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A DECISION PROCEDURE FOR FITCH'S PROPOSITIONAL CALCULUS

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In this paper[1] a *Sequenzenkalkül*, in the sense of Gentzen [3], will be formulated and shown equivalent (in a sense to be specified) to the propositional system (which we will term **F**) of Fitch's [2]. Naturally, the proof of equivalence requires an elimination theorem for the first system; the bulk of this paper, in fact, will concern itself with the task of establishing such a theorem. Finally, a decision method will be sketched for the *Sequenzenkalkül*, and thereby, indirectly, for Fitch's system. Though indirect and more complicated in some ways than the methods of James [4] and Resnik [7], this method has the advantage of applying to Fitch's full system of propositional calculus; the procedure of [4] does not take into account formulas containing nested implications, and that of [7] applies only to the implicational fragment of **F**.

1. *The System* **LF**. This is an **L**-system, in the sense of [3], designed to be equivalent to the system **F**.

1.1. *Wffs*. Any propositional variable p is well-formed (wf); furthermore, if A and B are wf, so are $(A \vee B)$, $(A \wedge B)$, $\sim A$, and $(A \supset B)$. Where α and β are strings of wffs separated by commas, $\alpha \vdash \beta$ is a (wf) *sequent*.

1.2. *Axioms*. There is one axiom-scheme, *identity* (**Id**): $A \vdash A$.

1.3. *Rules*.

1.3.1. *Structural rules*:

$$\vdash\mathbf{K}\ \frac{\alpha \vdash \beta}{\alpha \vdash A,\beta} \qquad \mathbf{K}\vdash\ \frac{\alpha \vdash \beta}{\alpha,A \vdash \beta}$$

$$\vdash\mathbf{C}\ \frac{\alpha \vdash \beta,A,B,\gamma}{\alpha \vdash \beta,B,A,\gamma} \qquad \mathbf{C}\vdash\ \frac{\alpha,A,B,\beta \vdash \gamma}{\alpha,B,A,\beta \vdash \gamma}$$

$$\vdash\mathbf{W}\ \frac{\alpha \vdash A,A,\beta}{\alpha \vdash A,\beta} \qquad \mathbf{W}\vdash\ \frac{\alpha,A,A \vdash \beta}{\alpha,A \vdash \beta}$$

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1.3.2. *Logical rules:*

$$\vdash\vee\ \frac{\alpha \vdash A,B,\beta}{\alpha \vdash A \vee B,\beta} \qquad \vee\vdash\ \frac{\alpha,A \vdash \beta \quad \gamma,B \vdash \delta}{\alpha,\gamma,A \vee B \vdash \beta,\delta}$$

$$\vdash\wedge\ \frac{\alpha \vdash A,\beta \quad \gamma \vdash B,\delta}{\alpha,\gamma \vdash A \wedge B,\beta,\delta} \qquad \wedge\vdash\ \frac{\alpha,A,B \vdash \beta}{\alpha,A \wedge B \vdash \beta}$$

$$\vdash\sim\vee\ \frac{\alpha \vdash \sim A,\beta \quad \gamma \vdash \sim B,\delta}{\alpha,\gamma \vdash \sim(A \vee B),\beta,\delta} \qquad \sim\vee\vdash\ \frac{\alpha,\sim A,\sim B \vdash \beta}{\alpha,\sim(A \vee B) \vdash \beta}$$

$$\vdash\sim\wedge\ \frac{\alpha \vdash \sim A,\sim B,\beta}{\alpha \vdash \sim(A \wedge B),\beta} \qquad \sim\wedge\vdash\ \frac{\alpha,\sim A \vdash \beta \quad \gamma,\sim B \vdash \delta}{\alpha,\gamma,\sim(A \wedge B) \vdash \beta,\delta}$$

$$\vdash\sim\sim\ \frac{\alpha \vdash A,\beta}{\alpha \vdash \sim\sim A,\beta} \qquad \sim\sim\vdash\ \frac{\alpha,A \vdash \beta}{\alpha,\sim\sim A \vdash \beta}$$

$$\sim\vdash\ \frac{\alpha \vdash A,\beta}{\alpha,\sim A \vdash \beta}$$

$$\vdash\supset\ \frac{\alpha,A \vdash B}{\alpha \vdash A \supset B} \qquad \supset\vdash\ \frac{\alpha,B \vdash \beta \quad \gamma \vdash A,\delta}{\alpha,\gamma,A \supset B \vdash \beta,\delta}$$

Schemes such as $\frac{\alpha \vdash \beta}{\alpha \vdash A,\beta}$ are, of course, metalinguistic. Any result of replacing the premiss(es) and conclusion of such a scheme by sequents is an *instance* of the scheme, or inference. Corresponding to the six primitive structural rules and thirteen primitive logical rules of **LF**, there are six sorts of primitive structural inferences in **LF**, and thirteen sorts of primitive logical inferences; $\vdash\vee$-inferences, **W** $\vdash$-inferences, etc.

The Greek letters used in a scheme are called *parameters*. A constituent A of a primitive inference is *parametric* if it results by the the substitution of a sequent $B_1, \ldots, A, \ldots, B_n$ for some parameter of the corresponding scheme.

The wff(s) introduced by a primitive inference is (are) the constituent(s) of the conclusion which result by substitution of wffs for the Roman letters of the corresponding scheme. E.g., $A \vee B$ is introduced by $\frac{C \vdash A,B,D}{C \vdash A \vee B,D}$ and both A and B are introduced by $\frac{C,A,B \vdash D}{C,B,A \vdash D}$.

Given a proof of $\alpha \vdash \beta$, this sequent is said to be *justified* by the last inference of the proof (which has $\alpha \vdash \beta$ as conclusion), and is also said to be justified by the scheme of which the inference is an instance.

Notice especially that the rule $\vdash\supset$ is unlike the others in that it has no parameters on the right. This asymmetry corresponds to a similar feature of Fitch's system; his rule of implication introduction will not permit, e.g., the proof of $A \vee (A \supset B)$.

2. *Preliminary lemmas*. In this section we establish a number of lemmas needed in our later proof of an elimination theorem for **LF**. Except for lemmas 11 and 12, these all have to do with the reversibility of various primitive rules of **LF**.

By 'α^{-C}' we represent schematically any result of deleting some (or perhaps none) of the occurrences of C as constituent of α. We say that a rule is *admissible* in **LK** if its addition to **LK** as a primitive rule would not extend the class of theorems.

Lemma 1. *The rule* $\dfrac{\alpha \vdash \beta}{\alpha \vdash A, B, \beta^{-(A \vee B)}}$ *is admissible in* **LF**.

Lemma 2. *The rules* $\dfrac{\alpha \vdash \beta}{\alpha^{-(A \vee B)}, A \vdash \beta}$ *and* $\dfrac{\alpha \vdash \beta}{\alpha^{-(A \vee B)}, B \vdash \beta}$ *are admissible in* **LF**.

Lemma 3. *The rules* $\dfrac{\alpha \vdash \beta}{\alpha \vdash A, \beta^{-(A \wedge B)}}$ *and* $\dfrac{\alpha \vdash \beta}{\alpha \vdash B, \beta^{-(A \wedge B)}}$ *are admissible in* **LF**.

Lemma 4. *The rule* $\dfrac{\alpha \vdash \beta}{\alpha^{-(A \wedge B)}, A, B \vdash \beta}$ *is admissible in* **LF**.

Lemma 5. *The rules* $\dfrac{\alpha \vdash \beta}{\alpha \vdash \sim A, \beta^{-\sim(A \vee B)}}$, *and* $\dfrac{\alpha \vdash \beta}{\alpha \vdash \sim B, \beta^{-\sim(A \vee B)}}$ *are admissible in* **LF**.

Lemma 6. *The rule* $\dfrac{\alpha \vdash \beta}{\alpha^{-\sim(A \vee B)}, \sim A, \sim B \vdash \beta}$ *is admissible in* **LF**.

Lemma 7. *The rule* $\dfrac{\alpha \vdash \beta}{\alpha \vdash \sim A, \sim B, \beta^{-\sim(A \wedge B)}}$ *is admissible in* **LF**.

Lemma 8. *The rules* $\dfrac{\alpha \vdash \beta}{\alpha^{-\sim(A \wedge B)}, \sim A \vdash \beta}$ *and* $\dfrac{\alpha \vdash \beta}{\alpha^{-\sim(A \wedge B)}, \sim B \vdash \beta}$ *are admissible in* **LF**.

Lemma 9. *The rule* $\dfrac{\alpha \vdash \beta}{\alpha \vdash A, \beta^{-\sim\sim A}}$ *is admissible in* **LF**.

Lemma 10. *The rule* $\dfrac{\alpha \vdash \beta}{\alpha^{-\sim\sim A}, A \vdash \beta}$ *is admissible in* **LF**.

Lemma 11. *The rule* $\dfrac{\alpha \vdash \beta}{\alpha, A \vdash \beta^{-\sim A}}$ *is admissible in* **LF**.

Lemma 12. *The rule* $\dfrac{\alpha \vdash \beta}{\alpha, A \vdash B, \beta^{-(A \supset B)}}$ *is admissible in* **LF**.

We say that $\alpha \leqslant \beta$, where α and β are sequences, if every constituent of α is a constituent of β. And we say that $\Vdash_{\mathbf{LF}} \alpha \vdash \beta$ (briefly, $\Vdash \alpha \vdash \beta$) if $\alpha \vdash \beta$ is a sequent provable in **LF**. Finally, the notation[2] $\dfrac{\alpha \vdash \beta}{\overline{\gamma \vdash \delta}}$ indicates that $\gamma \vdash \delta$ can be obtained from $\alpha \vdash \beta$ by a number of applications of structural rules.[3]

Proof of Lemma 11. We will show that if $\Vdash \alpha \vdash \beta_1, \sim A_1, \beta_2, \sim A_2, \ldots, \sim A_n, \beta_{n+1}$ and $\alpha, A_1, A_2, \ldots, A_n \leqslant \alpha^*$ and $\beta_1, \beta_2, \ldots, \beta_{n+1} \leqslant \beta^*$, then $\Vdash \alpha^* \vdash \beta^*$.

Case 1. $\alpha \vdash \beta_1, {\sim}A_1, \ldots, {\sim}A_n, \beta_{n+1}$ is an instance of **Id**, and so is ${\sim}A_1 \vdash {\sim}A_1$. Derive $\alpha^* \vdash \beta^*$ as follows:

$$\cfrac{\cfrac{A \vdash A}{A, {\sim}A \vdash}\ {\sim}\vdash}{\alpha^* \vdash \beta^*}$$

Case 2. The inference justifying $\alpha \vdash \beta_1, {\sim}A_1, \ldots, {\sim}A_n, \beta_{n+1}$ is structural. Because of the remark in footnote 3, this case is immediate.

Case 3. The constituents ${\sim}A_1, {\sim}A_2, \ldots, {\sim}A_n$ are all parametric in the inference which justifies $\alpha \vdash \beta_1, {\sim}A_1, \ldots, \beta_n, {\sim}A_n, \beta_{n+1}$. All of these cases are alike in their essentials; as an example we will present the case in which the rule is ${\sim}\vdash$. Here, α is $\alpha_1, {\sim}B$ and we have

$$\frac{\alpha_1 \vdash B, \beta_1, {\sim}A_1, \ldots, {\sim}A_n, \beta_{n+1}}{\alpha_1, {\sim}B \vdash \beta_1, {\sim}A_1, \ldots, {\sim}A_n, \beta_{n+1}}\ {\sim}\vdash$$

By the hypothesis of induction, $\Vdash \alpha^* \vdash B, \beta^*$. Then proceed as follows:

$$\cfrac{\cfrac{\alpha^* \vdash B, \beta^*}{\alpha^*, {\sim}B \vdash \beta^*}\ {\sim}\vdash}{\alpha^* \vdash \beta^*}$$

Case 4. ${\sim}A_1$ is introduced by the inference which justifies $\alpha \vdash \beta_1, {\sim}A_1, \ldots, {\sim}A_n, \beta_{n+1}$. There are three subcases.

4.1. The rule is $\vdash{\sim}{\sim}$. Here, ${\sim}A_1$ is ${\sim}{\sim}B$, and we have

$$\frac{\alpha \vdash B, \beta_2, {\sim}A_2, \ldots, {\sim}A_n, \beta_{n+1}}{\alpha \vdash {\sim}{\sim}B, \beta_2, {\sim}A_2, \ldots, {\sim}A_n, \beta_{n+1}}$$

By the hypothesis of induction, $\Vdash \alpha^* \vdash B, \beta^*$. Then proceed:

$$\cfrac{\cfrac{\alpha^* \vdash B, \beta^*}{\alpha^*, {\sim}A_1 \vdash \beta^*}\ {\sim}\vdash}{\alpha^* \vdash \beta^*}$$

4.2. The rule is $\vdash{\sim}\vee$. Here, A_1 is $B \vee C$, and one premiss is $\alpha \vdash {\sim}B, \beta_2, {\sim}A_2, \ldots, {\sim}A_n, \beta_{n+1}$ and the other $\alpha \vdash {\sim}B, \beta_2, {\sim}A_2, \ldots, {\sim}A_n, \beta_{n+1}$. By the hypothesis of induction $\Vdash \alpha^*, B \vdash \beta^*$ and $\Vdash \alpha^*, C \vdash \beta^*$. Then proceed:

$$\cfrac{\cfrac{\alpha^*, B \vdash \beta^* \quad \alpha^*, C \vdash \beta^*}{\alpha^*, \alpha^*, A_1 \vdash \beta^*, \beta^*}\ \vee\vdash}{\alpha^* \vdash \beta^*}$$

4.3. The rule is $\vdash{\sim}\wedge$. Here, A_1 is $B \wedge C$, and the premiss is $\alpha \vdash {\sim}A, {\sim}B, \beta_2, \ldots, \beta_{n+1}$. By the hypothesis of induction, $\Vdash \alpha^*, A, B \vdash \beta^*$. Then proceed:

$$\cfrac{\cfrac{\alpha^*, A, B \vdash \beta^*}{\alpha^*, A \wedge B \vdash \beta^*}\ \wedge\vdash}{\alpha^* \vdash \beta^*}$$

This completes the proof of lemma 11. The proofs of lemmas 1-10 and 12 are very much alike, though lemmas 6, 8, and 10 are complicated by the fact that a formula having the shape $\sim\sim A$, $\sim(A \vee B)$, or $\sim(A \wedge B)$ can be introduced on the left by either of two logical rules.

As a typical example, we will supply a complete proof of lemma 4, and also partial treatments (interesting cases only) of lemmas 6, 8, 10, and 12.

Proof of Lemma 4. We will show that if $\Vdash \alpha_1, A \wedge B, \alpha_2 \vdash \beta$ and $\alpha_1, A, B, \alpha_2 \leqslant \alpha^*$ and $\beta \leqslant \beta^*$, then $\Vdash \alpha^* \vdash \beta^*$.

Case 1. $\alpha_1, A \wedge B, \alpha_2 \vdash \beta$ is $A \wedge B \vdash A \wedge B$. Proceed as follows:

$$\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \wedge B}\ \vdash\wedge}{\overline{\alpha^* \vdash \beta^*}}$$

Case 2. $\alpha_1, A \wedge B, \alpha_2 \vdash \beta$ is justified by a structural rule. This is like case 2 of the previous proof.

Case 3. $A \wedge B$ is parametric in the inference which justifies $\alpha_1, A \wedge B, \alpha_2 \vdash \beta$. As an example we will consider the case in which the rule is $\vdash \supset$. Here β must consist of just one wff, say D, and we have

$$\frac{\alpha_1, A \wedge B, \alpha_2, C \vdash D}{\alpha_1, A \wedge B, \alpha_2 \vdash C \supset D}$$

so that $\Vdash \alpha^*, C \vdash D$ by the hypothesis of induction. Proceed as follows:

$$\dfrac{\dfrac{\alpha^*, C \vdash D}{\alpha^* \vdash C \supset D}\ \vdash\supset}{\overline{\alpha^* \vdash \beta^*}}$$

Case 4. $A \wedge B$ is introduced by the inference which justifies $\alpha_1, A \wedge B, \alpha_2 \vdash \beta$. This rule must be $\wedge\vdash$, so that we have $\dfrac{\alpha_1, A, B, \alpha_2 \vdash \beta}{\alpha_1, A \wedge B, \alpha_2 \vdash \beta}$. By the hypothesis of induction, $\Vdash \alpha^* \vdash \beta^*$

This completes the proof of lemma 4.

Proof of Lemma 6. We will show that if $\Vdash \alpha_1, \sim(A \vee B), \alpha_2 \vdash \beta$ and $\alpha_1, \sim A, \sim B, \alpha_2 \leqslant \alpha^*$ and $\beta \leqslant \beta^*$ then $\Vdash \alpha^* \vdash \beta^*$.

Case 1. $\alpha_1, \sim(A \vee B), \alpha_2 \vdash \beta$ is $\sim(A \vee B) \vdash \sim(A \vee B)$. Then:

$$\frac{\sim A \vdash \sim A \quad \sim B \vdash \sim B}{\sim A, \sim B \vdash \sim(A \vee B)}\ \vdash\sim\vee\,.$$

Case 4. $\sim(A \vee B)$ is introduced by the inference which justifies $\alpha_1, \sim(A \vee B), \alpha_2 \vdash \beta$.

4.1. The rule is $\sim\vee\vdash$. Then we have

$$\frac{\alpha_1, \sim A, \sim B \vdash \beta}{\alpha_1, \sim(A \vee B) \vdash \beta}\ \sim\vee\vdash\ ,$$

and so $\Vdash \alpha^* \vdash \beta^*$ by the hypothesis of induction.

4.2. The rule is $\sim\vdash$. Then we have

$$\frac{\alpha_1 \vdash A \vee B, \beta}{\alpha_1, \sim(A \vee B) \vdash \beta} \sim\vdash ,$$

and $\Vdash \alpha^* \vdash A \vee B, \beta^*$ by the hypothesis of induction. By lemma 2 (which can be established independently) $\Vdash \alpha^* \vdash A, B, \beta^*$. Then proceed:

$$\cfrac{\cfrac{\cfrac{\alpha^* \vdash A, B, \beta^*}{\alpha^*, \sim A \vdash B, \beta^*} \sim\vdash}{\alpha^*, \sim A, \sim B \vdash \beta^*} \sim\vdash}{\alpha^* \vdash \beta^*}$$

Proof of Lemma 8. We will show that if $\Vdash \alpha_1, \sim(A \wedge B), \alpha_2 \vdash \beta$ and $\alpha_1, \sim A, \alpha_2 \leqslant \alpha^*$ and $\beta \leqslant \beta^*$ then $\Vdash \alpha^* \vdash \beta^*$. (The other half of the proof is similar.)

Case 4. $\sim(A \wedge B)$ is introduced by the (logical) inference which justifies $\alpha_1, \sim(A \wedge B), \alpha_2 \vdash \beta$.

4.1. The rule is $\sim\wedge\vdash$. We have $\gamma, \sim A \vdash \delta$ as a premiss, then, where $\gamma \leqslant \alpha_1$ and $\delta \leqslant \beta$, so that $\Vdash \alpha^* \vdash \beta^*$ by the hypothesis of induction.

4.2. The rule is $\sim\vdash$. Then we have

$$\frac{\alpha_1 \vdash A \wedge B, \beta}{\alpha_1, \sim(A \wedge B) \vdash \beta} \sim\vdash .$$

Proceed as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\alpha_1 \vdash A \wedge B, \beta}{\alpha_1^* \vdash A \wedge B, \beta^*} \text{ hypothesis of induction}}{\alpha_1^* \vdash A, \beta^*} \text{ lemma 4}}{\alpha_1^*, \sim A \vdash \beta^*} \sim\vdash}{\alpha^* \vdash \beta^*}$$

This completes the proof of lemma 8.

Proof of Lemma 10. We will show that if $\Vdash \alpha_1, \sim\sim A, \alpha_2 \vdash \beta$ and $\alpha_1, A, \alpha_2 \leqslant \alpha^*$ and $\beta \leqslant \beta^*$ then $\Vdash \alpha^* \vdash \beta^*$.

Case 4. $\sim\sim A$ is introduced by the inference which justifies $\alpha_1, \sim\sim A, \alpha_2 \vdash \beta$.

4.1. The rule is $\sim\sim\vdash$. Then we have

$$\frac{\alpha_1, A \vdash \beta}{\alpha_1, \sim\sim A \vdash \beta} \sim\sim\vdash .$$

By the hypothesis of induction, $\Vdash \alpha^* \vdash \beta^*$.

4.2. The rule is $\sim\vdash$. Then we have

$$\frac{\alpha_1 \vdash \sim A, \beta}{\alpha_1, \sim\sim A \vdash \beta} \sim\vdash .$$

By the hypothesis of induction, $\Vdash \alpha^* \vdash \sim A, \beta$. Then proceed, remembering that lemma 11 has already been established:

$$\cfrac{\cfrac{\alpha^* \vdash {\sim}A,\beta}{\alpha^*,A \vdash \beta}\ \text{lemma 11.}}{\alpha^* \vdash \beta^*}$$

This is a sufficient sketch of the proof of lemma 10.

Proof of Lemma 12. We will show that if $\Vdash \alpha \vdash \beta_1,A \supset B,\beta_2$ and $\alpha,A \leqslant \alpha^*$ and $\beta_1,B,\beta_2 \leqslant \beta^*$ then $\Vdash \alpha^* \vdash \beta^*$.

Case 1. $\alpha \vdash \beta_1,A \supset B,\beta_2$ is $A \supset B \vdash A \supset B$. Then proceed as follows:

$$\cfrac{\cfrac{A \vdash A \quad B \vdash B}{A,A \supset B \vdash B}\ \supset\vdash}{\alpha^* \vdash \beta^*}$$

Case 4. $A \supset B$ is introduced by the inference which justifies $\alpha \vdash \beta_1$, $A \supset B,\beta_2$. Then we have $\dfrac{\alpha,A \vdash B}{\alpha \vdash A \supset B}$ since the rule must be $\vdash \supset$. By the hypothesis of induction, $\Vdash \alpha^* \vdash \beta^*$.

Since by now the method of proof of these lemmas must be clear, we will proceed to the next section.

3. *Elimination Theorem*

Theorem 1. *The rule* **mix**, $\dfrac{\alpha \vdash \beta \quad \gamma \vdash \delta}{\alpha,\gamma^{-C} \vdash \beta^{-C},\delta}$, *is admissible in* **LF**.

Proof. As usual,[4] the proof takes the form of a double induction on the rank and degree of inferences which are instances of **mix**. As hypotheses of induction we assume the following:

H_1: All **mix**-inferences with degree less than d are admissible, whatever their rank.

H_2: All **mix**-inferences with rank less than r and with degree d are admissible.

We must now suppose of an arbitrary **mix**-inference that it has degree d and rank r, and show that, under the hypotheses H_1 and H_2 it can be eliminated in favor of a proof in **LF**.

The argument falls into seven main cases, according to the form of the eliminated constituent C of the given inference.

Case 1. C is a propositional variable p, and the inference is:

$$\frac{\alpha \vdash \beta \quad \gamma \vdash \delta}{\alpha,\gamma^{-p} \vdash \beta^{-p},\delta}\ .$$

We distinguish subcases depending on how $\alpha \vdash \beta$ is justified.

1.1. $\alpha \vdash \beta$ is $p \vdash p$. Replace the inference by $\dfrac{\gamma \vdash \delta}{\alpha,\gamma^{-p} \vdash \beta^{-p},\delta}$.

1.2. $\alpha \vdash \beta$ is justified by a structural rule. In each of these six cases, mixing the premiss of $\alpha \vdash \beta$ with $\gamma \vdash \delta$ (by H_2) will produce the desired result—except for the case in which p is introduced by $\vdash$ **K** and does not already appear as a constituent of the right side of the premiss of $\alpha \vdash \beta$. Here, structural rules applied to this premiss will do the trick.

1.3. $\alpha \vdash \beta$ is justified by a logical rule. In this case, p is parametric, and so the rule cannot be $\vdash \supset$. Here, judicious use of H_2 will again yield the desired conclusion. We will supply one example: say, where the rule is $\vdash \vee$. Here, we have

$$\dfrac{\dfrac{\alpha \vdash A, B, \beta}{\alpha \vdash A \vee B, \beta} \quad \gamma \vdash \delta}{\alpha, \gamma^{-p} \vdash A \vee B, \beta^{-p}, \delta}\,. \qquad \text{Replace by} \qquad \dfrac{\dfrac{\alpha \vdash A, B, \beta \quad \gamma \vdash \delta}{\alpha, \gamma^{-p} \vdash A, B, \beta^{-p}, \delta}\;\textbf{mix}\ (H_2)}{\alpha, \gamma^{-p} \vdash A \vee B, \beta^{-p}, \delta}\;\vdash \vee\,.$$

Case 2. C has the form $\sim p$ or the form $\sim(A \supset B)$. These forms share with the preceding case the property that there is no logical rule for their introduction on the right side of a sequent. And so the argument used in Case 1 will apply here too, with no changes.

Case 3. C has the form $A \vee B$, and the inference is

$$\dfrac{\alpha \vdash \beta \qquad \gamma \vdash \delta}{\alpha, \gamma^{-(A \vee B)} \vdash \beta^{-(A \vee B)}, \delta}$$

Proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \beta}{\alpha \vdash A, B, \beta^{-A \vee B}}\;\text{lemma 1} \quad \dfrac{\gamma \vdash \delta}{\gamma^{-A \vee B}, A \vdash \delta}\;\text{lemma 2}}{\alpha, \gamma^{-A \vee B} \vdash B, \beta^{-A \vee B}, \delta}\;\textbf{mix}\ (H_1) \quad \dfrac{\gamma \vdash \delta}{\gamma^{-A \vee B}, B \vdash \delta}\;\text{lemma 2}}{\alpha, \gamma^{-A \vee B}, \gamma^{-A \vee B} \vdash \beta^{-A \vee B}, \delta}\;\textbf{mix}\ (H_1)}{\alpha, \gamma^{-A \vee B} \vdash \beta^{-A \vee B}, \delta}\,.$$

Case 4. C has the form $A \wedge B$, and the inference is:

$$\dfrac{\alpha \vdash \beta \qquad \gamma \vdash \delta}{\alpha, \gamma^{-A \wedge B} \vdash \beta^{-A \wedge B}, \delta}\,.$$

Proceed as follows:

$$\dfrac{\dfrac{\dfrac{\alpha \vdash \beta}{\alpha \vdash B, \beta^{-A \wedge B}}\;\text{lemma 3} \quad \dfrac{\dfrac{\alpha \vdash \beta}{\alpha \vdash A, \beta^{-A \wedge B}}\;\text{lemma 3} \quad \dfrac{\gamma \vdash \delta}{\gamma^{-A \wedge B}, A, B \vdash \delta}\;\text{lemma 4}}{\alpha, \gamma^{-A \wedge B}, B \vdash \beta^{-A \wedge B}, \delta}\;\textbf{mix}\ (H_1)}{\alpha, \alpha, \gamma^{-A \wedge B} \quad \beta^{-A \wedge B}, \beta^{-A \wedge B}, \delta}\;\textbf{mix}\ (H_1)}{\alpha, \gamma^{-A \wedge B} \quad \beta^{-A \wedge B}, \delta}\,.$$

Case 5. C has the form $\sim(A \vee B)$, and the inference is:

$$\dfrac{\alpha \vdash \beta \qquad \gamma \vdash \delta}{\alpha, \gamma^{-\sim(A \vee B)} \vdash \beta^{-\sim(A \vee B)}, \delta}\,.$$

Proceed as follows:

$$\dfrac{\dfrac{\dfrac{\alpha \vdash \beta}{\alpha \vdash \sim B, \beta^{-\sim(A \vee B)}}\;\text{lemma 5} \quad \dfrac{\dfrac{\alpha \vdash \beta}{\alpha \vdash \sim A, \beta^{-\sim(A \vee B)}}\;\text{lemma 5} \quad \dfrac{\gamma \vdash \delta}{\gamma^{-\sim(A \vee B)}, \sim A, \sim B \vdash \delta}\;\text{lemma 6}}{\alpha, \gamma^{-\sim(A \vee B)}, \sim B \vdash \beta^{-\sim(A \vee B)}, \delta}\;\textbf{mix}\ (H_1)}{\alpha, \alpha, \gamma^{-\sim(A \vee B)} \vdash \beta^{-\sim(A \vee B)}, \beta^{-\sim(A \vee B)}, \delta}\;\textbf{mix}\ (H_1)}{\alpha, \gamma^{-\sim(A \vee B)} \vdash \beta^{-\sim(A \vee B)}, \delta}$$

Case 6. C has the form $\sim(A \wedge B)$, and the inference is:

$$\frac{\alpha \vdash \beta \qquad\qquad \gamma \vdash \delta}{\alpha, \gamma^{-\sim(A \wedge B)} \vdash \beta^{-\sim(A \wedge B)}, \delta}$$

Proceed as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\alpha \vdash \beta}{\alpha \vdash \sim A, \sim B, \beta^{-\sim(A \wedge B)}}\text{ lemma 7} \qquad \cfrac{\gamma \vdash \delta}{\gamma^{-\sim(A \wedge B)}, \sim A \vdash \delta}\text{ lemma 8}}{\alpha, \gamma^{-\sim(A \wedge B)} \vdash \sim B, \beta^{-\sim(A \wedge B)}, \delta}\text{ mix }(\mathrm{H}_1) \qquad \cfrac{\gamma \vdash \delta}{\gamma^{-\sim(A \wedge B)}, \sim B \vdash \delta}\text{ lemma 8}}{\alpha, \gamma^{-\sim(A \wedge B)}, \gamma^{-\sim(A \wedge B)} \vdash \beta^{-\sim(A \wedge B)}, \delta, \delta}\text{ mix }(\mathrm{H}_1)}{\alpha, \gamma^{-\sim(A \wedge B)} \vdash \beta^{-\sim(A \wedge B)}, \delta}$$

Case 7. C has the form $A \supset B$, and the inference has the form

$$\frac{\alpha \vdash \beta \qquad\qquad \gamma \vdash \delta}{\alpha, \gamma^{-A \supset B} \vdash \beta^{-A \supset B}, \delta} .$$

This is the most complicated case, since our lemmas cannot be applied. We distinguish subcases:

7.1. Either $\alpha \vdash \beta$ or $\gamma \vdash \delta$ is justified by **Id** or a structural rule. These cases yield easily to applications of H_2; in some cases the hypotheses of induction do not need to be used at all.

7.2. $A \supset B$ is not introduced by the inference which justifies $\alpha \vdash \beta$.—Again, use H_2. As an example consider the case in which the rule is $\sim\vdash$. Here the **mix** is

$$\frac{\cfrac{\alpha \vdash D, \beta}{\alpha, \sim D \vdash \beta} \qquad \gamma \vdash \delta}{\alpha, \sim D, \gamma^{-A \supset B} \vdash \beta^{-A \supset B}, \delta} .$$

Proceed as follows:

$$\cfrac{\cfrac{\cfrac{\alpha \vdash D, \beta \qquad\qquad \gamma \vdash \delta}{\alpha, \gamma^{-A \supset B} \vdash D, \beta^{-A \supset B}, \delta}\text{ mix }(\mathrm{H}_2)}{\alpha, \gamma^{-A \supset B}, \sim D \vdash \beta^{-A \supset B}, \delta}\sim\vdash}{\alpha, \sim D, \gamma^{-A \supset B} \vdash \beta^{-A \supset B}, \delta} .$$

(Because of the restriction built into $\vdash\supset$, we could not use H_1 if $\alpha \vdash \beta$ were justified by this rule. But this situation cannot arise in the present case.)

7.3. $A \supset B$ is introduced by the rule-application which justifies $\alpha \vdash \beta$. Here, the **mix** is:

$$\frac{\cfrac{\alpha, A \vdash B}{\alpha \vdash A \supset B} \qquad \gamma \vdash \delta}{\alpha, \gamma^{-A \supset B} \vdash \delta} .$$

Now distinguish more subcases, according to the role $A \supset B$ plays in the inference which justifies $\gamma \vdash \delta$.

7.3.1. $A \supset B$ is not introduced by the inference which justifies $\gamma \vdash \delta$. Then $A \supset B$ is parametric in the inference, and H_2 can be applied as in 7.2.

7.3.2. $A \supset B$ is introduced by the inference which justifies $\gamma \vdash \delta$. Then we have

$$\cfrac{\cfrac{\alpha, A \vdash B}{\alpha \vdash A \supset B} \qquad \cfrac{\gamma_1 \vdash A, \delta_1 \quad \gamma_2 B \vdash \delta_2}{\gamma_1, \gamma_2, A \supset B \vdash \delta_1, \delta_2}}{\alpha, \gamma_1^{-A \supset B}, \gamma_2^{-A \supset B} \vdash \delta_1, \delta_2} .$$

We distinguish still more subcases.

7.3.2.1. $A \supset B$ occurs as constituent in neither γ_1 nor γ_2. Here, proceed as follows:

$$\cfrac{\cfrac{\cfrac{\gamma_1 \vdash A, \delta_1 \quad \alpha, A \vdash B}{\gamma_1, \alpha \vdash \delta_1, B}\ \text{mix}\ (H_1) \qquad \gamma_2, B \vdash \delta_2}{\gamma_1, \alpha, \gamma_2 \vdash \delta_1, \delta_2}\ \text{mix}\ (H_1)}{\alpha, \gamma_1^{-A \supset B}, \gamma_2^{-A \supset B} \vdash \delta_1, \delta_2} .$$

7.3.2.2. $A \supset B$ occurs as constituent in both γ_1 and γ_2. Proceed as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\alpha \vdash A \supset B \quad \gamma_1 \vdash A, \delta_1}{\alpha, \gamma_1^{-A \supset B} \vdash A, \delta_1}\ \text{mix}\ (H_2) \quad \alpha, A \vdash B}{\alpha, \gamma_1^{-A \supset B}\ \alpha \vdash \delta_1, B}\ \text{mix}\ (H_1) \qquad \cfrac{\alpha \vdash A \supset B \quad \gamma_2, B \vdash \delta_2}{\alpha, \gamma_2^{-A \supset B}, B \vdash \delta_2}\ \text{mix}\ (H_1)}{\alpha, \gamma_1^{-A \supset B}\ \alpha, \alpha, \gamma_2^{-A \supset B} \vdash \delta_1, \delta_2}\ \text{mix}\ (H_2)}{\alpha, \gamma_1^{-A \supset B}, \gamma_2^{-A \supset B} \vdash \delta_1, \delta_2}$$

Cases 7.3.2.3 and 7.3.2.4 are mixtures of the two cases above. This completes the proof of Theorem 1.

Corollary 1. *The rule* $\dfrac{\alpha \vdash A \quad \alpha \vdash A \supset B}{\alpha \vdash B}$ *is admissible in* **LF**.

Proof:

$$\cfrac{\cfrac{\alpha \vdash A \quad \cfrac{\alpha \vdash A \supset B}{\alpha, A \vdash B}\ \text{lemma 12}}{\alpha, \alpha \vdash B}\ \text{mix}}{\alpha \vdash B}$$

4. **LF** *and the Fitch System*. In this section the results of section 3 will be used to demonstrate the equivalence of **LF** and Fitch's system **F** of propositional calculus. Where β is the sequence $B_1, \ldots, B_n$ let $\mathbf{V}\beta$ be the disjunction $B_1 \vee (B_2 \vee \ldots \vee (B_{n-1} \vee B_n) \ldots)$. We will use the notation '$\alpha \Vdash_{\mathbf{F}} \beta$', or, more simply, '$\alpha \Vdash \beta$' to indicate that (where β is nonempty) there is a proof in **F** of $\mathbf{V}\beta$ on the hypotheses α: i.e., a hypothetical proof having the form

$$\begin{array}{|l} A_1 \\ \vdots \\ \underline{A_n} \\ \vdots \\ \mathbf{V}\beta \end{array}$$

, where α is $A_1, \ldots, A_n$.

Where β is empty, '$\alpha \Vdash \beta$' indicates that there is a proof in **F** of $p \wedge {\sim} p$ on the hypotheses α where p is a fixed propositional variable (say, the first alphabetically). For convenience, we set $\mathbf{V}\beta$ equal to $p \wedge {\sim} p$ where β is empty in $\alpha \Vdash \beta$.

Theorem 2. $\Vdash_{\mathbf{LF}} \alpha \vdash \beta$ *iff* $\alpha \Vdash \mathbf{V}\beta$.

Proof. *Part* 1: If $\Vdash \alpha \vdash \beta$ then $\alpha \Vdash \mathbf{V}\beta$.

We induce on the length of proof in **LF** of $\alpha \vdash \beta$ to show that $\alpha \Vdash \beta$. Corresponding to **Id** and the nineteen primitive rules of **LF**, there are one hypothetical proof and nineteen derived rules to be checked in **F**. We will present five of the most interesting of these cases.

Case 1. **Id**. $A \Vdash A$ as follows:

1.	A	hyp
2.	A	rep

Case 2. **W** $\vdash$. Suppose $B, A, A \Vdash C$. Then $B, A \Vdash C$ as follows:

1.	B	hyp
2.	A	hyp
3.	A	2, rep
	$\vdots$	
	C	

Case 3. $\sim \vdash$. Suppose $A \Vdash \mathbf{V}(B,\beta)$.

3.1. β is nonempty, so that $A \Vdash B \vee \mathbf{V}\beta$. Then $A, {\sim}B \Vdash \mathbf{V}\beta$, as follows:

1.	A	hyp
2.	${\sim}B$	hyp
3.	A	1, rep
	$\vdots$	
n.	$B \vee \mathbf{V}\beta$	3, $A \Vdash B \vee \mathbf{V}\beta$
n+1.	B	hyp
n+2.	${\sim}B$	2, reit
n+3.	$\mathbf{V}\beta$	n+1, n+2, $\sim$elim
n+4.	$\mathbf{V}\beta$	hyp
n+5	$\mathbf{V}\beta$	n+4, rep
n+6	$\mathbf{V}\beta$	n, n+1-n+3, n+4-n+5, $\vee$ elim

3.2. β is empty, so that $A \Vdash B$. Then $A, {\sim}B \Vdash p \wedge {\sim}p$, as follows:

1.	A	hyp
2.	${\sim}B$	hyp
3.	A	1, rep
	$\vdots$	
n.	B	3, $A \Vdash B$
n+1.	${\sim}B$	2, rep
n+2.	$p \wedge {\sim}p$	n, n+1, $\sim$elim

Case 4. $\vdash\mathbf{K}$. Suppose $A \Vdash \mathbf{V}\beta$.

4.1. β is nonempty. Then $A \Vdash B \vee \mathbf{V}\beta$, as follows:

$$\begin{array}{r|ll}
1. & \underline{A} & \text{hyp} \\
 & \vdots & \\
n. & \mathbf{V}\beta & 1, A \Vdash \mathbf{V}\beta \\
n{+}1. & B \vee \mathbf{V}\beta & n, \vee\ \text{int}
\end{array}$$

4.2. β is empty, so that $A \Vdash p \wedge \sim p$. Then $A \Vdash B$, as follows:

$$\begin{array}{r|ll}
1. & \underline{A} & \text{hyp} \\
2. & p \wedge \sim p & 1, A \Vdash p \wedge \sim p \\
3. & p & 2, \wedge\ \text{elim} \\
4. & \sim p & 2, \wedge\ \text{elim} \\
5. & B & 3, 4, \sim \text{elim}
\end{array}$$

Case 5. $\vdash\supset$. Suppose that $A, B \Vdash C$. Then $A \Vdash B \supset C$, as follows:

$$\begin{array}{r|l|ll}
1. & \underline{A} & & \text{hyp} \\
2. & & \underline{B} & \text{hyp} \\
3. & & A & 1, \text{reit} \\
 & & \vdots & \\
n. & & C & 3, A, B \Vdash C \\
n{+}1. & B \supset C & & 2\text{-}n, \supset \text{int}
\end{array}$$

Part 2: If $\alpha \Vdash_{\mathbf{F}} \mathbf{V}\beta$ then $\Vdash_{\mathbf{LF}} \alpha \vdash \beta$. It is known[5] that $\alpha \Vdash_{\mathbf{F}} B$ iff $\alpha \Vdash_{\mathbf{HF}} B$, where $\Vdash_{\mathbf{HF}}$ is the consequence-relation of the system given by the following twelve axiom-schemes and the sole rule of inference *modus ponens*:

1. $(A \supset \cdot B \supset C) \supset A \supset B \supset \cdot B \supset C$
2. $A \supset \cdot B \supset A$
3. $A \supset B \vee A$
4. $A \supset A \vee B$
5. $A \vee B \supset \cdot A \supset C \supset \cdot B \supset C \supset C$
6. $A \wedge B \supset A$
7. $B \wedge A \supset A$
8. $A \supset \cdot B \supset A \wedge B$
9. $A \supset \cdot \sim A \supset B$
10. $\sim\sim A \equiv A$
11. $\sim(A \vee B) \equiv \sim A \vee \sim B$
12. $\sim(A \wedge B) \equiv \sim A \wedge \sim B$

Here, $A \equiv B =_{df} (A \supset B) \wedge (B \supset A)$.

Since by Corollary 1 of the previous section, $\dfrac{\alpha \vdash A \quad \alpha \vdash A \supset B}{\alpha \vdash B}$ is an admissible rule in **LF**, it will suffice for part 2 to show that all the axioms of **HF** are provable in **LF**; i.e., that if A is an axiom of **HF**, then $\Vdash_{\mathbf{LF}} \vdash A$. In each case, this can easily be done. And this completes the proof of Theorem 2.

5. *The System* **LF'**. This is another **L**-system, differing slightly from **LF**.

5.1. *Axioms*. One axiom-scheme, generalized identity (**Id**): $\alpha_1, A, \alpha_2 \vdash \beta_1, A, \beta_2$.

5.2.1. *Rules*. All of the rules of **LF'** are logical rules.

$$\vdash'\vee\ \frac{\alpha \vdash \beta_1, A, B, \beta_2, A \vee B}{\alpha \vdash \beta_1, A \vee B, \beta_2} \qquad \vee\vdash'\ \frac{\alpha_1, A, \alpha_2, A \vee B \vdash \beta_1 \quad \alpha_1, B, \alpha_2, A \vee B, \vdash \beta}{\alpha_1, A \vee B, \alpha_2 \vdash \beta}$$

$$\vdash'\wedge\ \frac{\alpha \vdash \beta_1, A, \beta_2, A \wedge B \quad \alpha \vdash \beta_1, B, \beta_2, A \wedge B}{\alpha \vdash \beta_1, A \wedge B, \beta_2} \qquad \wedge\vdash'\ \frac{\alpha_1, A, B, \alpha_2, A \wedge B \vdash \beta}{\alpha_1, A \wedge B, \alpha_2 \vdash \beta}$$

$$\vdash'{\sim}\vee\ \frac{\alpha \vdash \beta_1, {\sim}A, \beta_2, {\sim}(A \vee B) \quad \alpha \vdash \beta_1, {\sim}B, \beta_2, {\sim}(A \vee B)}{\alpha \vdash \beta_1, {\sim}(A \vee B), \beta_2} \qquad {\sim}\vee\vdash'\ \frac{\alpha_1, {\sim}A, {\sim}B, d_2, {\sim}(A \vee B) \vdash \beta}{\alpha_1, {\sim}(A \vee B), \alpha_2 \vdash \beta}$$

$$\vdash'{\sim}\wedge\ \frac{\alpha \vdash \beta_1, {\sim}A, {\sim}B, \beta_2, {\sim}(A \wedge B)}{\alpha \vdash \beta_1, {\sim}(A \wedge B), \beta_2}$$

$${\sim}\wedge\vdash'\ \frac{\alpha_1, {\sim}A, \alpha_2, {\sim}(A \wedge B) \vdash \beta \quad \alpha_1, {\sim}B, \alpha_2, {\sim}(A \wedge B) \vdash \beta}{\alpha_1, {\sim}(A \wedge B), \alpha_2 \vdash \beta}$$

$${\sim}\vdash'\ \frac{\alpha_1, \alpha_2, {\sim}A \vdash \beta_1, A, \beta_2}{\alpha_1, {\sim}A, \alpha_2 \vdash \beta_1, \beta_2}$$

$$\vdash\supset'\ \frac{\alpha_1, A, \alpha_2 \vdash B}{\alpha_1, \alpha_2 \vdash \beta_1, A \supset B, \beta_2} \qquad \supset\vdash'\ \frac{\alpha_1, B, \alpha_2, A \supset B \vdash \beta \quad \alpha_1, \alpha_2, A \supset B \vdash A, \beta}{.\alpha_1, A \supset B, \alpha_2 \vdash \beta}$$

$$\vdash'{\sim}{\sim}\ \frac{\alpha \vdash \beta_1, A, \beta_2, {\sim}{\sim}A}{\alpha \vdash \beta_1, {\sim}{\sim}A, \beta_2} \qquad {\sim}{\sim}\vdash'\ \frac{\alpha_1, A, \alpha_2, {\sim}{\sim}A \vdash \beta}{\alpha_1, {\sim}{\sim}A, \alpha_2 \vdash \beta}$$

By the *length* of a proof in **LF'**, we mean the maximum number of steps in any branch of the proof. We write '$\Vdash^{m}_{\mathbf{LF'}} \alpha \vdash \beta$' to indicate that $\alpha \vdash \beta$ has a proof in **LF'** of length m.

Lemma 13. *If* $\Vdash^{m}_{\mathbf{LF'}} \alpha \vdash \beta$ *and* $\dfrac{\alpha \vdash \beta}{\alpha^* \vdash \beta^*}$ (*i.e., if* $\alpha \leqslant \alpha^*$ *and* $\beta \leqslant \beta^*$) *then for some* $n \leqslant m$, $\Vdash^{n}_{\mathbf{LF'}} \alpha^* \vdash \beta^*$.

Proof. By induction on m. If $m = 1$, $\alpha \vdash \beta$ is an instance of **Id'**, and so is $\alpha^* \vdash \beta^*$. We will present just two of the remaining cases.

Case 1. If $\alpha \vdash \beta$ is $\alpha \vdash \beta_1, A \vee B, \beta_2$ and is proved as follows:

$$\begin{array}{ll} m-1 & \alpha \vdash \beta_1, A, B, \beta_2, A \vee B \\ \hline m & \alpha \vdash \beta_1, A \vee B, \beta_2 \end{array} \ \vdash'\vee$$

then β^* has the form $\gamma_1, A, B, \gamma_2, A \vee B$. By the hypothesis of induction, $\Vdash^{m-1}_{\mathbf{LF'}}, \alpha^* \vdash \gamma_1, A, B, \gamma_2, A \vee B$.

Then proceed: $$\begin{array}{ll} m-1 & \alpha^* \vdash \gamma_1, A, B, \gamma_2, A \vee B \\ \hline m & \alpha^* \vdash \gamma_1, A \vee B, \gamma_2 \end{array} \ \vdash'\vee\ .$$

Case 2. If $\alpha \vdash \beta$ is $\alpha_1, \alpha_2 \vdash \beta_1, A \supset B, \beta_2$ and is proved:

$$\begin{array}{cc} m\text{-}1 & \dfrac{\alpha_1, A, \alpha_2 \vdash B}{\alpha_1, \alpha_2 \vdash \beta_1, A \supset B, \beta_2} \\ n & \end{array} \quad \vdash' \supset,$$

then $\Vdash^{m-1}_{\mathbf{LF}} \alpha^*, A \vdash B$ by the hypothesis of induction. Then proceed:

$$\begin{array}{cc} m\text{-}1 & \dfrac{\alpha^*, A \vdash B}{\alpha^* \vdash \beta^*} \\ m & \end{array} \quad \vdash' \supset.$$

Theorem 3. $\Vdash_{\mathbf{LF}} \alpha \vdash \beta$ *iff* $\Vdash_{\mathbf{LF}'} \alpha \vdash \beta$.

Proof. Part 1. If $\Vdash_{\mathbf{LF}} \alpha \vdash \beta$ then $\Vdash_{\mathbf{LF}'} \alpha \vdash \beta$. By lemma 13 all the structural rules of **LF** are admissible in **LF'**, and it follows directly that all of the logical rules of **LF** are likewise admissible in **LF'**: e.g., $\vdash \vee$, as follows:

$$\dfrac{\dfrac{\alpha \vdash A, B, \beta}{\alpha \vdash A, B, \beta, A \vee B}}{\alpha \vdash A \vee B, \beta} \quad \vdash' \vee$$

Part 2. If $\Vdash_{\mathbf{LF}'} \alpha \vdash \beta$ then $\Vdash_{\mathbf{LF}} \alpha \vdash \beta$. This is clear, since any instance of **Id'** is provable in **LF** and since all the rules of **LF'** are easily derivable in **LF**: $\vee \vdash'$, for instance, as follows:

$$\dfrac{\dfrac{\dfrac{\alpha_1, A, \alpha_2, A \vee B \vdash \beta}{\alpha_1, \alpha_2, A \vee B, A \vdash \beta} \qquad \dfrac{\alpha_1, B, \alpha_2, A \vee B \vdash \beta}{\alpha_1, \alpha_2, A \vee B, B \vdash \beta}}{\alpha_1, \alpha_2, A \vee B, \alpha_1, \alpha_2, A \vee B, A \vee B \quad \beta}}{\alpha_1, \alpha_2, A \vee B \vdash \beta}$$

This provides a sufficient sketch of the proof of Theorem 3.

6. *A Decision Procedure for* **LF'**.[6] By a *tree* we mean a discrete lower semilattice with least element, such that any two elements of the tree which have an upper bound are comparable. The least element of a tree is called its *origin*, the tree's elements *nodes*, and its maximal chains (under set-theoretical inclusion) *branches*. We diagram a tree by placing its origin at the bottom of the diagram, and by placing lines connecting each node with its immediate successors, which are placed on a level immediately above the node. Clearly, every proof in **LF'** is a tree whose nodes are sequents. A tree is *finite* if it has a finite number of nodes, and has the *finite branch property* if all of its branches are finite. It has the *finite fork property* if none of its nodes possesses an infinite number of immediate successors.

The *distinguished proof-search tree* (**dpst**) $p_{\alpha \vdash \beta}$ of a sequent $\alpha \vdash \beta$ is the tree defined as follows:

i) $\alpha \vdash \beta$ is the origin of $p_{\alpha \vdash \beta}$.

ii) Where $\gamma \vdash \delta$ is an axiomatic node (i.e., is an instance of **Id'**), $\gamma \vdash \delta$ has no successors, and the branch terminates.

iii) Where $\gamma \vdash \delta$ is a nonaxiomatic node of $p_{\alpha \vdash \beta}$, the immediate successors of $\gamma \vdash \delta$ consist of all those sequents $\gamma' \vdash \delta'$ which

a) can count as premisses for $\gamma \vdash \delta$ under the rules of **LF'**, and

b) are such that it is not the case that $\dfrac{\gamma \vdash \delta}{\gamma' \vdash \delta'}$, or that indeed $\dfrac{\gamma^* \vdash \delta^*}{\gamma' \vdash \delta'}$ for any $\gamma^* \vdash \delta^*$ preceding $\gamma \vdash \delta$ in $p_{\alpha \vdash \beta}$.

In view of note 3, the stipulation that it is not the case that $\dfrac{\gamma^* \vdash \delta^*}{\gamma' \vdash \delta'}$ is equivalent to the condition that $\gamma^* \nleq \gamma'$ or $\delta^* \nleq \delta'$.

Lemma 14. *$p_{\alpha \vdash \beta}$ is complete, in the sense that if $\Vdash_{\mathbf{LF'}} \alpha \vdash \beta$ then some subtree of $p_{\alpha \vdash \beta}$ is a proof in* **LF'** *of $\alpha \vdash \beta$.*

Proof. In view of the construction of $p_{\alpha \vdash \beta}$, the lemma follows immediately from lemma 13.

We will say that $\alpha \vdash \beta$ and $\gamma \vdash \delta$ are *cognate* if $\dfrac{\alpha \vdash \beta}{\gamma \vdash \delta}$ and $\dfrac{\gamma \vdash \delta}{\alpha \vdash \beta}$. The class of all sequents cognate with $\alpha \vdash \beta$ is called *cognation class* of $\alpha \vdash \beta$. A cognation class is said to appear in a branch of a **dpst** if any of its members occurs in the branch.

Lemma 15. *Only a finite number of cognation classes can appear in any branch of $p_{\alpha \vdash \beta}$.*

Proof. By inspection of the rules of **LF'**, it is easily verified that a wff A is a well-formed part of a constituent of some premiss of an inference only if A is a well-formed part of some constituent of the conclusion. And since the constituents of $\alpha \vdash \beta$ can have only a finite number of subformulas, and only a finite number of cognation classes can be constructed out of these, the desired result follows.

Lemma 16. *Every* **dpst** *$p_{\alpha \vdash \beta}$ has the finite branch property.*[7]

Proof. Given Lemma 15, it will suffice to show that only a finite number of sequents from any given cognation class appear in a branch of $p_{\alpha \vdash \beta}$. Then let M consist of those members of a given cognation class which appear in a specified branch. We may order M under the relation $<$ such that $\gamma_1 \vdash \delta_1 < \gamma_2 \vdash \delta_2$ iff every wff in γ_1 has at least as many occurrences as constituent in γ_2 as it does in γ_1, and every wff in δ_1 has at least as many occurrences as constituent in δ_2 as it does in δ_1.

Since there are only a finite number of sequents $\gamma_1 \vdash \delta_1$ such that $\gamma_1 \vdash \delta_1 < \gamma_2 \vdash \delta_2$, there must be minimal elements under $<$ in M. And since only a finite number of different constituents can appear in the members of cognation class, there must be only a finite number of such minimal elements.

But given condition iii(b) of the definition of **dpst**, it follows that any node of the branch which succeeds all of these minimal elements of M cannot itself be a member of M. And since it follows that no member of M can appear above a certain finite level of the branch, M is finite.

Lemma 17. *Every* **dpst** *$p_{\alpha \vdash \beta}$ is finite.*

Proof. It is clear that $p_{\alpha \vdash \beta}$ satisfies the finite fork property. Then our result follows from the general result, proved by D. König [5] (on the basis of the axiom of choice), that every tree possessing both the finite fork property and the finite branch property is finite.

Theorem 4. **LF'** *is decidable.*

Proof. It is evident that the construction from $\alpha \vdash \beta$ of $p_{\alpha \vdash \beta}$ is effective. Then, since $p_{\alpha \vdash \beta}$ is finite (lemma 17) and hence possesses only a finite number of subtrees, one of which must be a proof of $\alpha \vdash \beta$ if $\Vdash_{\mathbf{LF'}} \alpha \vdash \beta$ (lemma 14), and since it is also clear that the property of being a proof in **LF'** is effective, it follows that there is an effective way of finding a proof in **LF'** of $\alpha \vdash \beta$ if there is any such proof, and of verifying that there is no such proof in case $\alpha \vdash \beta$ is not provable.

NOTES

1. I am grateful to Nuel D. Belnap, Jr., Hughes Leblanc, and Michael D. Resnik for helpful comments and suggestions. This research was supported in part by National Science Foundation Grant GS-190.

2. We will abuse this notation in much the same way that the meta-linguistic assertion-sign '$\vdash$' is sometimes mistreated. That is, we will use $\dfrac{`\alpha \vdash \beta'}{\gamma \vdash \delta}$ sometimes to indicate that *there is* a structural proof of $\gamma \vdash \delta$ on the hypothesis $\alpha \vdash \beta$, and sometimes as an *abbreviation* of such a proof on hypotheses.

3. It is not difficult to show that $\dfrac{\alpha \vdash \beta}{\gamma \vdash \delta}$ iff $\alpha \leqslant \gamma$ and $\beta \leqslant \delta$.

4. The *degree* of a wff is simply the number of occurrences of connectives in that wff. The *rank* of a wff eliminated in an instance of **mix** depends on the number of steps leading back to the points at which the eliminated wff is first introduced in the proofs of the premisses of the instance. For a definition of *rank*, see Gentzen [3].

5. As far as I know, a proof of this result has not been published. But the methods of the second chapter of Fitch's [2] can easily be used to establish the equivalence of **F** and **HF**.

6. The strategy and terminology of this section is modeled on that of Belnap-Wallace [1], pp. 24-29.

7. The idea behind this lemma is due to Kripke. His abstract [6] announces a result obtained by a similar method.

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