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ADDENDUM TO MY ARTICLE "PROOF OF SOME THEOREMS ON RECURSIVELY ENUMERABLE SETS"

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In the mentioned previous paper I proved the theorem that every recursively enumerable set could already be enumerated by a lower elementary function (see Df. 1 on p. 65 in [3]). On pp. 71-72 in the same paper I gave a hint of another possible proof of this statement. I have found later a version of this second proof which is particularly simple and which I should like to present here.

It follows from a result of E. L. Post that it will be sufficient to prove that every canonical set in a normal system (see [1], p. 287 and [2], p. 170) can be lower elementary enumerated. This can be done as follows. In a normal language we are dealing with strings of the two symbols *1* and *b*. One axiom is given, say the string γ. Further there are say m rules of production of the form

$$\sigma_{1,r}\,\alpha \to \alpha\,\sigma_{2,r}, \quad r = 1, \ldots, m$$

where α is an arbitrary string, the $\sigma_{1,r}$, $\sigma_{2,r}$ given strings. To any string β with n symbols we now let correspond the integer

$$p_0^{\epsilon_0}\, p_1^{\epsilon_1} \cdots p_n^{\epsilon_n},$$

where $\epsilon_r = 1$ or 2 according as the r^{th} symbol in β is *1* or *b*, with p_0, p_1, p_2, . . . being the sequence of natural primes. Obviously this yields a one to one correspondence $\mathfrak{F}$ between the strings and the subset of the natural numbers consisting of the cubefree integers.

Let a correspond to the axiom γ. Further let us consider a production rule

$$\sigma_1 \alpha \to \alpha\, \sigma_2,$$

while a_1 and a_2 correspond to σ_1 and σ_2 respectively, say

$$a_1 = p_0^{\epsilon_0}\, p_1^{\epsilon_1} \cdots p_c^{\epsilon_c}, \quad a_2 = p_0^{\tau_0}\, p_1^{\tau_1} \cdots p_d^{\tau_d}$$

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Then the x corresponding to $\sigma_1\,\alpha$ will for any α possess the form

$$x = a_1 z, \text{ where } z = p_{c+1}^{\epsilon_{c+1}} \ldots p_n^{\epsilon_n} \text{ corresponds to } \alpha\,.$$

Further, the y corresponding to $\alpha\sigma_2$ will be

$$y = uv\ ,$$

where

$$u = p_0^{\epsilon_{c+1}} \ldots p_{n-c-1}^{\epsilon_n}\,, \qquad v = p_{n-c}^{\tau_c} \ldots p_{n-c+d}^{\tau_d}\,.$$

Now u is a lower elementary function of z. Indeed n is such a function of z, because $n-c$ is the number of different primefactors of z and c a given constant. The number of different primefactors of z is namely

$$\chi(z) = \sum_{r=0}^{z} (1 \dot{-} \mathbf{P}(r))\, \mathbf{d}(r,\ z)\ ,$$

where $\mathbf{P}(r)$ is the l.el. function which is 0 or 1 according as r is a prime or not, while $\mathbf{d}(r,s)$ is 1 or 0 according as r divides s or not (see the previous paper p. 67). Then it is seen that

$$u = \sum_{s=0}^{z} s\,(1 - \sum_{t=n-c}^{z} \mathbf{e}(s,\ t))\,(1 \dot{-} \sum_{r=c+1}^{n} \bar{\delta}(\mathbf{e}(s,\ r-c-1),\ \mathbf{e}(z,\ r)))\ .$$

Further v is obviously a l.el. function of n and therefore of z. Finally $z = \left[\frac{x}{a_1}\right]$. Thus y is a lower elementary function of x.

To each of the m rules of production

$$\sigma_{1,r}\,\alpha = \alpha\,\sigma_{2,r}$$

we obtain in this way a lower elementary function l_r such that $y = l_r(x)$ corresponds to $\alpha\sigma_{2,r}$ as often as x corresponds to $\sigma_{1,r}\alpha$. Then it is clear that the set S of numbers corresponding to the set of strings generated from γ by use of the production rules will consist of $\mathbf{a}$ and the numbers we get by repeated insertions of already obtained numbers into the functions l_r, that is

$$a,\ l_1(a), \ldots, l_m(a),\ l_1 l_1(a),\ l_2 l_1(a), \ldots, l_m l_1(a),\ l_1 l_2(a), \ldots, l_m l_2(a), \ldots$$

However, this set S will be just the values of the following function ϕ:

$$\phi(0) = a, \quad \phi(n+1) = \sum_{r=1}^{m} l_r(\phi\left[\frac{n}{m}\right])\ \delta(\mathbf{rm}(n+1,\ m),\ r)\ ,$$

where $\mathbf{rm}(x,\ m)$ is the least positive remainder of x divided by m. This is a recursive definition of ϕ of the kind considered in Theorem 1 in my previous paper. Thus according to this theorem the set S can be enumerated by some lower elementary function.

Lemma. The intersection of two l.el. enumerable sets S_1 and S_2 is l.el. enumerable if it is not empty.

Proof: Let S_1 and S_2 be the set of values of the l.el. functions $f_1(t)$ and $f_2(t)$ respectively and let c belong to $S_1 \cap S_2$ so that for certain c_1 and c_2

$$f_1(c_1) = f_2(c_2) = c\ .$$

Then the l.el. function

$$g(x, y) = f_1(x)\,\delta(f_1(x), f_2(y)) + c\,\overline{\delta}(f_1(x), f_2(y))$$

takes the value $f_1(x)$ for every x, y such that $f_1(x) = f_2(y)$ and otherwise the value c. Therefore it is clear that $g(\varepsilon_1^{(2)}(z), \tau_2^{(2)}(z))$ which is a l.el. function of z takes for $z = 0, 1, 2, \ldots$ successively all the values of $f_1(x)$ which are also values of $f_2(y)$.

Now let $\mathbf{q}(n)$ be the n^{th} squarefree number, that is an integer not divisible by the square of any number > 1. It is seen at once that the l.el. function

$$\kappa(a) = \sum_{r=0}^{a} \mathbf{d}((r+1)^2, a)$$

is 0 or > 0 according as a is squarefree or not. Since every prime is squarefree, we have

$$\mathbf{q}(n) \leqq \mathbf{p}_n < (n+1)^2, \text{ (l.c.p. 67)}$$

whence

$$\mathbf{q}(n) = \sum_{r=0}^{(n+1)^2} r(1 \dot{-} \kappa(r))\,\delta\left(\sum_{s=0}^{r-1}(1 \dot{-} \kappa(s)),\ n-1\right)$$

so that $\mathbf{q}(n)$ is l.el. Since both S and the set K of squarefree numbers are l.el. enum., we have according to the lemma that $S \cap K$ is l.el. enum., if it is not empty. Now according to Post every recursively enumerable set of integers may be obtained as the integers represented by the strings, of symbols 1 only, existing in one of the diverse normal languages. The integers corresponding by $\mathfrak{F}$ to these strings are just the elements of $S \cap K$ when S by $\mathfrak{F}$ corresponds to the strings altogether in the normal system. The elements of $S \cap K$ are the diverse values of the l.el. function $\psi(t)$ say. Now if N corresponds to the string with n symbols 1, n is the number of different primes dividing N, that is

$$n = \chi(N)\ .$$

Since the integers N are the diverse values of $\psi(t)$, we obtain, putting

$$n = \chi\psi(t)\ ,$$

all n represented by the strings built up of symbols 1 only in our arbitrarily chosen normal system by putting successively $t = 0, 1, 2, \ldots$ into the l.el. function $\chi\psi(t)$. Thus we have got a second proof of our theorem, that every recursively enumerable set is already lower elementary enumerable.

REFERENCES

[1] E. L. Post: Recursively enumerable sets of positive integers and their decision problems. *Bulletin of the American Mathematical Society*, v. 50, (1944) pp. 284-316.

[2] Paul C. Rosenbloom: *The elements of mathematical logic*. Dover Publications, 1950.

[3] Th. Skolem: Proof of some theorems on recursively enumerable sets. *Notre Dame Journal of Formal Logic*, v. III (1962), pp. 65-74.

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