Notre Dame Journal of Formal Logic
Volume VIII, Numbers 1 and 2, April 1967

NOTE ON AN INEQUALITY OF TIBOR RADO

MICHAEL ANDERSON

In his papers [1], [2] and [3] on non-computable functions Tibor Rado has mentioned the inequality $\mathsf{S}(n) \leqslant (n+1)\Sigma(5n)2^{\Sigma(5n)}$, where $\mathsf{S}(n)$ is the maximum number of shifts that can be made by an n card (state) Turing machine—under certain restrictive conditions—and $\Sigma(n)$ is the maximum number of strokes which can be printed by an n card Turing machine. It is the purpose of this note to show that in fact $\mathsf{S}(n) \leqslant (n+1)\Sigma(3n)2^{\Sigma(3n)}$.*

Throughout his papers and, in particular, in defining the functions he is concerned with, Rado uses two-symbol Turing machines which do not have a stay shift i.e., the machine must shift to the right or to the left after each print. The program of each machine is displayed in the form of a set of numbered cards. Each card has two rows of information; the first row having the instructions for the case when the machine scans a blank, the second row having the instructions for the case when the machine scans a stroke. Each row has three pieces of information: the first piece, a *1* or a *0*, determines whether the machine will print a stroke or a blank; the second piece, a *1* or a *0*, determines whether the machine will shift to the right or to the left; and the third piece, a non-negative integer, determines what card will be used for the next set of instructions. If the last number of the set of instructions is *0*, the machine stops. The *1* card is the first card used by the machine.

We are now prepared to define Rado's functions $\Sigma(n)$, $\mathsf{S}(n)$, and the range function $\mathsf{R}(n)$. $\Sigma(n)$ is the maximum number of strokes left by an n card Turing machine starting with a blank tape and stopping after a finite number of shifts. $\mathsf{S}(n)$ is similarly defined as the maximum number of shifts an n card Turing machine can make beginning on a blank tape and stopping after a finite number of shifts. $\mathsf{R}(n)$ is the maximum number of distinct cells an n card Turing machine can scan beginning with a blank tape and stopping after a finite number of shifts.

Rado has stated that $\mathsf{S}(n) \leqslant (n+1)\Sigma(5n)2^{\Sigma(5n)}$. This is a result of the fact that $\mathsf{S}(n) \leqslant (n+1)\mathsf{R}(n)2^{\mathsf{R}(n)}$, since the number of shifts a machine makes

*I obtained this result while attending Prof. Hans Zassenhaus' Seminar in Experimental Number Theory at Ohio State University, Columbus, Ohio, during the Summer of 1966.

Received September 3, 1966

before stopping is less than or equal to the total number of possible configurations, and that $\mathsf{R}(n) \leqslant \Sigma(5n)$. It will be shown that $\mathsf{R}(n) \leqslant \Sigma(3n)$ and thus that $\mathsf{S}(n) \leqslant (n+1)\Sigma(3n)2^{\Sigma(3n)}$. First we state the following lemma.

Lemma: Given an n card Turing machine which eventually stops when given a blank tape and which has a range of k cells on this tape, a Turing machine of less than 3n cards can be constructed which eventually stops when given a blank tape and which prints out at least k-1 strokes on this tape.

A sketch of the proof will be given. Call the given machine $\mathcal{M}$ and from the cards of this machine construct the cards of another Turing machine, $\overline{\mathcal{M}}$, as follows: For each card of $\mathcal{M}$ we have three, two, or one card of $\overline{\mathcal{M}}$ depending on whether the card takes $\mathcal{M}$ to the stop state in none, one, or two of the scanning alternatives. For c_q and $c_q^{\prime}$ not 0, the construction is as follows.

$$\boxed{\begin{array}{c} q \\ p_q\ s_q\ c_q \\ p_q^{\prime}\ s_q^{\prime}\ c_q^{\prime} \end{array}} \longrightarrow \boxed{\begin{array}{c} \overline{q}_1 \\ p_q\ s_q\ \overline{q}_2 \\ p_q^{\prime}\ s_q^{\prime}\ \overline{q}_3 \end{array}} \quad \boxed{\begin{array}{c} \overline{q}_2 \\ 1\ s_q\ \overline{c}_q \\ 1\ s_q\ \overline{c}_q \end{array}} \quad \boxed{\begin{array}{c} \overline{q}_3 \\ 1\ s_q^{\prime}\ \overline{c}_q^{\prime} \\ 1\ s_q^{\prime}\ \overline{c}_q^{\prime} \end{array}}$$

If c_q (or $c_q^{\prime}$) is 0, the $\overline{q}_2$ ($\overline{q}_3$) card is omitted and the state instruction for the alternative of scanning a blank (stroke) in $\overline{q}_1$ is left a 0.

By associating the configuration of $\mathcal{M}$ after m shifts with the configuration of $\overline{\mathcal{M}}$ after $2m$ shifts and proceeding by induction, it can be shown that $\overline{\mathcal{M}}$ stops. Also $\overline{\mathcal{M}}$ has a range of $2k-1$ cells and prints a stroke on each of the $k-1$ extra cells of its range. The number of cards used for $\overline{\mathcal{M}}$ is less than or equal to $3n-1$ (there is at least one stop instruction in $\mathcal{M}$) and thus $\overline{\mathcal{M}}$ is the machine we are looking for.

Theorem: $\mathsf{R}(n) \leqslant \Sigma(3n)$

Proof: Assume $\mathsf{R}(n) > \Sigma(3n)$. *Then* $\mathsf{R}(n) \geqslant \Sigma(3n)+1$. Thus by the lemma we can construct a stopping machine with less than $3n$ cards ($\leqslant 3n-1$) which prints out at least $\mathsf{R}(n)-1$ ($\geqslant \Sigma(3n)$) strokes. But then $\Sigma(3n-1) \geqslant \Sigma(3n)$ which is a contradiction since $\Sigma(n)$ is a strictly increasing function. Thus $\mathsf{R}(n) \leqslant \Sigma(3n)$.

Corollary: $\mathsf{S}(n) \leqslant (n+1)\Sigma(3n)2^{\Sigma(3n)}$

BIBLIOGRAPHY

[1] T. Rado: "On Non-Computable Functions." *Bell System Technical Journal*, vol. XLI, No. 3, May, 1962. Pages 1-10.

[2] T. Rado: "*On Non-Computable Functions.*" Bell System Monograph 4199, 1962.

[3] T. Rado: "On a Simple Source for Non-Computable Functions." *Proceedings of the Symposium on Mathematical Theory of Automata*. Polytechnic Institute of Brooklyn, April 24-26, 1962.

Seminar in Symbolic Logic
University of Notre Dame
Notre Dame, Indiana