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A NOTE ON THE AXIOMATIZATIONS OF CERTAIN MODAL SYSTEMS[1]

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In this note I prove

a) that the proper axiom of S2°, viz.,

B8 $\mathfrak{C}MKpqMp$

can be substituted by the following thesis

V1 $\mathfrak{C}MKpNpMp$

which shows that not only $\{S2°\} \rightleftarrows \{S1°;B8\} \rightleftarrows \{S1°;V1\}$ but also, *a fortiori,* that $\{S2\} \rightleftarrows \{S1;B8\} \rightleftarrows \{S1;V1\}$; and
b) that a result of Hállden who has proved in [2], p. 128, Lemma 4, that the addition of

P1 $\mathfrak{C}MKpNpKpNp$

to S3 as a new axiom generates a system equivalent to S4 can be strengthened as follows:

The addition of a thesis

V2 $\mathfrak{C}MKpNpp$[2]

which, clearly, is an elementary consequence of *P1* as a new axiom to S3° gives a system equivalent to S4°, i.e., $\{S4°\} \rightleftarrows \{S3°;V2\}$ and, therefore, $\{S4\} \rightleftarrows \{S3;V2\}$.

On the other hand I show that a result of Yonemitsu who has proved in [9] that $\{S2;P1\} \rightleftarrows \{T\}$ can also be strengthened; as follows: $\{S1;V2\} \rightleftarrows \{T\}$. It may be remarked that this shows that *V2* is a weaker formula than the proper axiom of S4° or S4, i.e.,

C10 $\mathfrak{C}MMpMp$

This remark—that *V2* is weaker than C10—was made by Sobociński in [7].

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1. I am indebted to Professor Bolesław Sobociński for helpful suggestions.
2. The structural similarity of *V1* and *V2* may be noted.

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In this paper we use the notation of [6]. An acquaintance with the modal systems of Lewis and, especially, with the systems S1°-S4°, S3* and T which are defined e.g., in [1], pp. 43-144 and [6], pp. 52-53, is presupposed.

1. Theorem 1. $\{S2°\} \rightleftarrows \{S1°; B8\} \rightleftarrows \{S1°; V1\}$.

Since $V1$ is a substitution instance of $B8$, it remains only to prove that $\{S1°; V1\} \rightarrow \{S2°\}$.

$Z1$	$\mathfrak{C}\mathfrak{C}KpqpCMKpqMp$	[S1°; *cf.* 35.32 in [1]]
$Z2$	$\mathfrak{C}CKpqpCpp$	[S1°; *cf.* 34.1 in [1]]
$Z3$	$\mathfrak{C}\mathfrak{C}KpqpLCpp$	[$Z2$; S1°; *cf.* 34.421 in [1]]
$Z4$	$\mathfrak{C}LCppCMKpqMp$	[$Z1$; $Z3$; S1°]
$Z5$	$\mathfrak{C}NMpLCpp$	[$V1$; S1°]
$Z6$	$\mathfrak{C}NMpCMKpqMp$	[$Z5$; $Z4$; S1°]
$B8$	$\mathfrak{C}MKpqMp$	[$Z6$; S1°]

Thus Theorem 1 is proved.

2. Corollary 1. $\{S2\} \rightleftarrows \{S1; B8\} \rightleftarrows \{S1; V1\}$

It follows immediately from Theorem 1.

3. Theorem 2. $\{S4°\} \rightleftarrows \{S3°; V2\}$

3.1. First we show that $\{S4°\} \rightarrow \{V2\}$[3]

$Z1$	$\mathfrak{C}NMNp\mathfrak{C}qp$	[S2°]
$Z2$	$\mathfrak{C}NMMKpNp\mathfrak{C}qNMKpNp$	[$Z1$, $p/NMKpNp$; S1°]
$Z3$	$\mathfrak{C}NMpNMMp$	[S4°]
$Z4$	$\mathfrak{C}NMKpNp\mathfrak{C}qNMKpNp$	[$Z3$, $p/KpNp$; $Z2$; S1°]
$Z5$	$NMKpNp$	[S1°]
$Z6$	$\mathfrak{C}qNMKpNp$	[$Z4$; $Z5$]
$V2$	$\mathfrak{C}MKpNpp$	[$Z6$, q/Np; S1°]

Thus $V2$ is a consequence of S4°.

3.2. Now, we prove that $\{S3°; V2\} \rightarrow \{S4°\}$.

$Z1$	$\mathfrak{C}MKpNpNp$	[$V2$, p/Np; S1°]
$Z2$	$\mathfrak{C}MKpNpKpNp$	[$V2$; $Z1$; S1°]
$Z3$	$\mathfrak{C}MKpNpq$	[$Z2$; S1°]
$Z4$	$\mathfrak{C}qLCpp$	[$Z3$, q/Nq; S1°]
$Z5$	$CLqLLCpp$	[$Z4$; S1°]
$Z6$	$LLCpp$	[$Z5$, q/Cpp; S1°]
$Z7$	$\mathfrak{C}Lp\mathfrak{C}qp$	[S2°]
$Z8$	$\mathfrak{C}\mathfrak{C}pq\mathfrak{C}LpLq$	[S3°]
$Z9$	$\mathfrak{C}Lp\mathfrak{C}LqLp$	[$Z7$; $Z8$, p/q, q/p; S1°]
$Z10$	$\mathfrak{C}LpCLLqLLp$	[$Z9$; S1°]
$Z11$	$\mathfrak{C}LLqCLpLLp$	[$Z10$; S1°]
$Z12$	$\mathfrak{C}LLLq\mathfrak{C}LpLLp$	[$Z11$; S2°]
$Z13$	$LLLCpp$	[$Z11$, q/Cpp, $p/LCpp$; $Z6$; $Z6$; S1°]

3. The proof in 3.1 is a modification of the proof of Theorem 7, p. 126, [4].

Z14 $\mathfrak{C}LpLLp$ [*Z12*,*q*/*Cpp*;*Z13*]
C10 $\mathfrak{C}MMpMp$ [*Z14*; S1°]

This completes the proof of Theorem 2. Note that the deductions presented in 3.2 above show that Parry's Theorem *cf.* [5], p. 148 that an addition of any formula possessing a form *LLα* to S3 yields S4 can be reformulated for S3° and S4°.

4. Corollary 2. $\{S4\} \rightleftarrows \{S3; V2\}$

This follows immediately from Theorem 2.

5. It is reasonable to investigate now whether $\{S4°\} \rightleftarrows \{S3^*; V2\}$ or $\{S4\} \rightleftarrows \{S3^*; V2\}$. In fact, both are false. That the former is false is seen by Group IV., p. 494, [3] taking *1* alone as the designated value. The matrix verifies S4° but does not verify *Z5* [6], an axiom of S3*. The falsity of the latter is again established by the above-mentioned matrix taking *1* and *2* both as designated values. It verifies $\{S3^*, V2\}$ but does not verify $\mathfrak{C}pMp$, an axiom of S4.

6. In [7], p. 176, Sobociński has proved that *P1* is a consequence of T. Hence, clearly, *V2* is also provable in T. On the other hand, since in 3.2 *Z6* follows from *V2* and S1°, by a result of Yonemitsu [8] we can conclude that $\{S1; V2\} \rightleftarrows \{T\}$.

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