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CONTRARIETY

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This paper is an attempt to make philosophical capital out of an important difference between the Aristotelian logic of terms and the Stoic, or 'modern', logic of propositions. This difference is, that although both logics include and give formal recognition to the relation of contradiction, only the former, and not the latter, takes account of the relation of contrariety. Here I do not refer to the relation of contrariety as extending between terms (thus for example the terms 'pleasure' and 'pain', 'black' and 'white' denote contraries), but as extending between propositions.

The most common definition of *contrariety* is as follows: two propositions are contraries if they cannot both be true. For comparison, the definition of *contradiction* states that two propositions are contradictories if they can neither both be true nor both be false, and that of *sub-contrariety*, that they cannot both be false. As examples from the Aristotelian square of opposition, 'All A is B' and 'No A is B' are contraries, while 'All A is B' and 'Some A is not B' are contradictories, and 'Some A is B' and 'Some A is not B' are sub-contraries. In the modernized Stoic logic, p and Np[1] are contradictories, but there is no formal analogue for, nor logical role played by, the contrary of p. The fact that there is not seems *prima facie* to be a consequence of Stoic logic's being a logic of unanalysed propositions, while Aristotelian logic is not. Notwithstanding this seemingly irreconcilable difference between the two logics, there may still be ways of introducing the notion of contrariety into propositional logic. For example we might, analogously with Np, write Rp for the contrary of p. This device is adopted by L. Goddard[2] in order to give a satisfactory analysis of exclusive disjunction: he points out that what makes disjunctions exclusive is not use of the exclusive 'or', but an internal opposition between the disjuncts which we can express by saying that they are contraries. The aim of this paper will be to investigate the

1. The logical notation of Łukasiewicz will be used throughout.

2. 'The Exclusive "Or"', *Analysis* 1960.

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formal properties of the notion of contrariety: that is, to provide means of introducing the operator R into the logic of propositions.

1. Intuitive notions. There are difficulties about the very expression Rp. In the first place, it is not a truth-function of p. If p is true, then Rp is false, but nothing about Rp follows from p's falsehood. Secondly, there may seem to be a vagueness about Rp which Np lacks. Thus if p denotes that X is white, Np denotes that X is not white, but what does Rp denote? That X is black, grey, blue? This question as to whether there is a unique contrary of p will receive different answers in this paper, each of which will be accommodated in the formal part of our investigation. Thirdly, it may be thought that the concept of contrariety is essentially a *relational* one; that it is really a function of two propositional variables rather than one. What could it mean to say that p is contrary (Rp)? Must we not rather say that p is contrary *to something* (Rpq)? This objection is not a strong one, since it applies equally to the notion of contradiction. As is the case with Np, it is sometimes very useful to have a way of denoting the contrary of p, in addition to saying *that* p is contrary to something else. For example: 'It is not the case that p implies its own contrary' ($NCpRp$). This paper will explore the possibility of using R as a propositional function of one variable, analogous to N.

Let us now examine some concrete examples of contrary propositions. In the *De interpretatione*, where he is mainly interested in investigating the various types of opposition among propositions, Aristotle gives us sufficient information to construct the following table:[3]

Man is just	Man is not-just
Man is not not-just	Man is not just

Here the lines join contradictories. The two propositions at the top are contraries. But are they true contraries? To answer this we must examine Aristotle's doctrine of contrariety more closely. In his inquiry into change in the *Physics*, Aristotle points out that when something becomes white, it does so only from being 'not-white', and furthermore not from *any* 'not-white' (not, for example, from 'musical'), but from black or some intermediate colour (188a35 ff.). Change is, for Aristotle, from one contrary to its twin within the same genus, although there may be many intermediate states between the two. It is, however, the idea of contraries as existing in *pairs* that interests us here. This idea implies that if q is the contrary of p, q must be the sole contrary of p. In that case p would be the sole contrary of q, in other words the contrary of the contrary of p. In symbols, $EpRRp$. For clarity, let us speak of contraries which necessarily exist in pairs as *strong contraries*, while those which need not be paired will be *weak contraries*. For Aristotle, who regarded all the colours as

3. 19b19 ff. See also the *Prior Analytics*, 51b36 ff. Aristotle's own table has contraries and sub-contraries diagonally opposed, rather than contradictories.

intermediate to, and composed of, black and white,[4] only the latter two would be strong contraries.

There is evidence in Aristotle's writings that he favoured the strong over the weak concept of contrariety. In the fragment of his work *On Contraries* which remains to us (*Arist. Frag.* ed. W. D. Ross p. 110) he defines contraries as 'the things which differ most from one another in the same genus'. This seems to limit the number of mutually correlative contraries to two. However in the *Nicomachean Ethics*, 1108b13, we read that 'the extreme states are contrary both to the intermediate state and to each other'. But, further on, 'the greatest contrariety is that of the extremes to each other' (1108b27), and 'contraries are defined as the things that are furthest from each other'.

The formalization of the notion of strong contrariety will be seen to present the greater logical challenge. But even among weak contraries interesting logical connexions hold. For example, the propositions 'X is white' and 'X is red' would be weak contraries, and we may construct for them a square of opposition analogous to the one above, with the addition of arrows denoting implications:

X is white	X is red
↓	↓
X is not red	X is not white.

From this square we may extract a principle fundamental to any theory of contrariety, whether strong or weak, namely that the contrary of a proposition implies the negation of that proposition. In symbols, *CRpNp*. By substitution, transposition and double negation we obtain from this law an indefinitely large number of derivative laws, *CRNpp*, *CpNRp*, *CNpNRNp*, *CRNRpRp*, etc., all of which may be exhibited in an extended square of opposition:[5]

etc.

RNRNp	*RNRp*
RNp	*Rp*
p	*Np*
NRp	*NRNp*
NRNRp	*NRNRNp*

etc.

Do we see all these implications in concrete form, in ordinary speech? Doubtless not, but we see the beginnings of them. Taking 'John adores Mary' as the contrary (not the contradictory) of 'John does not love

4. See the Metaphysics, 1057a17 ff., and J. P. Anton, *Aristotle's Theory of Contrariety*, p. 94.

5. In this and in subsequent tables, formulae occurring one above another in columns will be understood to be joined by arrows of implication reading downwards.

Mary' ("John, you don't love me". "Don't be ridiculous. I adore you"), we get:

John adores Mary	John hates Mary
John loves Mary	John does not love Mary
John does not hate Mary	John does not adore Mary.

This table could be continued upward and downward as long as there existed a supply of increasingly strong terms such as 'worships', 'detests', etc. At the median line, the pair of contradictories 'John loves Mary' and 'John does not love Mary' separate columns of (weak) contraries above, and columns of sub-contraries below. Thus ordinary language provides at least the core of our indefinitely extended logical square.

2. *Formalization of these notions.* The two logical laws based on intuition, *CRpNp* and *EpRRp* (the latter for strong contraries only), provide us with sufficient material for exhibiting the 'logic' of the concept of contrariety in various alternative formal axiomatic systems. The first thing that strikes one is the analogy between the operator *R* and the modal operator *LN*, 'it is impossible that . . .'. Under this interpretation the law *CRpNp* becomes the familiar *CLpp* ('if *p* is necessarily true, then *p* is true'), basic to all modal logic. The extended square of contraries then becomes:

etc.

LLp	*LLNp*
Lp	*LNp*
p	*Np*
Mp	*MNp*
MMp	*MMNp*

etc.

This analogy between contrariety and impossibility suggests some already existing modal system as a means of introducing the operator *R*. It may not be wholly satisfactory, as we shall see, to introduce *R* in this way, but at least any contrariety logic will be a modal logic in the sense of containing the law *CRNpp*. Let us, therefore, as a start, define the operator *R* as follows:

Df. *R*: $R = LN$,

and attach this definition to the best-known group of modal systems, the Lewis systems. It is characteristic of the Lewis systems that they all contain theses of the form 'it is necessary that. . .'. Therefore our definition of *R* will give us corresponding theses of the form 'it is contrary for it not to be the case that. . .'. If we can stomach grammar of this sort then nothing further will prevent us from accepting at least the early Lewis systems as basic contrariety logic; if not, then we must look for systems not containing theses of the form $L\alpha$.[6] In defence of the former

6. The Ł-modal system of Łukasiewicz, and Prior's system Q, suggest themselves here, but will not be dealt with in this paper.

alternative we may note that the only propositions asserted as necessary in the Lewis systems are already-demonstrated theses, including theses of the propositional calculus. Hence it would be plausible to translate R in the Lewis systems as 'it is contrary to the logical rules of this system that . . .'.

3. The system S2 with R added. We shall start with S2, which Lewis eventually designated *the* system of strict implication, as the basic Lewis system to which to add our definition of R. This is not to deprecate the weaker system S1 as a base, but simply takes account of the fact that nothing radically new is added to the theory of contrariety in progressing from S1 to S2. Something radically new, however, *is* added when we progress from S2 to S3. For S2 contains an infinite number of non-equivalent modalities while S3 does not.[7] By a 'modality' is meant a sequence of monadic operators, i.e. a sequence containing only N's, L's, M's or, of course, R's. Now our extended square of opposition contains an infinite number of modalities of the form RN, $RNRN$, $RNRNRN$, . . . (L, LL, LLL, . . .) alone, apart from other forms. Hence S3 cannot serve as the formal system corresponding to our intuitions as reflected in that square. As, however, all the implications contained in that square are to be found in S2, we may take the latter system as providing a satisfactory formalization of the notion of weak contrariety.

When we come to strong contrariety, we note that S2's infinite square of opposition is quite consistent with the law $EpRRp$. This square requires the irreducibility of infinite strings of RN's, while strong contrariety merely eliminates every pair RR. But neither S2, nor any stronger Lewis system, can contain the reduction thesis $EpRRp$ without destroying that distinction between p and Lp which makes it a modal logic. This may be most easily seen by noting that

$$RRp = LNLNp = LMp,$$

and that while the system S5 contains one half of the reduction theses $EpRRp$, namely

(a) $\quad CpLMp$,

no Lewis system contains

(b) $\quad CLMpp$.

In fact we find in Prior, *Formal Logic* p. 207, an argument showing that adding $CLMpp$ to S2 will enable us to prove the thesis $CpLp$, destructive of the distinction between necessity and actuality, or, in its alternative form $CNpRp$, of the distinction between contrariety and contradiction. The proof proceeds as follows, C denoting material implication and E' strict equivalence:

7. McKinsey, 'Proof that there are infinitely many modalities in Lewis's system S2', *The Journal of Symbolic Logic* 1940. See Prior, *Time and Modality*, appendix B, for a summary discussion of modalities in Lewis systems.

1.	*CLMpp*	
2.	*E'AMpMqMApq*	(thesis of S2)
3.	*LCpMp*	(thesis of S2)
4.	*LCLpMLp*	(3)
5.	*LANLpMLp*	(4 Df. C)
6.	*LAMNpMLp*	(5 Df. *L*, double negation)
7.	*LMANpLp*	(6, 2 Replacement of strict equivalents)
8.	*ANpLp*	(1,7 Detachment for *C*)
9.	*CpLp*	(8 Df. *C*).

In view of this difficulty, it will be best to attempt a formalization of strong contrariety by constructing a system of modal logic from scratch. In the next section I examine what such a system, let us call it R5, would have to look like, and in sections 5-6 I make a start at constructing it.

4. *Requirements for a logic R5 of strong contrariety.* Transposing proposition (a) of the previous section we obtain:

		CNLMpNp,
i.e.		*CNLNLNpNp*,
whence by substitution		*CNLNLNNpNNp*,
and double negation		*CNLNLpp*,
i.e.	(c)	*CMLpp*;

and a similar argument yields from (b):

(d) *CpMLp*.

Hence in any system of strong contrariety containing transposition, double negation and a rule for the replacement of equivalents we shall find that the three expressions *p*, *LMp*, and *MLp* are equivalent and replaceable by one another. This provides a means of reducing strings of iterated modalities different from that of the Lewis systems. The reduction theses of R5 may be compared with those of S5 as follows:

In both R5 and S5:	*CpLMp*
	CMLpp,
In R5 but not S5:	*CLMpp*
	CpMLp,
In S5 but not R5:	*CMpLMp*
	CMLpLp.

Thus in S5 strings of *L*'s and *M*'s reduce to the last *L* or *M*, while in R5 the combinations *LM* and *ML* vanish. Concerning the relative degree of intuitiveness of R5 and S5, in the former we have that if *p* is true, then it is possible that *p* is necessary; in the latter, that if *p* is possible, then it is necessary that *p* is possible. The reader will choose which he prefers:

considered purely as a modal system, R5 provides at least an interesting alternative to S5.[8]

In R5, however, we cannot reduce the number of nonequivalent modalities to S5's six, or indeed to any finite number. R5 must contain an infinite number of non-equivalent modalities of the form p, Lp, LLp, . . . , as may be seen by the fact that if we identified any two of them we would have to identify them all. Writing $L^n p$, $n \geqslant 0$, for the expression $LL \ldots Lp$ containing n L's, we would have, for some n and $r > 0$, and assuming that R5 allows for the replacement of equivalents,

	$CL^n pL^{n+r}p$
hence by substitution	$CL^n MpL^{n+r}Mp$
hence by reduction	$CL^{n-1}pL^{n+r-1}p$,
hence eventually	$CL^0 pL^r p$,
i.e.	$CpL^r p$,

and hence, by the law $CL^r pLp$, derived from $CLpp$ and the transitivity of implication:

$$CpLp.$$

In other words, weak contrariety's infinite square of opposition must be a feature of the logic of strong contrariety. This fact provides us with a significant piece of information concerning R5. It is known that a finite matrix exists for all the Lewis systems which is adequate, in the sense that it satisfies their axioms, but which fails to satisfy the formula $CpLp$. With R5 this is not the case, as will be shown. No finite matrix allows for the assignment of one of more than a finite number of different possible sequences of truth-values to any formula. In particular this is true of formulae containing only one variable; in the classical two-valued propositional calculus, for example, there are open to formulae of one variable only the possible sequences TT, TF, FT and FF. Again, a three-valued system will allow only 9 different sequences for formulae comprising one variable and implication and negation functions alone ('C-N-p formulae'). We may express this by saying that a three-valued system allows for a maximum of 9 'C-N-p modalities' (in an extended sense of 'modality'). Any C-N-p formula will have a truth-value sequence identical with one of these 9, hence be equivalent to another formula of the same sequence (assuming that the system contains the thesis Cpp), and hence be replaceable by that formula (assuming that the system allows for the replacement of equivalents).

8. Another interesting alternative to S5, again no less plausible, is provided by the eight-modality system R8M, in which LMp and MLp are equivalent to each other though not to p. See Storrs McCall, 'A Modal logic with eight modalities', in *Contributions to Logic and Methodology in honor of T. M. Bocheński*, Amsterdam 1965, pp. 84-90.

To return to R5, we have seen that it must contain an infinite number of modalities of the form *p*, *Lp*, *LLp*, ... If it does not, then *CpLp* can be be proved in it. Hence there can be no finite matrix, adequate for R5, which does not at the same time satisfy *CpLp*, since such a matrix would satisfy theses permitting reduction of the modalities *p*, *Lp*, *LLp*, . . . to a finite number. In the terminology of Harrop,[9] R5 lacks the *finite model property* in that, in the case of one of its non-theorems, namely *CpLp*, there exists no finite matrix which fails to satisfy it. It would be convenient, in attempting to construct an infinite matrix adequate for R5 which rejects *CpLp*, if we could confine ourselves to matrices for implication and negation only. This is not difficult to arrange, in virtue of the fact that it is possible to define necessity in terms of implication and negation. We find in Lewis and Langford's *Symbolic Logic*, theorem 18.14, the equivalence of *Lp* and *CNpp* (*C* denoting strict implication) and with this definition of necessity the law *CLpp* becomes *CCNppp*, the *consequentia mirabilis* of the Scholastics. With this definition it is possible to regard modal logic as a species of propositional logic, a species which asserts *CCNppp* and rejects *CpCNpp*, the latter being a substitution of *CpCqp*, one of the paradoxes of material implication. It is I think more natural to view the Lewis systems S1-5 in this way: phrased in terms of (strict) implication, negation and conjunction alone, and with *L* defined as above, they become progressively larger proper parts of the classical calculus, and the distinction between strict and material implication is no longer to be found *within* the Lewis systems, but *between* each of them and classical logic. We shall construct R5 in this way, defining *Rp* to be *CpNp* (= *LNp*).

To summarize these requirements which we make of it, R5 must contain as theses:

(a)	*CpNNp*	
(b)	*CNNpp*	
(c)	*CCNqNpCpq*	
(d)	*CCNppp*	(= *CLpp*, whence *CRpNp*)
(e)	*CpCCpNpNCpNp*	(= *CpLMp* = *CpRRp*)
(f)	*CCCpNpNCpNpp*	(= *CLMpp* = *CRRpp*).

plus a rule for the replacement of equivalents. Finally, R5 must *not* contain a thesis:

CpCNpp (= *CpLp* = *CNpRp*).

Propositions (a)-(f) being all classical, it follows that what we seek is a particular fragment of two-valued logic.

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9. R. Harrop, 'On the existence of finite models and decision procedures for propositional calculi', *Proc. Camb. Phil. Soc.*, 1958, pp. 1-13. See also Harrop's paper, 'Some structure results for propositional calculi', *The Journal of Symbolic Logic*, 1965, p. 286, where he refers to the system R5. Harrop's remark about R5 is not quite accurate, since the axiomatic system presented below in section 6 lacks the finite model property.

5. *A matrix for the system R5.* As we saw in the previous section, proof of the independence of *CpCNpp* in R5 requires an infinite matrix. Such a matrix will now be given, and certain propositions listed which it satisfies. As will be seen, not all propositions satisfied by it will be acceptable in R5.
Field = The signed integers.
Designated Values = All values less than or equal to *1*.
Implication and Negation Functions

$Cpq = 1, \quad p \geq q, p \neq -q$
$Cpq = q + 1, \quad p = -q$
$Cpq = 1 + \max(|p|, |q|), q > p$, where $|p|$ is the absolute value of p.
$Np = -p$.

The Central Part of the Matrix for Implication

		q: -3	-2	-1	0	1	2	3
	3	-2	1	1	1	1	1	1
	2	1	-1	1	1	1	1	4
	1	1	1	0	1	1	3	4
p	0	1	1	1	1	2	3	4
	-1	1	1	1	2	2	3	4
	-2	1	1	3	3	3	3	4
	-3	1	4	4	4	4	4	4

Functions with Designated Values

(a) $CpNNp = Cp--p = Cpp = 1$
(b) $CNNpp = C--pp = Cpp = 1$
(c) $CCNqNpCpq$

Case I $\quad p \neq -q, p \geq q \quad$ (c) $= C11 = 1$
Case II $\quad p \neq -q, q > p, \; |q| > |p| \quad = C(q+1)(q+1) = 1$
Case III $\quad p \neq -q, q > p, \; |p| > |q| \quad = C(|p|+1)(|p|+1) = 1$
Case IV $\quad p = -q \quad = C(-p+1)(q+1) = C(q+1)(q+1) = 1.$

(d) $CCNppp = C(p+1)p = 1$
(e) $CpCCpNpNCpNp = CpC(-p+1)(p-1) = Cpp = 1$
(f) $CCCpNpNCpNpp = CC(-p+1)(p-1)p = Cpp = 1$
(g) $CCpCpqCpq$

Case I $\quad p \neq -q, p \geq q, p \geq 1 \quad$ (g) $= CCp11 = C11 = 1$
Case II $\quad p \neq -q, p \geq q, p = 0 \quad = C21 = 1$
Case III $\quad p \neq -q, p \geq q, p \leq -1 \quad = C(|p|+1)1 = 1$
Case IV $\quad p \neq -q, q > p, |q| > |p| \quad = CCp(q+1)(q+1) = C(q+2)(q+1) = 1$

Case V	$p \neq -q, q > p, \lvert p\rvert > \lvert q\rvert$	$= CCp(\lvert p\rvert + 1)\ (\lvert p\rvert + 1)$
		$= C(\lvert p\rvert + 2)\ (\lvert p\rvert + 1) = 1$
Case VI	$p = -q, p > q, q \neq -2$	$= CCp(q+1)\ (q+1) = C1(q+1) = 1$
Case VII	$p = -q, p > q, q \neq -2$	$= 0$
Case VIII	$p = -q, q \geqslant p$	$= C(q+2)\ (q+1) = 1.$

Functions with Undesignated Values

$$CpCNpp = Cp(p+1) = p+2,\ p \geqslant 0$$

Functions with Designated Values not acceptable in F5

In addition to satisfying formulae (a)-(g), our matrix also satisfies their negations. This follows from the fact that these formulae always take the values *1* or *0*, and *-1* is designated. However these negations, which are all non-classical, may be excluded from R5, in the axiomatic development of that system, by not assuming any non-classical formulae as axioms, or any non-classical rules. No non-classical formula can be proved from classical axioms by classical rules, and hence none of the negations of (a)-(g) can be.

The Rule of Modus Ponens

Our matrix satisfies the rule of modus ponens, which states that if formulae X and CXY have designated values ('are designated'), then Y is designated:

$$\vdash X,\ \vdash CXY \rightarrow\ \vdash Y.$$

I shall use x to denote the sequence of values assigned by our (denumerable) matrix to the formula X for all possible values of the variables of X. The proof that the matrix satisfies the rule than follows upon noting that CXY is designated if and only if $x \geqslant y$. Hence if $x \leqslant 1$ then $y \leqslant 1$, i.e. Y is designated.

The Rule for the Replacement of Equivalents

Although we have no sign for conjunction in R5, so that we cannot define equivalence in the usual way, we can show that our matrix satisfies a rule providing in effect for the replacement of equivalents:

$$\vdash CXY,\ \vdash CYX,\ \vdash F(X) \rightarrow\ \vdash F(Y)$$

where $F(X)$ is a formula containing an occurrence of X. The proof that the matrix satisfies this rule involves first of all the fact that if CXY and CYX are both designated, then $x = y$. For suppose at one point that $x > y$. Then we will have at that point, $CYX > 1$, contrary to the hypothesis that CYX is designated. Similarly if $y > x$. The proof of the rule of replaceability, as stated syntactically above, then follows from a

Matrix rule of replaceability: If X and Y are formulae such that $x = y$, $F(X)$ and $F(Y)$ have identical sequences of values.

To see that this matrix rule holds we observe that evaluation of any formula F by appeal to the matrix consists in substituting matrix values for its sub-formulae F_1, F_2, . . . F_n. F will continue to take the same values if a different sub-formula, G, having the same sequence of values as F_j, $1 \leqslant j \leqslant n$, be substituted for F_j. This completes the proof of our syntactic rule of replaceability, since if $F(X)$ is designated, $F(Y)$ will be.

The Rule of Transitivity

The proposition asserting the transitivity of implication usually called 'Syl', namely

CCpqCCqrCpr,

is not satisfied by the matrix, since for example

$CC34CC44C34 = C5C15 = C56 = 7.$

However, we find that the corresponding rule *is* satisfied:

$\vdash CXY,\ \vdash CYZ \rightarrow \vdash CXZ.$

Proof: if $\vdash CXY$, then $x \geqslant y$, and if $\vdash CYZ$, then $y \geqslant z$. Hence $x \geqslant z$, i.e. $\vdash CXZ$.

6. *R5 developed axiomatically*. In this section I present certain formulae and rules, and make certain deductions from them, without attempting to prove that they provide a complete axiomatization of the system of classically valid propositions satisfied by the matrix. The system is in C and N.

Definitions

Df. L: $Lp = CNpp$
Df. R: $R = LN$
Df. M: $M = NLN$

Axioms

1. *CCNqNpCpq*
2. *CCNppp*
3. *CpCCpNpNCpNp*
4. *CCCpNpNCpNpNNp*
5. *CCpCpqCpq*,

where 4 is a trivial variant of formula (f) of the previous section.

Rules

(1) **Substitution**
(2) **MP**: $\vdash X,\ \vdash CXY \rightarrow \vdash Y$ (Modus Ponens)
(3) **RE**: $\vdash CXY,\ \vdash CYX,\ \vdash F(X) \rightarrow \vdash F(Y)$ (Replacement of Equivalents)
(4) **TR**: $\vdash CXY,\ \vdash CYZ \rightarrow \vdash CXZ$ (Transitivity).

Deductions

6.	*CpNNp*	(3,4, **Tr**)
7.	*CNNpp*	(1,6, **MP**)
8.	*Cpp*	(6,7, **TR**)
9.	*CLNpCNNpNp*	(8, Df. L)
10.	*CRpCpNp*	(9,6,7, **RE**, Df. R)

11.	$CCNNpNpLNp$	(8, Df. L)
12.	$CCpNpRp$	(11,6,7, **RE**, Df. R)
13.	$CpRRp$	(3,10,12, **RE**)
14.	$CpLMp$	(13, Df. R, Df. M)
15.	$CRRpp$	(4,6,7, **RE**, 10,12, **RE**)
16.	$CLMpp$	(15, Df. R, Df. M)
17.	$CLpp$	(2, Df. L)
18.	$CRpNp$	(17, Df. R)

So far, in proving all the strong contrariety theses we set out to prove, we have made no use of 5. We may, however, use its substitution $CCpCpNpCpNp$ to define further contrariety operators, of increasing strength, in the following way:

Df. R': $R'p = CpRp$

Df. R'': $R''p = CpR'p$ etc.

By 5, $R'p$ implies Rp, but $CRpR'p$ (= $CCpNpCpCpNp$) is rejected by the matrix. We have:

	for $p \leqslant 0$	for $p > 0$
Np	$= \|p\|$	$= -p$
$Rp = CpNp$	$= \|p\| + 1$	$= 1$
$R'p = CpRp$	$= \|p\| + 2$	$= 1$
$R''p = CpR'p$	$= \|p\| + 3$	$= 1$

etc.

Thus we have, in addition to the infinite string of irreducible modalities p, Lp, LLp, $LLLp$, . . . , each of which is implied by the next in line to it, another similar string Np, Rp, $R'p$, $R''p$, ... Of these operators, only R obeys the law of double contrariety; $R'R'p$ (= $CCpCpNpCCpCpNpNCpCpNp$) not having the same matrix value as p.

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