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A STANDARD FORM FOR ŁUKASIEWICZ MANY-VALUED LOGICS

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The purpose of this paper is to establish a relationship between a simple set of arithmetic functions and the functions in the Łukasiewicz many-valued logics. We first define the set of arithmetic functions to be used and derive a standard form for them. Then by an obvious restriction on the variables and a natural restriction on the function values these functions prove to be those generated in the above mentioned logics.

D1 $\mathcal{M}$ *is the set of functions defined on the integers and generated by substitution from the constant functions* o *and* n, *and the binary operations* +, −, max *and* min.[1]

T1 $\mathcal{L} = \left\{ a_o n + \sum_{k=1}^{m} a_k p_k \mid a_o \text{ and the } a_k \text{ are integers and the } p_k \text{ are variables} \right\}$
is the totality of functions generated by +, −, *o*, and *n*.

Proof: Algebraically obvious.

D2 *If* $f \in \mathcal{L}$, *then* f *is called a linear form.*[2]

P1 (a) $\max_i(a_i) + \max_j(b_j) = \max_{i,j}(a_i + b_j)$.[3]

(b) $\min_i(a_i) + \min_j(b_j) = \min_{i,j}(a_i + b_j)$

Proof of a.

$a_i \leq \max_i(a_i)$, $b_j \leq \max_j(b_j)$ for all i, j. So

$a_i + b_j \leq \max_i(a_i) + \max_j(b_j)$ for all i, j. Therefore

1. n, −, and max would suffice.
2. When convenient we shall identify f with $f(p_i, \cdots, p_n)$
3. The index sets hereafter will be sections of the natural numbers. We adopt the convention, $\max(a) = a = \min(a)$

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$\max_{i,j}(a_i + b_j) \leq \max_i(a_i) + \max_j(b_i)$.

There exist i_o and j_o so that

$a_{i_o} = \max_i(a_i)$, $b_{j_o} = \max_j(b_j)$. Therefore

$\max_i(a_i) + \max_j(b_j) = a_{i_o} + b_{j_o} \leq \max_{i,j}(a_i + b_j)$. *QED*

P2 (a) $-\max_i(a_i) = \min_i(-a_i)$

(b) $-\min_i(a_i) = \max_i(-a_i)$

Proof of a.

$a_i \leq \max_i(a_i)$ for all i.

$-\max_i(a_i) \leq -a_i$ for all i. Therefore

$-\max_i(a_i) \leq \min_i(-a_i)$.

There exists an i_o so that $a_{i_o} = \max_i(a_i)$

$-\max_i(a_i) = -a_{i_o} \geq \min_i(-a_i)$ *QED*

P3 (a) $a + \max_i(a_i) = \max_i(a + a_i)$ [*P1a*]

(b) $a + \min_i(a_i) = \min_i(a + a_i)$ [*P1b*]

P4 (a) $a - \max_i(a_i) = \min_i(a - a_i)$ [*P2a*, *P3b*]

(b) $a - \min_i(a_i) = \max_i(a - a_i)$ [*P2b*, *P3a*]

P5 (a) $\min_i(a_i) + \max_j(b_j) = \max_j(\min_i(a_i + b_j))$

(b) $\min_i(a_i) + \max_j(b_j) = \min_i(\max_j(a_i + b_j))$

Proof of a.[4]

$\min_i(a_i) + \max_j(b_j) = [P3a]\ \max_j(\min_i(a_i) + b_j) = [P3b]\ \max_j(\min_i(a_i + b_j))$

Thus max and min can always be moved exterior to + by *P1*, *P5*, and to − by *P1*, *P2*, *P5*. This together with *T1* proves

T2 If $f(p_i, \ldots, p_k) \in \mathcal{M}$, then either f is a linear form or $f(p_i, \ldots, p_k) = F(\cdots, L_i(p_i, \cdots, p_k), \cdots)$, where F is composed only of max's and min's and the L_i are linear forms. If $f \in \mathcal{L}$, F may be considered as vacuous.

4. We shall insert reasons into a proof by means of [1].

D3 *If* $f(p_i, \cdots, p_k) \in \mathcal{M}$ *and* $f(p_i, \cdots, p_k) = F(\cdots, L_i(p_i, \cdots, p_k), \cdots)$ *then* $F(\cdots, L_i(p_i, \cdots p_k), \cdots)$ *is called a standard form.*

T3 *Every function in* $\mathcal{M}$ *can be represented by a standard form and vice-versa.*

Proof. [*T2, D1, D3*]

Note, as *P5* shows, that this representation is not unique.

P6 (a) $\max_j(\max_i(a_i), b_j) = \max_{i,j}(a_i, b_j)$

(b) $\min_j(\min_i(a_i), b_j) = \max_{i,j}(a_i, b_j)$

To bring the functions of $\mathcal{M}$ into the realm of $n + 1$ – valued logic, we first restrict the domains of all the variables to $\{0, 1, 2, \cdots, n\}$,[5] and then truncate their function values by means of the definition given below. We shall then prove that the functions in $\mathcal{M}$ thus modified are precisely the functions in Lukasiewicz $\{C,N\}$–logics.

D4 *For any integer a, the truncation of a, denoted by* $\mathbf{T}(a)$ is defined by

$$\mathbf{T}(a) = \begin{cases} n, & a \geq n \\ a, & 0 \leq a \leq n \\ 0, & a \leq 0 \end{cases}$$

P7 $0 \leq \mathbf{T}(a) \leq n$ [*D4*][6]

P8 $a \leq b \rightarrow \mathbf{T}(a) \leq \mathbf{T}(b)$

PF 1) $b \geq n.$ $\mathbf{T}(b) = n \geq \mathbf{T}(a)$ [*P7*]

2) $a \leq o.$ $\mathbf{T}(a) = o \leq \mathbf{T}(b)$ [*P7*]

3) $a \geq o$ and $b \leq n.$ $o \leq a \leq b \leq n \rightarrow \mathbf{T}(a) = a$ and $\mathbf{T}(b) = b.$

P9 $\mathbf{T}(\mathbf{T}(a)) = \mathbf{T}(a)$ [*P7*]

P10 (a) $a \geq o \rightarrow \mathbf{T}(a) = \min(n, a)$

(b) $a \leq n \rightarrow \mathbf{T}(a) = \max(o, a)$

P11 $a \geq o$ and $b \geq o \rightarrow \mathbf{T}(a + b) = \mathbf{T}(\mathbf{T}(a) + \mathbf{T}(b))$

Proof:

(1) $a \geq n.$ (Similarly for $b \geq n$)

$a + b \geq n \rightarrow \mathbf{T}(a + b) = n$

$\mathbf{T}(a) = n \rightarrow$ [*P7*] $\mathbf{T}(a) + \mathbf{T}(b) \geq n \rightarrow \mathbf{T}(\mathbf{T}(a) + \mathbf{T}(b)) = n.$

5. Note that this is different from restricting the arguments of the various operations to this set.

6. In subsequent proofs we shall not refer explicitly to *D4*.

(2) $o \leq a \leq n$ and $o \leq b \leq n$.

$\mathbf{T}(a) = a$ and $\mathbf{T}(b) = b \to \mathbf{T}(a + b) = \mathbf{T}(\mathbf{T}(a) + \mathbf{T}(b))$.

P12 $o \leq a \leq n$ and $o \leq b \to \mathbf{T}(a - b) = \mathbf{T}(a - \mathbf{T}(b))$

Proof:

(1) $b \leq n$. $\mathbf{T}(b) = b \to \mathbf{T}(a - b) = \mathbf{T}(a - \mathbf{T}(b))$

(2) $b \geq n$.

$a - b \leq o \to \mathbf{T}(a - b) = o$.

$\mathbf{T}(b) = n \to a - \mathbf{T}(b) \leq o \to \mathbf{T}(a - \mathbf{T}(b)) = o$

P13 (a) $\mathbf{T}(\max(a, b)) = \max(\mathbf{T}(a), \mathbf{T}(b))$

(b) $\mathbf{T}(\min(a, b)) = \min(\mathbf{T}(a), \mathbf{T}(b))$.

Proof of a

(1) $a \geq n$. (Similarly for $b \geq n$)

$\max(a, b) \geq n \to \mathbf{T}(\max(a, b)) = n$.

$\mathbf{T}(a) = n$ and [*P7*] $\mathbf{T}(b) \leq n \to \max(\mathbf{T}(a), \mathbf{T}(b)) = n$.

(2) $o \leq a, b \leq n$.

$o \leq \max(a, b) \leq n \to \mathbf{T}(\max(a, b)) = \max(a, b)$

$\mathbf{T}(a) = a$ and $\mathbf{T}(b) = b \to \max(\mathbf{T}(a), \mathbf{T}(b)) = \max(a, b)$

(3) $a \leq o$ and $b \leq o$.

$\mathbf{T}(a) = o = \mathbf{T}(b) \to \max(\mathbf{T}(a), \mathbf{T}(b)) = o$

$\max(a, b) \leq o \to \mathbf{T}(\max(a, b)) = o$

(4) $a \leq o$ and $o \leq b \leq n$. (Similarly for $b \leq o$ and $o \leq a \leq n$)

$\mathbf{T}(a) = o \leq b = \mathbf{T}(b) \to \max(\mathbf{T}(a), \mathbf{T}(b)) = \mathbf{T}(b)$

$a \leq b \to \max(a, b) = b \to \mathbf{T}(\max(a, b)) = \mathbf{T}(b)$

P14 $\mathbf{T}(\mathbf{T}(b) - \mathbf{T}(a)) = \min(\mathbf{T}(b), \mathbf{T}(b - a), \mathbf{T}(n - a))$

Proof:

(1) $b \leq o$.

$\mathbf{T}(b) = o \to$ [*P7*] $\min(\mathbf{T}(b), \mathbf{T}(b - a), \mathbf{T}(n - a)) = o$.

$\mathbf{T}(b) = o \to$ [*P7*] $\mathbf{T}(b) - \mathbf{T}(a) \leq o \to \mathbf{T}(\mathbf{T}(b) - \mathbf{T}(a)) = o$

(2) $a \leq o$.

$\mathbf{T}(a) = o \to \mathbf{T}(\mathbf{T}(b) - \mathbf{T}(a)) = \mathbf{T}(\mathbf{T}(b)) =$ [*P9*] $\mathbf{T}(b)$

$b \leq b - a \to$ [*P8*] $\mathbf{T}(b) \leq \mathbf{T}(b - a)$

$n - a \geq n \to \mathbf{T}(n - a) = n \geq \mathbf{T}(b)$

$\therefore \min(\mathbf{T}(b), \mathbf{T}(b - a), \mathbf{T}(n - a)) = \mathbf{T}(b)$.

(3) $a \geq o$ and $b \geq n$.

$\mathbf{T}(b) = n \to \mathbf{T}(\mathbf{T}(b) - \mathbf{T}(a)) = \mathbf{T}(n - \mathbf{T}(a)) = [P12]\ \mathbf{T}(n - a)$.

$\mathbf{T}(b) = n \geq \mathbf{T}(n - a)$

$n - a \leq b - a \to [P8]\ \mathbf{T}(n - a) \leq \mathbf{T}(b - a)$

$\therefore \min(\mathbf{T}(b), \mathbf{T}(b - a), \mathbf{T}(n - a)) = \mathbf{T}(n - a)$

(4) $a \geq o$ and $o \leq b \leq n$.

$\mathbf{T}(b) = b \to \mathbf{T}(\mathbf{T}(b) - \mathbf{T}(a)) = \mathbf{T}(b - \mathbf{T}(a)) = [P12]\ \mathbf{T}(b - a)$

$b - a \leq b \to [P8]\ \mathbf{T}(b - a) \leq \mathbf{T}(b)$.

$b - a \leq n - a \to [P8]\ \mathbf{T}(b - a) \leq \mathbf{T}(n - a)$

$\therefore \min(\mathbf{T}(b), \mathbf{T}(b - a), \mathbf{T}(n - a)) = \mathbf{T}(b - a)$.

We now enter the $n + 1$-valued logic based on C and N and using the values $\{o, 1, \ldots, n\}$. We shall use p, q, possibly with subscripts, for the variables. To recall

$Cpq = \max(o, q - p)$; $Np = n - p$.

$\mathfrak{a}_2 pq = \min(n, p + q)$ is also in this logic [1].

P15 $Cpq = \mathbf{T}(q - p)$

PF $q \leq n$ and $p \geq o \to q - p \leq n \to \mathbf{T}(q - p) = \max(o, q - p) = Cpq$

T4 *The truncation of any linear form, i.e.* $\mathbf{T}(a_o n + \sum_{i+1}^{k} a_i p_i)$ *is generated by C and N. Let* $n = p_o$, *then this becomes* $\mathbf{T}\left(\sum_{i=o}^{k} a_i p_i\right)$

Proof. By induction on $\sum_{i=o}^{k} |a_i|$.

1. $\sum_{i=o}^{k} |a_i| = o$.

   then $a_i = o$ for all $o \leq i \leq k$, and $\sum_{i=o}^{k} a_i p_i = o = Cpp$.

2. Suppose the theorem is valid for all linear forms

   $\sum_{i=o}^{k} a_i p_i$ for which $\sum_{i=o}^{k} |a_i| < l$.

   Consider an arbitrary linear form $\sum_{i=o}^{k} a_i p_i$ for which $\sum_{i=o}^{k} |a_i| = l$.

2.1. $a_i \leq o$ for all $o \leq i \leq k$.

   then $\sum_{i=o}^{k} a_i p_i \leq o \to \mathbf{T}\ \sum_{i=o}^{k} a_i p_i = o = Cpp$.

2.2. There is a j, $o \leq j \leq k$ so that $a_j > o$.

Then $|a_j - 1| < |a_j|$; and since $|a| = |-a|$,

$$|-(a_j - 1)| + \sum_{\substack{i=o \\ i \neq j}}^{k} |-a_j| = |a_j - 1| + \sum_{\substack{i=o \\ i \neq j}}^{k} |a_i| < l. \text{ Therefore}$$

by the induction hypothesis, and since the negative of a linear form is a linear form,

for $A = (a_j - 1)\, p_j + \sum_{\substack{i=o \\ i \neq j}}^{k} a_i p_i$, $\mathbf{T}(A)$ and $\mathbf{T}(-A)$ are generated.

Consider

$C(\mathbf{T}(-A), \alpha_2(p_j, \mathbf{T}(A))) = \max(o, \min(n, p_j + \mathbf{T}(A)) - \mathbf{T}(-A))$

For values of the variables for which $A \geq o$

$\max(o, \min(n, p_j + \mathbf{T}(A)) - \mathbf{T}(-A)) = [-A \leq o]\ \max(o, \min(n, p_j + \mathbf{T}(A)) = [p_j \geq o,\ \mathbf{T}(A) \geq o]\ \min(n, p_j + \mathbf{T}(A)) = [P10]\ \mathbf{T}(p_j + \mathbf{T}(A)) = [p_j = \mathbf{T}(p_j)]\ \mathbf{T}(\mathbf{T}(p_j) + \mathbf{T}(A)) = [P11]\ \mathbf{T}(p_j + A) = \mathbf{T}(p_j + (a_j - 1)\, p_j + \sum_{\substack{i=o \\ i \neq j}}^{k} a_i p_i) = \mathbf{T}(\sum_{\substack{i=o \\ i \neq j}}^{k} a_i p_i)$.

For values of the variables for which $A \leq o$.

$\max(o, \min(n, p_j + \mathbf{T}(A)) - \mathbf{T}(-A)) = \max(o, \min(n, p_j) - \mathbf{T}(-A)) = \max(o, p_j - \mathbf{T}(-A)) = [P10]\ \mathbf{T}(p_j - \mathbf{T}(-A)) = [P12, -A \geq o]\ \mathbf{T}(p_j - (-A)) = \mathbf{T}(p_j + A) = \mathbf{T}(\sum_{\substack{i=o \\ i \neq j}}^{k} a_i p_i)$.

Thus the induction step is proved and the proof is complete.

T5 *If* $f \in \mathcal{M}$, $\mathbf{T}(f)$ *is generated by* C *and* N.

Proof:

Let $F(\cdots, L_i, \cdots)$ be a standard form representing f.

Then $\mathbf{T}(f) = \mathbf{T}(F(\cdots, L_i, \cdots)) = [T2, P13]\ F(\cdots, \mathbf{T}(L_i), \cdots)$.

By *T4*, the $\mathbf{T}(L_i)$ are generated. Now

$$\max(p, q) = \alpha_2 pCpq;\ \min(p, q) = CCpqq.$$

Therefore by *P6*, max and min for any finite number of arguments are generated by C and N.

Therefore, $F(\cdots, \mathbf{T}(L_i), \cdots) = \mathbf{T}(f)$ is generated.

T6 *Every function generated by C and N is the truncation of a function in* $\mathcal{M}$.

Proof:

$Cpq =$ [*P15*] $\mathbf{T}(q-p)$ and $Np = \mathbf{T}(n-p)$. Therefore, Cpq and Np are truncations of standard forms. Since $Np = Cpn$, it remains to prove that the truncations of standard forms are closed under the application of C.

Take any two standard forms $F(\cdots, L_i, \cdots)$ and $G(\cdots, M_j, \cdots)$, then by *P13*

$\mathbf{T}(F(\cdots, L_i, \cdots)) = F(\cdots, \mathbf{T}(L_i), \cdots) = F_t$ and

$\mathbf{T}(G(\cdots, M_j, \cdots)) = G(\cdots, \mathbf{T}(M_j), \cdots) = G_t$.

There are four possible forms that $G_t - F_t$ may take,

I. $\min_k(\alpha_k) - \max_l(\beta_l)$

II. $\max_k(\alpha_k) - \min_l(\beta_l)$

III. $\max_k(\alpha_k) - \max_l(\beta_l)$

IV. $\min_k(\alpha_k) - \min_l(\beta_l)$

From *P1*, *P2*, and *P5* we know

1) $\min_k(a_k) - \max_l(b_l) = \min_{k,l}(a_k - b_l)$

2) $\max_k(a_k) - \min_l(b_l) = \max_{k,l}(a_k - b_l)$

3) $\max_k(a_k) - \max_l(b_l) = \max_k(\min_l(a_k - b_l))$

4) $\min_k(a_k) - \min_l(b_l) = \min_k(\max_l(a_k - b_l))$

We note that minus sign gives rise to only minus signs in all four cases. Therefore, by repeated application[7] of 1), 2), 3) and 4) we deduce that $G_t - F_t = H(\cdots, \mathbf{T}(M_j) - \mathbf{T}(L_i), \cdots)$ for some values of i and j, and where H is composed only of max's and min's.

Therefore

$C(\mathbf{T}(F), \mathbf{T}(G)) =$ [*P15*] $\mathbf{T}(\mathbf{T}(G) - \mathbf{T}(F)) = \mathbf{T}(G_t - F_t) = \mathbf{T}(H) = H(\cdots, \mathbf{T}(\mathbf{T}(M_j) - \mathbf{T}(L_i)), \cdots) =$ [*P14*] $H(\cdots, \min(\mathbf{T}(M_j), \mathbf{T}(M_j - L_i), \mathbf{T}(n - L_i)), \cdots) =$ [*P13*] $\mathbf{T}(H(\cdots, \min(M_j, M_j - L_i, n - L_i), \cdots)) = \mathbf{T}(H^*(\cdots, M_j, M_j - L_i, n - L_i, \cdots))$

where H^* is again composed only of max's and min's. Since L, M, and n are linear forms, clearly so are $M_j - L_i$ and $n - L_i$. Thus $H^*(\cdots, M_j, M_j - L_i, n - L_i, \cdots)$ is a standard form.

7. A trivial but notationally cumbersome induction.

So

$C(\mathbf{T}(F), \mathbf{T}(G)) = \mathbf{T}(H^*)$, the truncation of a standard form. Therefore the truncations of standard forms are closed under the application of C.

T7 *The functions generated by C and N are precisely the truncations of the functions of* $\mathcal{M}$.

Proof by *T3, T5, T6*.

If desired, any standard form $F(\cdot\cdot\cdot, L_i, \cdot\cdot\cdot)$ may be transformed to have the following alternating property:

T8 *If* A_i *is the main functor of F and* A_{i+1} *is the main functor of any argument of* A_i, *then any sequence* $A_1, A_2, \ldots, A_k$ *that can be formed in F is an alternation of* max's *and* min's.

Proof. If A_i and A_{i+1} are both max's, then *P6a* reduces them to a single max. Similarly for min.

It is of interest to note that although C and N are formed from n and the truncation of subtraction, that the truncations of the functions in $\mathcal{L}$, i.e. those generated by n and $-$, are not sufficient to yield the $\{C,N\}$–logics. To be assured of this insufficiency we shall prove that

T9 max(similarly min) *is not the truncation of a linear form unless the logic is two-valued,* i.e. $n = 1$.

Proof. If $n = 1$, $\max(p,q) = \mathbf{T}(p + q) = \alpha_2(p,q)$

Let $n > 1$, suppose that $\max(p,q)$ is the truncation of a linear form, i.e.

$$\max(p,q) = \mathbf{T}(an + bp + cq)$$
$$1 = \max(1,0) = \mathbf{T}(an + b) \to an + b = 1$$
$$1 = \max(1,1) = \mathbf{T}(an + b + c) \to an + b + c = 1$$

$\therefore$ By subtraction, $c = 0$. Similarly $b = 0$

Therefore $\max(p,q) = \mathbf{T}(an)$. By this makes max a constant function. Contradiction.

REFERENCE

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