z) = \mathsf{g}^*(x, x \circ y, x \circ z) .$$

Hence the operations f^* and g^* must be different if f and g are. But

$$\mathsf{f}^*(a, a \circ b, a \circ c) = \mathsf{f}(a,b,c) = \mathsf{g}(a,b,c) = \mathsf{g}^*(a, a \circ b, a \circ c) ,$$

thus a, $a \circ b$, $a \circ c$ are dependent.

Lemma 2. *If the elements a, b, c are distinct and do not form a Steiner triple, then a, b, c, are dependent if and only if a, $a \circ b$, c are dependent.*

Proof. The elements a, $a \circ b$, c are different because of the assumption that they do not form a Steiner triple. The proof is analogous to that of

Lemma 1. We use the operations

$$\mathsf{f}^{**}(x,y,z) = \mathsf{f}(x,\ x \circ y,\ z)\ ,$$
$$\mathsf{g}^{**}(x,y,z) = \mathsf{g}(x,\ x \circ y,\ z)$$

instead of f^* and g^* in this case.

4. Let us now describe the algebra $\langle A,\circ\rangle$ associated with the Steiner triple system constructed by Frascella. The set A is the union of three copies of the real line. The elements of A will be represented as pairs $x = (\xi, i)$, where ξ is a real number and $i = 0, 1$ or 2. We shall call ξ the coordinate of x, and i - the level of x. Addition, when applies to levels, will be understood as addition modulo 3.

The fundamental operation "$\circ$" is commutative and idempotent. With this in mind, it is fully described by the following formulae:
For distinct elements on the same level

$$(5) \qquad (\xi,i) \circ (\eta,i) = \left(\frac{\xi+\eta}{2}, i+1\right) .$$

For elements on two different levels

$$(6) \qquad (\xi,i) \circ (\eta,i+1) = (2\eta - \xi, i) \quad \text{if } \xi \neq \eta\ ,$$
$$(7) \qquad (\xi,i) \circ (\xi,i+1) = (\xi, i+2)\ ,$$

4. Now consider the following two ternary operations in the algebra $\langle A,\circ\rangle$

$$\mathsf{f}(x,y,z) = y \circ \{[x\circ(y\circ z)] \circ [z\circ(y\circ x)]\}\ ,$$
$$\mathsf{g}(x,y,z) = \ y \circ [x\circ(y\circ z)]\} \circ \{y\circ[z\circ(y\circ x)]\}$$

Lemma 3. *The operations* f *and* g *are different algebraic operations.*

Proof. A substitution

$$x = (-4,\ 0)\ , \qquad y = (8,\ 0)\ , \qquad z = (0,\ 1)$$

gives $\mathsf{f} = (8,\ 2)$, $\mathsf{g} = (-55,\ 2)$, hence $\mathsf{f} \neq \mathsf{g}$.

Lemma 4. *Let* $\alpha < \beta < \gamma$ *be real numbers and let* $a = (\alpha, i)$, $b = (\beta, i)$, $c = (\gamma, i)$, *be three elements on the same level* i. *Then*

$$\mathsf{f}(a,b,c) = \mathsf{g}(a,b,c).$$

Proof: By (5)

$$b \circ c = \left(\frac{\beta+\gamma}{2}\ ,\ i+1\right), \qquad b \circ a = \left(\frac{\alpha+\beta}{2}\ ,\ i+1\right)$$

and

$$\alpha < \frac{\alpha+\beta}{2} < \beta < \frac{\beta+\gamma}{2} < \gamma\ .$$

Hence, by (6)

$$a\circ(b\circ c) = (\beta+\gamma-\alpha, i)\ ,$$
$$c\circ(b\circ a) = (\beta+\alpha-\gamma, i)\ .$$

Further, by (5)

$$[a \circ (b \circ c)] \circ [c \circ (b \circ a)] = (\beta, i+1) \quad ,$$

and by (7)

$$\mathsf{f}(a, b, c) = (\beta, i+2)$$

On the other hand, by (5)

$$\{b \circ [a \circ (b \circ c)]\} = \left(\beta + \frac{\gamma - \alpha}{2}, i+1\right) \quad ,$$

$$\{b \circ [c \circ (b \circ a)]\} = \left(\beta + \frac{\alpha - \gamma}{2}, i+1\right) \quad ,$$

and again by (5)

$$\mathsf{g}(a, b, c) = (\beta, i+2) \quad .$$

Lemmas 3 and 4 imply at once the

Corollary. *Any three elements on the same level are dependent.*

5. We are now ready to prove

Theorem 2. *Every set of three elements of the algebra* $\langle A, \circ \rangle$ *is dependent.*

Proof. If the elements a, b, c form a Steiner triple they are obviously dependent since, in this case,

$$c = a \circ b$$

If they do not form a Steiner triple we shall consider the following cases.

1. All three elements a, b, c are on the same level. Then they are dependent by the corollary of the preceding section.

2. a and c are on level i, b on level $i+1$. We can assume without loss of generality that the coordinates of a and b are different. But then a, $a \circ b$, c are different elements on the level i. Lemma 2 reduces the proof to case 1.

3. a is on level i, b and c on level $i+1$ and all coordinates are different. Then a, $a \circ b$, $a \circ c$ are three elements on level i, and the proof is reduced to case 1 by lemma 1.

4. b is on level i, a and c on level $i+1$ and the coordinates of a and b coincide. Then $a \circ b$ is on level $i+2$, and we have the two elements a and c on level $i+1$ and $a \circ b$ on level $i+2$. By lemma 2 this reduces to case 2 with $i+1$ instead of i.

5. Finally, if a, b, c are on three different levels, then a, $a \circ b, c$ are on two levels only, and therefore lemma 2 reduces the proof to one of the previous cases.

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