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ON STRENGTHENING INTUITIONISTIC LOGIC

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Leblanc and Belnap [2] have shown that standard Gentzen rules of inference (N-version) for intuitionistic propositional calculus (PC_I) become rules for classical propositional calculus (PC_C) upon strengthening the '≡'–elimination rule. They conjecture that PC_I can be strengthened to PC_C only by strengthening rules for '∼' or '⊃' or '≡'.

We show that the addition of a clause (c) to their two part 'v'–introduction rule turns their formulation of PC_I into one of PC_C. The new rule is:

DI_C: (a) $A \vdash A \vee B$, (b) $B \vdash A \vee B$, (c) *If* $\vdash A^*$ *and* $A, P \vdash Q$, *then* $\vdash A \vee P$,

where for (c) the restrictions hold: (i) P and Q are (metamathematical variables for) distinct proposition letters (using the terminology of [1]); (ii) A is a wff containing no proposition letter other than P; (iii) A^* is an instance of A (i.e. there is some wff B such that A^* results from A upon substitution of B for P).

Lemma 1: *In the system obtained from* PC_I *by replacing DI by* DI_C: $\vdash A \vee \sim A$.

Proof: For any wff A, let $P_1, P_2, \ldots, P_m$ be the proposition letters occurring in A. In PC_I for each P_i, $i = 1, \ldots, m$, (and for any proposition letter Q): $\vdash \sim(P_i \ \& \sim P_i)$ and $\sim P_i, P_i \vdash Q$. Hence by DI_C (c) (with $\sim P_i$ as A): $\vdash \sim P_i \vee P_i$. Thence (*cf.* [1] §29 Remark 1 (b)): $\vdash A \vee \sim A$.

Lemma 2: DI_C (c) *is a derivable rule of inference for* PC_C.

Proof: Assume in PC_C (with (i) - (iii) above): (a) $\vdash A^*$ and (b) $A, P \vdash Q$. By (a), (iii) and the consistency of PC_C, there is some assignment of truth values to the proposition letters of A which makes t the value of A. By (ii), the only proposition letter of A is P. Then by (b) (with (i)) and consistency, the assignment which gives the value t to A is exactly the assignment of f to P. Similarly from (b) and consistency, the assignment of t to P yields f for A. Hence by completeness: $\vdash \sim A \equiv P$. Then from $\vdash A \vee \sim A$ we can deduce $\vdash A \vee P$.

Theorem: *If in* PC_I *the rule DI is replaced by* DI_C, *the system obtained is* PC_C.

Proof: By Lemmas 1 and 2.

REFERENCES

[1] S. C. Kleene, *Introduction to metamathematics*, New York, 1952.

[2] Hugues Leblanc and Nuel D. Belnap, Jr., "Intuitionism reconsidered," *Notre Dame Journal of Formal Logic*, vol. III (1962), pp. 79-82.

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