THE TRUNCATION OF TRUTH-FUNCTIONAL CALCULATION

DESMOND PAUL HENRY

§1 The precise determination of those combinations of truth-values which verify a sentence (Δ) of the two-value propositional calculus normally requires the exhaustive serial consideration of 2^n such combinations when Δ involves n variables. Hereunder are enunciated principles which obviate the invariable necessity for such consideration, without at the same time sacrificing exactitude of calculation.

PART I. *Theses on Abbreviational Arrays*

§2.1 In this Part theses of the two-value propositional calculus will be expressed in a notation designed for the problem in hand.

§2.2 The 2^n possible combination of truth-values of n variables may be generated by reducing to the scale of 2 the integers 0 to $2^n - 1$, and adding 0's leftwards where necessary so that each resulting number has n digits: then, assuming that "1" and "0" represent the truth-constants "true" and "false" respectively, and that "p", "q" etc., alphabetically ordered, are the propositional variables used, the successive application to p of the series of values given in the unit places of those equivalents (i.e. $0, 1, 0, 1, \ldots.$) and of the series given in the radix place (i.e. $0, 0, 1, 1, \ldots.$) to q, and so on, ensures that no combination is neglected. In respect of a given Δ the successive outcomes of such allocations may be recorded by means of a row of 2^n digits (called a "selector") each corresponding (from left to right) to the combinations of truth-values given by 0 to $2^n - 1$ (in that order) in the scale of 2; e.g. for $n = 3$, a 1-digit in the right-hand (2^nth) place indicates that Δ has the value 1 for the set of truth-values given by $2^n - 1$ in the scale of 2, viz, 1, 1, and 1, for p, q and r respectively. The explicitation of rules for the economical determination of selectors, as thus described, corresponding to sentences involving large numbers (e.g. > 6) of

Received December 4, 1960

diverse variables, is the object of the present study. Now if, from each combination of truth-values which verifies (i.e. gives the outcome *1*) Δ is formed a conjunction of the propositional variables of Δ such that the varibles corresponding to the *1*-digits of that combination appear as unnegated conjuncts, and those corresponding to its *0*-digits as negated conjuncts, and if all such conjunctions then become alternants of a disjunction, the resulting expression is logically equivalent to Δ. Hence, in view of the fixed order described above, in which sets of truth-combinations are herein considered as generated, allocated to variables, and having their outcomes recorded, each selector may be considered as a summary statement of a certain conjunctive-disjunctive form constructed as described, and as such as a possible argument of propositional functors in sentences of the propositional calculus. Variables for such selectors, and for abbreviational schemata involving those variables may therefore figure as arguments of propositional functors in theses of the propositional calculus such as those which will be formulated below. It may here be noted that a fixed order of truth-combination allocation and outcome-recording has only herein been adopted for the purpose of stating the theses of the present Part, and that the rules which emerge in Part II can of course be applied to selectors correlated to any order of truth-combination allocation.

§3. "*a*", "*b*", "*c*", etc. with or without subscripts, being variables whose values are selectors, a type of abbreviation of selectors (i.e. an "abbreviation array") in terms of other selectors may be schematically expressed:

$$\left|\begin{matrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_e \end{matrix}\right|^{i} \tag{1}$$

Each a_x of the abbreviation array (1) is composed of 2^m digits; $e = n/m$, where n/m equals some positive integer greater than *1*. This columnar arrangement can be used to indicate that the selector of which (1) is an abbreviation is such that where $i = 0$ or $i = 1$, the pattern of a_1 occupies a position corresponding to each *1*-digit of a_2, and $(2^m)^i$ digits, each of value i, occupy a position corresponding to each *0*-digit of a_2; in its turn, the pattern thus constituted by a_1 and a_2 occupies a position corresponding to each *1*-digit of a_3, and $(2^m)^2$ digits, each of value i, occupy a position corresponding to each *0*-digit of a_3, and so on until finally the pattern thus constituted by $a_1 \ldots . a_{e-1}$ occupies a position corresponding to each *1*-digit of a_e, and $(2^m)^{e-1}$ digits, each of value i, occupy a position corresponding to each *0*-digit of a_e.

Example 1

$$n = 6,\ m = 2,\ a_1 = 0101,\ a_2 = 0011,\ a_3 = 1001,\ i = 0$$

$$\left|\begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix}\right|^{i} = \left|\begin{matrix} 0101 \\ 0011 \\ 1001 \end{matrix}\right|^{0}$$

$$= 0000000001010101000000000000000000000000000000000000000001010101$$

Example 2

Suppose $n = 9$, $m = 3$. Then the 2^n successive allocations of truth-values of p, q, r etc. which would be made in accordance with §2.2 can be abbreviated for $i = 0$ or $i = 1$, as follows:

For $i = 0$:

$p: \left|\begin{matrix} 01010101 \\ 11111111 \\ 11111111 \end{matrix}\right|^{0}$ $\quad u: \left|\begin{matrix} 11111111 \\ 00001111 \\ 11111111 \end{matrix}\right|^{0}$

$q: \left|\begin{matrix} 00110011 \\ 11111111 \\ 11111111 \end{matrix}\right|^{0}$ $\quad v: \left|\begin{matrix} 11111111 \\ 11111111 \\ 01010101 \end{matrix}\right|^{0}$

$r: \left|\begin{matrix} 00001111 \\ 11111111 \\ 11111111 \end{matrix}\right|^{0}$ $\quad w: \left|\begin{matrix} 11111111 \\ 11111111 \\ 00110011 \end{matrix}\right|^{0}$

$s: \left|\begin{matrix} 11111111 \\ 01010101 \\ 11111111 \end{matrix}\right|^{0}$ $\quad x: \left|\begin{matrix} 11111111 \\ 11111111 \\ 00001111 \end{matrix}\right|^{0}$

$t: \left|\begin{matrix} 11111111 \\ 00110011 \\ 11111111 \end{matrix}\right|^{0}$

For $i = 1$:

$p: \left|\begin{matrix} 01010101 \\ 11111111 \\ 11111111 \end{matrix}\right|^{1}$ $\quad u: \left|\begin{matrix} 00000000 \\ 11110000 \\ 11111111 \end{matrix}\right|^{1}$

$q: \left|\begin{matrix} 00110011 \\ 11111111 \\ 11111111 \end{matrix}\right|^{1}$ $\quad v: \left|\begin{matrix} 00000000 \\ 11111111 \\ 10101010 \end{matrix}\right|^{1}$

$r: \left|\begin{matrix} 00001111 \\ 11111111 \\ 11111111 \end{matrix}\right|^{1}$ $\quad w: \left|\begin{matrix} 00000000 \\ 11111111 \\ 11001100 \end{matrix}\right|^{1}$

$s: \left|\begin{matrix} 00000000 \\ 10101010 \\ 11111111 \end{matrix}\right|^{1}$ $\quad x: \left|\begin{matrix} 00000000 \\ 11111111 \\ 11110000 \end{matrix}\right|^{1}$

$t: \left|\begin{matrix} 00000000 \\ 11001100 \\ 11111111 \end{matrix}\right|^{1}$

The arrays so far described are "regular" in the sense that the selectors of which they are composed each consist of 2^m digits. There is, however, no reason why arrays of diverse types of irregularity (i.e. having 2^x digits at each level, for various values of x) should not be used, provided always that the selector pattern of a_1 occupies a position corresponding to each 1-digit of a_2, and that a number of digits equal in number to those of a_1, and of value i occupy a position corresponding to each 0-digit of a_2, and so on. The theses evolved in §5 apply to both regular and irregular arrays. Using the values of i to indicate the form the abbreviation is taking, "$0\alpha\beta$" and "$1\alpha\bar{\beta}$" will be used as alternative forms of

$$\left|\begin{array}{c}\alpha\\ \beta\end{array}\right|^{0} \qquad \text{and} \qquad \left|\begin{array}{c}\alpha\\ \beta\end{array}\right|^{1} \tag{3}$$

respectively. Since, for example,

$$\left|\begin{array}{c}\left|\begin{array}{c}\alpha\\ \beta\end{array}\right|^{1}\\ \gamma\end{array}\right|^{1} = \left|\begin{array}{c}\alpha\\ \beta\\ \gamma\end{array}\right|^{1} \qquad \left|\begin{array}{c}\left|\begin{array}{c}\alpha\\ \beta\end{array}\right|^{0}\\ \gamma\end{array}\right|^{0} = \left|\begin{array}{c}\alpha\\ \beta\\ \gamma\end{array}\right|^{0} \tag{4}$$

such two-argument forms may be used to express abbreviation-arrays involving any number of selectors, provided always that β in (3) is taken to represent a single selector only. This is because, in contrast to what then holds in the cases shown in (4), members of each of the following pair need not be equivalent to each other:

$$\left|\begin{array}{c}\alpha\\ \left|\begin{array}{c}\beta\\ \gamma\end{array}\right|^{1}\end{array}\right|^{1} \qquad \left|\begin{array}{c}\alpha\\ \beta\\ \gamma\end{array}\right|^{1} \tag{5}$$

Because of their similiarity in appearance to two-argument functors and their arguments, $0\alpha\beta$ and $1\alpha\beta$ produce syntactical clarity in theses such as those of §5 without the necessity of introducing new conventions. In principle *0* or *1* could of course be followed by variables representing the full set of e selectors, where $e > 2$.

§4 That forms such as $0\alpha\beta$ and $1\alpha\beta$ may figure in sentences of the propositional calculus is plain from the considerations of §2.2. The thesishood or otherwise of such sentences is made more easily testable in view of the fact that where v is a selector having 2^m digits and $v\,(\alpha)$ represents that function of the *first* ordered set of m variables of the series p, q, r etc. to which the selector v considered in isolation, is equivalent (cf. §2.2), $v\,(\beta)$ represents the same function but now of corresponding members of the *second* ordered set of m variables in that same alphabetical series, and so on, then it will be found that:

$$E0abKa(\alpha)b(\beta) \tag{6}$$

$$E1abAa(\alpha)Nb(\beta) \tag{7}$$

hold, provided that the scheme of truth-value allocation outlined in §2.2 is adhered to. For instance, the array in example 1 above is equivalent to *KKpsEtu*. (6) and (7) are hence, incidentally, of use when lengthy selectors have to be "decoded" so as to show the Δ to which they are equivalent.

§5 Now follow theses of the propositional calculus involving the abbreviation-arrays described above: these theses may be verified simply in the light of (6) and (7). Hereunder ϵ, η, ι, and κ (with or without accents) are functorial variables having as instances the functors listed above the theses in which those variables figure; "A", "K", "C", "B" and "N" are used as the signs for alternation, conjunction, implication, converse implication and negation respectively, and X=NA, D=NK, L=NC, M=NB. x, y and z are i-values, z being that truth-value produced by ϵxy, ηxy, etc. in truth-functional decision procedure. R is one or other of the binary functors A and K such that when $z = 0$, $R = A$, and when $z = 1$, $R = K$. The point of formulating and diversifying theses in this manner is the practical one described in Part II.

For respective ϵ, x and y values: $A, 1, 1;\ K, 0, 0;\ X, 1, 1;\ D, 0, 0;\ L, 0, 1;\ C, 0, 1;\ M, 1, 0;\ B, 1, 0:$

$$E\epsilon xabycdz\epsilon acKbd \tag{8}$$

For respective η, x and y values: $K, 0, 1;\ L, 0, 0;\ A, 1, 0;\ B, 1, 1:$

$$E\eta xabycdRz\eta acKbdzaLbd \tag{9}$$

For respective $\acute{\eta}$, x and y values: $D, 0, 1;\ C, 0, 0;\ X, 1, 0;\ M, 1, 1:$

$$E\acute{\eta}xabycdRz\acute{\eta}acKbdzNaLbd \tag{10}$$

For respective ι, x and y values: $C, 1, 1;\ A, 0, 1;\ K, 1, 0;\ M, 0, 0:$

$$E\iota xabycdRz\iota acKbdzcMbd \tag{11}$$

For respective $\grave{\iota}$, x and y values: $L, 1, 1;\ X, 0, 1;\ D, 1, 0;\ B, 0, 0:$

$$E\grave{\iota}xabycdRz\grave{\iota}acKbdzNcMbd \tag{12}$$

For respective κ, x and y values: $A, 0, 0;\ K, 1, 1;\ E, 1, 1;\ J, 0, 0:$

$$E\kappa xabycdRRz\kappa acKbdzaLbdzcMbd \tag{13}$$

For respective $\acute{\kappa}$, x and y values: $C, 1, 0;\ E, 1, 0;\ J, 0, 1;\ M, 0, 1:$

$$E\acute{\kappa}xabycdRRz\acute{\kappa}acKbdzNaLbdzcMbd \tag{14}$$

For respective $\grave{\kappa}$, x and y values: $L, 1, 0;\ E, 0, 1;\ J, 1, 0;\ B, 0, 1:$

$$E\grave{\kappa}xabycdRRz\grave{\kappa}acKbdzaLbdzNcMbd \tag{15}$$

For respective $\hat{\kappa}$, x and y values: $X, 0, 0;\ D, 1, 1;\ E, 0, 0;\ J, 1, 1:$

$$E\hat{\kappa}xabycdRRz\hat{\kappa}acKbdzNaLbdzNcMbd \tag{16}$$

To these may be added the theses on negation:

$$EN1ab0Nab \tag{17}$$

$$EN0ab1Nab \tag{18}$$

In connection with 17 and 18 it may be noted that mechanically speaking, the negation of any selector is simply given by replacing its *0*-digits by *1*-digits, and *vice versa.*

PART II: *Decision Procedure using Abbreviational Arrays*

§6.1 In this part the theses stated in Part I will be exploited in order to develop a decision procedure which allows the selector which abbreviates the digital equivalent of a Δ involving large values of n to be accurately arrived at without the invariable necessity of considering individually the 2^n alternative truth-combinations to which that selector has reference (cf. §2.2).

§6.2 If, instead of allocating the full series of 2^n successive sets of truth-values to the variables of Δ, one allocates to them in the first place the series of values given by the *e*th selector of abbreviational arrays corresponding to variables such as those given in Example 2, then ordinary truth-value calculations in accordance with the parts of (8) - (16) which involve "b" and "d" (i.e. using K, L, or M only, according to the Φ, x and y values involved at each step (cf. *S2* below)) produce one selector as the *e*-level outcome when the Φ, x and y values throughout are those covered by (8) (cf. (22), (26) below), otherwise mutually exclusive selectors (in the sense that $D\delta\theta$ is true of any two such selectors δ and θ) are the outcome (e.g. (27) below). Such mutually exclusive selectors may be treated in calculation as components of a single outcome, provided some means of distinguishing between them is available (see *S4* below). Hence, to ensure economy (i.e. maximum possible appeal to the *Kbd* part of (8), which yields only one selector as outcome) one has the rule:

> *S1*: In respect of Δ, the i-value allocation formula (θ) in which i-values are allocated to the variables of Δ should be such that θ and its consequences demand maximum possible appeal to thesis (8), and to theses (9) - (12) in preference to theses (13) - (16). The i-values of this first θ should be retained by those variables at all subsequent stages of calculation.

Thus, for

$$\Delta = DMACpqCrsKLtuMvwx = DMA\phi_1\phi_2K\phi_3\phi_4x = DM\psi_1\psi_2x = D\chi_1x \quad (19)$$

the optimum i-allocations in accordance with *S1* are:

$$\theta = DMAC01C01KL01M100 = DMA11K000 = DM100 = D00 = 1 \quad (20)$$

(Here the consequences of the allocations shown are obtained as in normal truth-functional calculation, in view of the way in which z-values are derived from x and y values in (8) - (16)). The optimum nature of the allocations shown becomes most evident when the thesis-reference formula (Λ) which is plainly required in order to summarily indicate which of K, L, and M should, given θ, be used in the calculations relating to the eth selector of the abbreviational arrays, is constructed in accordance with the following rule:

> *S2*: By reference to θ and its consequences replace the functor-signs of by the appropriate Greek letters (with accents, where shown) which are given against the Φ, x, y combinations listed above each of theses (8) - (16), so as to produce a thesis-reference formula (Λ).

For, by *S2*, one has from (19) and (20):

$$\Lambda = \epsilon\epsilon\epsilon\epsilon p q \epsilon r s \epsilon\epsilon t u \epsilon v w x \tag{21}$$

And situating the appropriate (i.e. $i = 0$ or $i = 1$) eth level selectors (left-hand digits uppermost) from Example 2 under the guidance of Δ and θ (as obtained in (19) and (20)), and calculating according to the b, d level of the ϵ-thesis (8) only (since Λ here involves only ϵ), i.e., using only K, one has:

$$\Delta = DMACpqCrsKLtuMvwx = DMA\phi_1\phi_2K\phi_3\phi_4x = DM\psi_1\psi_2x = D\chi_1x$$

$$\theta = DMAC01C01KL01M100 = DMA11K000 = DM100 = D00 = 1$$

$$\Lambda = \epsilon\epsilon\epsilon\epsilon pq\epsilon rs\epsilon\epsilon tu\epsilon vwx = \epsilon\epsilon\epsilon\phi_1\phi_2\epsilon\phi_3\phi_4x = \epsilon\epsilon\psi_1\psi_2x = \epsilon\chi_1x$$

$$\left.\begin{array}{l}
KKKK11K11KK11K100 = KKK11K100 = KK100 = K00 = 0 \\
KKKK11K11KK11K000 = KKK11K100 = KK100 = K00 = 0 \\
KKKK11K11KK11K110 = KKK11K110 = KK110 = K10 = 0 \\
KKKK11K11KK11K010 = KKK11K100 = KK100 = K00 = 0 \\
KKKK11K11KK11K101 = KKK11K101 = KK101 = K01 = 0 \\
KKKK11K11KK11K001 = KKK11K101 = KK101 = K01 = 0 \\
KKKK11K11KK11K111 = KKK11K111 = KK111 = K11 = 1 \\
KKKK11K11KK11K011 = KKK11K101 = KK101 = K01 = 0
\end{array}\right. \tag{22}$$

The outcome of this calculation, each line of which proceeds as in normal truth-functional decision procedure, is the e-level selector of the abbreviational array for (19). In more complex cases, i.e. where, given Δ, θ can only be such that Λ involves Greek thesis-reference letters other than ϵ, the indications of (9) - (16) in respect of b and d are plainly automatically satisfied by:

> *S3: Rule of Anticipatory Functors.*
>
> (η) At those steps in the calculation governed by a corresponding η (with or without accent) in Λ, prefix L to all 1 - 0 combinations, K to all other combinations, and calculate accordingly.

(ι) At those steps in the calculation governed by a corresponding ι (with or without accent) in Λ, prefix M to all 0 - 1 combinations, K to all other combinations, and calculate accordingly.

(κ) At those steps in the calculation governed by a corresponding κ (with or without accent) in Λ, prefix L to all 1 - 0 combinations, M to all 0 - 1 combinations, K to all other combinations, and calculate accordingly.

S4: Distinction of outcome selector-components.

Those *1*'s of an outcome which terminate rows of calculation each having a like sequence of one or other or all of K, L and M constitute, when completed by *0*'s, a distinct component of that outcome.

(For examples, see (26), (27) below).

§7.1 The indications involving "a" and "c" of theses (8) - (16) show the connection of the determination of the outcome selector or its components at the eth level (§6.2) with their determination at the e-1th level; for the "a" and "c" parts of those theses (e.g. ϵac, ηac, ιac, a, c, Na, Nc) indicate the pattern which must be taken by what may be called a "residual expression" ($\mathbf{P}$) according to the Greek letter thesis-indication; hence the rule, which may upon inspection be found to fulfill the requirements of (8) - (16):

S5: Determination of residual expression ($\mathbf{P}$)

In respect of calculations ranged under Δ, θ, and Λ, as in (22), and their consequences, for an outcome or outcome-component the *1*-digits of which have preceding calculation-rows

(a) involving K only, $\mathbf{P} = \Delta$;

(b) involving at least one of L or M, $\mathbf{P} = \Delta$ diminished by those segments of Δ appearing above the *0*'s which are the arguments of L or M, and having negated the segments of Δ which appear above the *1*-digits of the arguments of

  (i) L (for Greek thesis-letters bearing an acute accent)

  (ii) M (for Greek thesis-letters bearing a grave accent)

  (iii) L and M (for Greek thesis-letters bearing a circumflex accent).

(For examples, see (27) below). On the negations which may be encountered as a result of $S5$, see (17) and (18). The residual expressions having been determined in accordance with $S5$, calculation in respect of each of them proceeds in accordance with rules $S1$ - $S5$ for Δ, only with the e-1th selectors from the variable-arrays now being allocated to the variables, the i-values of the original θ being retained throughout (cf. $S1$). This process is continued until the upper level selectors of the variable-arrays are to be considered, and then in accordance with (8)-(16) the following rule operates:

S6: Calculations in respect of the uppermost selectors of the abbreviation-arrays for variables proceed in accordance with the rules which would hold for the normal truth-functional evaluation of the **P** in question.

i.e. the selectors for variables having been chosen under the guidance of the θ for **P**, and duly situated, calculation proceeds by direct reference to the functors of **P**, a corresponding **Λ** hence not being required. (For examples, see (28 - 31)). The outcome components yielded by the original *e*th-level selectors are each correlated with the outcome components yielded by their corresponding **P**, and so on throughout the course of the process described, the final result being either one abbreviation-array (e.g. (25) below) or a disjunction (when the final z-value is *0*) or conjunction (when the final z-value is *1*) of mutually exclusive (cf. §6.2) abbreviation-arrays (e.g. (31) below). The following obvious §6.2 (*S3*) economy rule may also be invoked:

S7: (*a*) When calculations involving any but the final (upper selectors of arrays for variables yield an *outcome-selector* involving only *0*-digits, then that selector and its **P** may thenceforward be ignored: when all calculations at the levels mentioned have such an outcome, then the selector for Δ consists of 2^n digits of value equal to that of the final z-value in θ.

(*b*) When calculations involving the final (upper) selectors of arrays for variables yield an outcome-selector involving only digits of the same value as the final z-value in θ, then that selector may be ignored: when all calculations at the level mentioned have such an outcome, then the selector for Δ consists of 2^n digits of value equal to that of the final z-value in θ.

Hence the final results represent perspicuously the situation of those parts (of 2^m digits each in the case of regular arrays) of the full selector for Δ which consist of other than mere stretches of z-value, within the range of the 2^n digits of that full selector. The mutual non-interference of such parts (§6.2) also of course permits the simple writing down or mechanical printing of those 2^n digits in full.

§7.2 *Example 3*. From (22) by *S5* (*a*), $\mathbf{P} = \Delta$, so that θ and **Λ** are the same as those shown in (22); hence, allocating the second (middle) selectors indicated in Example 2, one has:

$$\mathbf{P} = DMACpqCrsKLtuMvwx = DMA\phi_1\phi_2K\phi_3\phi_4x = DM\psi_1\psi_2x = D\chi_1x$$

$$\theta = DMAC01C01KL01M100 = DMA11K000 = DM100 = D00 = 1$$

$$\mathbf{\Lambda} = \epsilon\epsilon\epsilon p q\epsilon r s\epsilon\epsilon t u\epsilon v w x = \epsilon\epsilon\epsilon\phi_1\phi_2\epsilon\phi_3\phi_4x = \epsilon\epsilon\psi_1\psi_2x = \epsilon\chi x$$

$$
\begin{aligned}
&KKKK11K11KK01K111 = KKK11K011 = KK101 = K01 = 0\\
&KKKK11K10KK01K111 = KKK10K011 = KK001 = K01 = 0\\
&KKKK11K11KK11K111 = KKK11K111 = KK111 = K11 = 1\\
&KKKK11K10KK11K111 = KKK10K111 = KK011 = K01 = 0\\
&KKKK11K11KK00K111 = KKK11K011 = KK101 = K01 = 0\\
&KKKK11K10KK00K111 = KKK10K011 = KK001 = K01 = 0\\
&KKKK11K11KK10K111 = KKK11K011 = KK101 = K01 = 0\\
&KKKK11K10KK10K111 = KKK10K011 = KK001 = K01 = 0
\end{aligned}
\tag{23}
$$

From (23), by $S5(a)$, **P** $= \Delta$, but by $S6$, one now has for the last (upper) selectors of Example 2:

$$\mathbf{P} = DMACpqCrsKLtuMvwx$$

$$\theta = DMAC01C01KL01M100 = DMA11K000 = DM100 = D00 = 1$$

$$
\begin{aligned}
&DMAC00C00KL10M011 = DMA11K111 = DM111 = D01 = 1\\
&DMAC10C00KL10M011 = DMA01K111 = DM111 = D01 = 1\\
&DMAC01C00KL10M011 = DMA11K111 = DM111 = D01 = 1\\
&DMAC11C00KL10M011 = DMA11K111 = DM111 = D01 = 1\\
&DMAC00C10KL10M011 = DMA10K111 = DM111 = D01 = 1\\
&DMAC10C10KL10M011 = DMA00K111 = DM011 = D11 = 0\\
&DMAC01C10KL10M011 = DMA10K111 = DM111 = D01 = 1\\
&DMAC11C10KL10M011 = DMA10K111 = DM111 = D01 = 1
\end{aligned}
\tag{24}
$$

Thus, since the single-selector outcome of (22) is correlated with the **P** of (23), and the outcome of the latter with the **P** of (24), the three outcomes of (22), (23) and (24) can be placed in their order so that, adding the final z-value given by θ throughout, the complete abbreviation of the selector for Δ is:

$$\begin{vmatrix} 11111011 \\ 00100000 \\ 00000010 \end{vmatrix}^{1} \tag{25}$$

Expanded into a single one-line selector in accordance with $S3$, (25) would yield the much less perspicuous result involving $2^9 = 512$ digits. In practice, should the final z-value given by θ be 1, it is clearly best in cases which involve only ϵ in their **Λ** to evaluate the case of the upper selectors first (as in (24)), for should this evaluation produce the truth-value 1 exclusively, then the unabbreviated selector for Δ must, by $S7(b)$, also consist entirely of 1's. And although the full apparatus of **P**, θ and **Λ** formulae has been shown for example's sake at every step, it is evident that for a Δ which has a **Λ** like (21), i.e. involving only ϵ, the exclusive use of K in the non-final stages of evaluation, and the constant equiformity of **P** with **Λ** throughout, enables the outcome at each such stage to be determined merely by alignment and inspection of the appropriate selectors from Example 2; in such a case, therefore, the pencil and paper evaluation of a Δ involving as many as several dozen variables is clearly possible.

§8 *Example 4*. When the appropriate (by θ) eth-level selectors of the abbreviational arrays for variables (Example 2) are used in accordance with *S3* for the Δ shown, one has:

$$\Delta = LBCprDtuAAKqsDtvAwx = LB\phi_1\phi_2AA\phi_3\phi_4\phi_5 = L\psi_1A\psi_2\phi_5 = L\psi_1\chi_1$$

$$\theta = LBC01D00AAK00D00A11 = LB11AA011 = L1A11 = L11 = 0$$

$$\boldsymbol{\Lambda} = \grave{\iota}\eta\epsilon pr\epsilon tu\epsilon\iota\epsilon qs\epsilon tv\epsilon wx = \grave{\iota}\eta\phi_1\phi_2\epsilon\iota\phi_3\phi_4\phi_5 = \grave{\iota}\psi_1\epsilon\psi_2\phi_5 = \grave{\iota}\psi_1\chi_1$$

$$\begin{array}{l} \grave{\iota}\eta K11K11\epsilon\iota K11K10K11 = \grave{\iota}K11\epsilon K101 = \grave{\iota}1K01 = K10 = 0 \\ \grave{\iota}\eta K11K11\epsilon\iota K11K11K11 = \grave{\iota}K11\epsilon K111 = \grave{\iota}1K11 = K11 = 1 \\ \grave{\iota}\eta K11K11\epsilon\iota K11K10K01 = \grave{\iota}K11\epsilon K100 = \grave{\iota}1K00 = K10 = 0 \\ \grave{\iota}\eta K11K11\epsilon\iota K11K11K01 = \grave{\iota}K11\epsilon K110 = \grave{\iota}1K10 = K10 = 0 \\ \grave{\iota}\eta K11K11\epsilon\iota K11K10K10 = \grave{\iota}K11\epsilon K100 = \grave{\iota}1K00 = K10 = 0 \\ \grave{\iota}\eta K11K11\epsilon\iota K11K11K10 = \grave{\iota}K11\epsilon K110 = \grave{\iota}1K10 = K10 = 0 \\ \grave{\iota}\eta K11K11\epsilon\iota K11K10K00 = \grave{\iota}K11\epsilon K100 = \grave{\iota}1K00 = K10 = 0 \\ \grave{\iota}\eta K11K11\epsilon\iota K11K11K00 = \grave{\iota}K11\epsilon K110 = \grave{\iota}1K10 = K10 = 0 \end{array} \qquad (26)$$

By *S4*, the outcome of (26) is not divided into components, and by *S5*, $\mathbf{P} = \Delta$. Hence, use of the e-1th level selectors from Example 2 for this $\mathbf{P}$ yields, by *S3*, the outcome shown in (27). Each *1* of that outcome happens, by *S4*, to distinguish a separate component; the values of $\mathbf{P}$ which, by *S5*, are appropriate to those components are shown in full in (28) - (31) below.

$$\mathbf{P} = LBCprDtuAAKqsDtvAwx = LB\phi_1\phi_2AA\phi_3\phi_4\phi_5 = L\psi_1A\psi_2\phi_5 = L\psi_1\chi_1$$

$$\theta = LBC01D00AAK00D00A11 = LB11AA011 = L1A11 = L11 = 0$$

$$\boldsymbol{\Lambda} = \grave{\iota}\eta\epsilon pr\epsilon tu\epsilon\iota\epsilon qs\epsilon tv\epsilon wx = \grave{\iota}\eta\phi_1\phi_2\epsilon\iota\phi_3\phi_4\phi_5 = \grave{\iota}\psi_1\epsilon\psi_2\phi_5 = \grave{\iota}\psi_1\chi_1$$

$$\begin{array}{ll} \grave{\iota}\eta K11K00\epsilon\iota K10K01K11 = \grave{\iota}L10\epsilon K001 = \grave{\iota}1K01 = K10 = 0 & \\ \grave{\iota}\eta K11K00\epsilon\iota K11K01K11 = \grave{\iota}L10\epsilon K101 = \grave{\iota}1K01 = K10 = 0 & \\ \grave{\iota}\eta K11K10\epsilon\iota K10K11K11 = \grave{\iota}L10\epsilon M011 = \grave{\iota}1K11 = K11 = 1 & (\mathbf{P}_1) \\ \grave{\iota}\eta K11K10\epsilon\iota K11K11K11 = \grave{\iota}L10\epsilon K111 = \grave{\iota}1K11 = K11 = 1 & (\mathbf{P}_2) \\ \grave{\iota}\eta K11K01\epsilon\iota K10K01K11 = \grave{\iota}L10\epsilon K001 = \grave{\iota}1K01 = K10 = 0 & \\ \grave{\iota}\eta K11K01\epsilon\iota K11K01K11 = \grave{\iota}L10\epsilon K101 = \grave{\iota}1K01 = K10 = 0 & \\ \grave{\iota}\eta K11K11\epsilon\iota K10K11K11 = \grave{\iota}K11\epsilon M011 = \grave{\iota}1K11 = K11 = 1 & (\mathbf{P}_3) \\ \grave{\iota}\eta K11K11\epsilon\iota K11K11K11 = \grave{\iota}K11\epsilon K111 = \grave{\iota}1K11 = K11 = 1 & (\mathbf{P}_4) \end{array} \qquad (27)$$

The four residual expressions of (27), i.e. $\mathbf{P}_1$ - $\mathbf{P}_4$ now govern the use of the appropriate (by θ) upper-level selectors of the arrays of Example 2; evaluation here proceeds, by *S6*, as in normal truth-functional calculation (as opposed to the reference to $\boldsymbol{\Lambda}$ hitherto called for by *S3*) thus:

$$\mathbf{P}_1 = LCprADtuAwx$$

$$\theta = LC01AD00A11 = L1A11 = L11 = 0$$

$$\begin{aligned}
&LC00AD11A00 = L1A00 = L10 = 1\\
&LC10AD11A00 = L0A00 = L00 = 0\\
&LC00AD11A00 = L1A00 = L10 = 1\\
&LC10AD11A00 = L0A00 = L00 = 0\\
&LC01AD11A00 = L1A00 = L10 = 1\\
&LC11AD11A00 = L1A00 = L10 = 1\\
&LC01AD11A00 = L1A00 = L10 = 1\\
&LC11AD11A00 = L1A00 = L10 = 1
\end{aligned} \tag{28}$$

$$\mathbf{P}_2 = LCprAAKqsDtvAwx$$

$$\theta = LC01AAK00D00A11 = L1AA011 = L1A11 = L11 = 0$$

$$\begin{aligned}
&LC00AAK01D11A00 = L1AA000 = L1A00 = L10 = 1\\
&LC10AAK01D11A00 = L0AA000 = L0A00 = L00 = 0\\
&LC00AAK11D11A00 = L1AA100 = L1A10 = L11 = 0\\
&LC10AAK11D11A00 = L0AA100 = L0A10 = L01 = 0\\
&LC01AAK01D11A00 = L1AA000 = L1A00 = L10 = 1\\
&LC11AAK01D11A00 = L1AA000 = L1A00 = L10 = 1\\
&LC01AAK11D11A00 = L1AA100 = L1A10 = L11 = 0\\
&LC11AAK11D11A00 = L1AA100 = L1A10 = L11 = 0
\end{aligned} \tag{29}$$

$$\mathbf{P}_3 = LBCprDtuADtvAwx$$

$$\theta = LBC01D00AD00A11 = LB11A11 = L11 = 0$$

$$\begin{aligned}
&LBC00D11AD11A00 = LB10A00 = L10 = 1\\
&LBC10D11AD11A00 = LB00A00 = L10 = 1\\
&LBC00D11AD11A00 = LB10A00 = L10 = 1\\
&LBC10D11AD11A00 = LB00A00 = L10 = 1\\
&LBC01D11AD11A00 = LB10A00 = L10 = 1\\
&LBC11D11AD11A00 = LB10A00 = L10 = 1\\
&LBC01D11AD11A00 = LB10A00 = L10 = 1\\
&LBC11D11AD11A00 = LB10A00 = L10 = 1
\end{aligned} \tag{30}$$

$$\mathbf{P}_4 = \Delta = LBCprDtuAAKqsDtvAwx$$

$$\theta = LBC01D00AAK00D00A11 = LB11AA011 = L1A11 = L11 = 0$$

$$\begin{aligned}
&LBC00D11AAK01D11A00 = LB10AA000 = L1A00 = L10 = 1\\
&LBC10D11AAK01D11A00 = LB00AA000 = L1A00 = L10 = 1\\
&LBC00D11AAK11D11A00 = LB10AA100 = L1A10 = L11 = 0\\
&LBC10D11AAK11D11A00 = LB00AA100 = L1A10 = L11 = 0\\
&LBC01D11AAK01D11A00 = LB10AA000 = L1A00 = L10 = 1\\
&LBC11D11AAK01D11A00 = LB10AA000 = L1A00 = L10 = 1\\
&LBC01D11AAK11D11A00 = LB10AA100 = L1A10 = L11 = 0\\
&LBC11D11AAK11D11A00 = LB10AA100 = L1A10 = L11 = 0
\end{aligned} \tag{31}$$

Hence, correlating the outcomes of (28) - (31) with the selector-components indicated (by *S4*) of the outcome of (27), and then correlating each of the four results thus obtained with the single-selector outcome of (26) which originally gave rise to the rest, one has:

$$AAA \quad \begin{vmatrix} 10101111 \\ 00100000 \\ 01000000 \end{vmatrix}^{0} \quad \begin{vmatrix} 10001100 \\ 00010000 \\ 01000000 \end{vmatrix}^{0}, \quad \begin{vmatrix} 11111111 \\ 00000010 \\ 01000000 \end{vmatrix}^{0} \quad \begin{vmatrix} 11001100 \\ 00000001 \\ 01000000 \end{vmatrix}^{0} \qquad (28)$$

This is the abbreviation, in terms of a disjunction of non-interfering arrays, of the $2^9 = 512$ digits of the full selector equivalent to the Δ shown in (26).

§9 Without going into full details, it may finally be remarked that theses (8) - (16) and the rules therefrom derived in Part II have built into them all that is required for calculations of the sort described to run smoothly; e.g. in respect of a given Δ final z-values will be constant for each **P**; also, the negations (*Na*, *Nc*, or both) demanded by (10), (12), (14), (15) and (16) (cf. *S5*(*b*)), when performed in accordance with (17) - (18), also yield results coincident with the appropriate final z-values.

§10 Acknowledgements are due to Dr. W. Mays.

Manchester University
Manchester, England