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A NOTE ON A PROBLEM CONCERNING THE AXIOMATIC FOUNDATIONS OF MEREOLOGY

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In 1948 Sobociński established that mereology could be based on the following single axiom:

$\mathfrak{A}$ $\quad [A\ B] \text{ ⁙ } A \ \varepsilon\ \text{el}\,(B)\ .\equiv \text{ ⁖: } B\ \varepsilon\ B \text{ ⁖: } [f\ a] :: [C] :: C\ \varepsilon\ f\,(a)\ .\equiv\ \therefore [D] : D\ \varepsilon\ a\ .\supset .\ D\ \varepsilon\ \text{el}\,(C) \therefore [D] : D\ \varepsilon\ \text{el}\,(C)\ .\supset . [\exists\ E\ F]\ .\ E\ \varepsilon\ a\ .\ F\ \varepsilon\ \text{el}\,(D)\ .\ F\ \varepsilon\ \text{el}\,(E) \text{ :·: } B\ \varepsilon\ \text{el}\,(B)\ .\ B\ \varepsilon\ a \text{ :·: } \supset .\ A\ \varepsilon\ \text{el}\,(f\,(a))$[1]

Since then a number of single axioms for other mereological constant terms have been found.[2] All of them involve, as Sobociński's axiom does, quantification over two types of variable, *viz.*, over nominal variables represented in $\mathfrak{A}$ by 'A', 'B', 'C', 'D', 'E', 'F', and 'a', and over functorial variables represented in $\mathfrak{A}$ by 'f'. The latter belong to the semantical category of name-forming functors for one nominal argument. Naturally, it was desirable to have a single mereological axiom with quantification over nominal variables only, and in 1961 Sobociński set his logical seminar in the University of Notre Dame the problem of finding such an axiom for the term 'el'.

In 1960 I found a thesis, a little longer than $\mathfrak{A}$, which could be used as a single axiom of mereology and which involved quantification over nominal variables only. It presupposed, however, a different primitive term. Given the definition

AD1 $\quad [A\ a] :: A\ \varepsilon\ A \therefore [B] : B\ \varepsilon\ a\ .\supset .\ B\ \varepsilon\ \text{el}\,(A) \therefore [B] : B\ \varepsilon\ \text{el}\,(A)\ .\supset . [\exists\ C\ D]\ .\ C\ \varepsilon\ a\ .\ D\ \varepsilon\ \text{el}\,(B)\ .\ D\ \varepsilon\ \text{el}\,(C) \therefore \equiv .\ A\ \varepsilon\ \text{Kl}\,(a)$

the term I required could be defined as follows:

AD2 $\quad [A\ B\ a] : A\ \varepsilon\ \text{el}\,(B)\ .\ B\ \varepsilon\ \text{Kl}\,(a)\ .\equiv .\ A\ \varepsilon\ \text{elKl}\,(B\ a)$

And the thesis to serve as a single axiom had the following form:

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$\mathfrak{B}$ $[A\ b\ a] \therefore A\ \varepsilon\ \mathrm{elKl}\ (B\ a) . \equiv \vdots\ B\ \varepsilon\ B \vdots \sim (B\ \varepsilon\ \mathrm{elKl}\ (B\ B)) . \vee \therefore [b] :: B\ \varepsilon\ b . \supset \therefore [\exists\ C] \therefore A\ \varepsilon\ \mathrm{elKl}\ (C\ b) \therefore [D] : D\ \varepsilon\ \mathrm{elKl}\ (D\ b) . \supset . C\ \varepsilon\ D \therefore [C] : C\ \varepsilon\ a . \supset . C\ \varepsilon\ \mathrm{elKl}\ (B\ B) \therefore [C] : C\ \varepsilon\ \mathrm{elKl}\ (B\ B) . \supset . [\exists\ D\ E] . D\ \varepsilon\ a . E\ \varepsilon\ \mathrm{elKl}\ (C\ C) . E\ \varepsilon\ \mathrm{elKl}\ (D\ D)$

The theses

BD1 $[A\ B] : A\ \varepsilon\ \mathrm{elKl}\ (B\ B) . \equiv . A\ \varepsilon\ \mathrm{el}\ (B)$

BD2 $[A\ a] : A\ \varepsilon\ \mathrm{elKl}\ (A\ a) . \equiv . A\ \varepsilon\ \mathrm{Kl}\ (a)$

defined 'el' and 'Kl' in terms of 'elKl'.

It is interesting to note that $\mathfrak{B}$ has 13 ontological units, *i.e.*, it exceeds $\mathfrak{A}$ by one unit only. Yet the system {$\mathfrak{B}$, *BD1*, *BD2*} involves the use of 17 units, while 23 units occur in the system {$\mathfrak{A}$, *AD1*, *AD2*}.[3]

The proof that the two systems are inferentially equivalent is given below in the form of a condensed outline. We assume $\mathfrak{A}$, *AD1*, and *AD2*, and we proceed as follows.

T1. $[A\ B] : A\ \varepsilon\ \mathrm{el}\ (B) . \supset . B\ \varepsilon\ B$ [follows from $\mathfrak{A}$]

T2. $[A\ a] : A\ \varepsilon\ a . \supset . A\ \varepsilon\ \mathrm{el}\ (A)$ [from $\mathfrak{A}$]

T3. $[A\ B] : A\ \varepsilon\ \mathrm{el}\ (B) . \supset . [\exists\ C\ D] . C\ \varepsilon\ B . D\ \varepsilon\ \mathrm{el}\ (A) . D\ \varepsilon\ \mathrm{el}\ (C)$ [*T1*, *T2*]

T4. $[A\ a] : A\ \varepsilon\ a . \supset . A\ \varepsilon\ \mathrm{Kl}\ (A)$ [*T2*, *T3*, *AD1*]

T5. $[A\ B] : A\ \varepsilon\ \mathrm{el}\ (B) . \supset . A\ \varepsilon\ \mathrm{elKl}\ (B\ B)$ [*T1*, *T4*, *AD2*]

T6. (= *BD1*) . [*AD2*, *T5*]

T7. $[A\ a] : A\ \varepsilon\ \mathrm{Kl}\ (a) . \supset . A\ \varepsilon\ \mathrm{elKl}\ (A\ a)$ [*T2*, *AD2*]

T8. (= *BD2*) [*AD2*, *T7*]

T9. $[A\ B\ a] : A\ \varepsilon\ \mathrm{elKl}\ (B\ a) . \supset . B\ \varepsilon\ B$ [*AD2*, *T1*]

T10. $[A\ a] : A\ \varepsilon\ a . \supset . A\ \varepsilon\ \mathrm{elKl}\ (A\ A)$ [*T2*, *T4*, *AD2*]

T11. $[E\ a] \therefore E\ \varepsilon\ a . \supset :: [A] :: A\ \varepsilon\ \mathrm{Kl}\ (a) . \equiv \therefore [B] : B\ \varepsilon\ a . \supset . B\ \varepsilon\ \mathrm{el}\ (A) \therefore [B] : B\ \varepsilon\ \mathrm{el}\ (A) . \supset . [\exists\ C\ D] . C\ \varepsilon\ a . D\ \varepsilon\ \mathrm{el}\ (B) . D\ \varepsilon\ \mathrm{el}\ (C)$ [*AD1*, *T1*]

T12. $[A\ B\ a] : A\ \varepsilon\ \mathrm{el}\ (B) . B\ \varepsilon\ a . \supset . A\ \varepsilon\ \mathrm{el}\ (\mathrm{Kl}\ (a))$ [*T11*, *T2*, $\mathfrak{A}$]

T13. $[D\ b] : D\ \varepsilon\ \mathrm{elKl}\ (D\ b) . \supset . \mathrm{Kl}\ (b)\ \varepsilon\ D$ [*AD2*, *T2*, *T12*, *T1*]

T14. $[A\ B\ a] :: A\ \varepsilon\ \mathrm{elKl}\ (B\ a) . B\ \varepsilon\ b . \supset \therefore [\exists\ C] \therefore A\ \varepsilon\ \mathrm{elKl}\ (C\ b) \therefore [D] : D\ \varepsilon\ \mathrm{elKl}\ (D\ b) . \supset . C\ \varepsilon\ D$ [*AD2*, *T12*, *T1*, *T13*]

T15. $[A\ B\ a\ C] : A\ \varepsilon\ \mathrm{elKl}\ (B\ a) . C\ \varepsilon\ a . \supset . C\ \varepsilon\ \mathrm{elKl}\ (B\ B)$ [*AD2*, *AD1*, *T4*]

T16. $[A\ B\ a\ C] : A\ \varepsilon\ \mathrm{elKl}\ (B\ a) . C\ \varepsilon\ \mathrm{elKl}\ (B\ B) . \supset . [\exists\ D\ E] . D\ \varepsilon\ a . E\ \varepsilon\ \mathrm{elKl}\ (C\ C) . E\ \varepsilon\ \mathrm{elKl}\ (D\ D)$ [*AD2*, *T4*, *AD1*]

T17. $[A\ B\ a] : B\ \varepsilon\ B . \sim (B\ \varepsilon\ \mathrm{elKl}\ (B\ B)) . \supset . A\ \varepsilon\ \mathrm{elKl}\ (B\ a)$ [*T10*]

T18. $[a\ B\ D] :: [C] : C\ \varepsilon\ a . \supset . C\ \varepsilon\ \mathrm{elKl}\ (B\ B) \therefore D\ \varepsilon\ a \therefore \supset . D\ \varepsilon\ \mathrm{el}\ (B)$ [*AD2*]

T19. $[a\ B\ F] :: [C] : C \varepsilon \mathrm{elKl}\,(B\ B) . \supset . [\exists\ D\ E] . D \varepsilon a . E \varepsilon \mathrm{elKl}\,(C\ C) . E \varepsilon \mathrm{elKl}\,(D\ D) \therefore F \varepsilon \mathrm{el}\,(B) \therefore \supset . [\exists\ D\ E] . D \varepsilon a . E \varepsilon \mathrm{el}\,(F) . E \varepsilon \mathrm{el}\,(D)$ [*T1*, *T4*, *AD2*]

T20. $[A\ B\ a] \vdots\vdots B \varepsilon B :\cdot: [b] :: B \varepsilon b . \supset \therefore [\exists\ C] \therefore A \varepsilon \mathrm{elKl}\,(C\ b) \therefore [D] : D \varepsilon \mathrm{elKl}\,(D\ b) . \supset . C \varepsilon D :\cdot: [C] : C \varepsilon a . \supset . C \varepsilon \mathrm{elKl}\,(B\ B) \therefore [C] : C \varepsilon \mathrm{elKl}\,(B\ B) . \supset . [\exists\ D\ E] . D \varepsilon a . E \varepsilon \mathrm{elKl}\,(C\ C) . E \varepsilon \mathrm{elKl}\,(D\ D) :\cdot: \supset . A \varepsilon \mathrm{elKl}\,(B\ a)$ [*T10*, *AD2*, *T18*, *T19*, *AD1*]

T21. $(= \mathfrak{B})$ [*T9*, *T14*, *T15*, *T16*, *T17*, *T20*]

It is evident from *T21*, *T6*, and *T8* that the system $\{\mathfrak{B}$, *BD1*, *BD2*$\}$ is implied by the system $\{\mathfrak{A}$, *AD1*, *AD2*$\}$. To complete the outline of our proof we now assume $\mathfrak{B}$, *BD1*, and *BD2*, from which we derive the following theses.

*T21*1.* $[A\ a] : A \varepsilon a . \supset . A \varepsilon \mathrm{elKl}\,(A\ A)$ [$\mathfrak{B}$]

*T21*2.* $[A\ a\ B] : A \varepsilon \mathrm{Kl}\,(a) . B \varepsilon a . \supset . B \varepsilon \mathrm{el}\,(A)$ [*T21*1*, *BD2*, $\mathfrak{B}$, *BD1*]

*T21*3.* $[A\ a\ B] : A \varepsilon \mathrm{Kl}\,(a) . B \varepsilon \mathrm{el}\,(A) . \supset . [\exists\ C\ D] . C \varepsilon a . D \varepsilon \mathrm{el}\,(B) . D \varepsilon \mathrm{el}\,(C)$ [*T21*1*, *BD2*, *BD1*, $\mathfrak{B}$]

*T21*4.* $[A\ a\ C] :: [B] : B \varepsilon a . \supset . B \varepsilon \mathrm{el}\,(A) \therefore C \varepsilon a \therefore \supset . C \varepsilon \mathrm{elKl}\,(A\ A)$ [*BD1*]

*T21*5.* $[A\ a\ E] :: [B] : B \varepsilon \mathrm{el}\,(A) . \supset . [\exists\ C\ D] . C \varepsilon a . D \varepsilon \mathrm{el}\,(B) . D \varepsilon \mathrm{el}\,(C) \therefore E \varepsilon \mathrm{elKl}\,(A\ A) \therefore \supset . [\exists\ C\ D] . C \varepsilon a . D \varepsilon \mathrm{elKl}\,(E\ E) . D \varepsilon \mathrm{elKl}\,(C\ C)$ [*BD1*]

*T21*6.* $[A\ a] :: A \varepsilon A \therefore [B] : B \varepsilon a . \supset . B \varepsilon \mathrm{el}\,(A) \therefore [B] : B \varepsilon \mathrm{el}\,(A) . \supset . [\exists\ C\ D] . C \varepsilon a . D \varepsilon \mathrm{el}\,(B) . D \varepsilon \mathrm{el}\,(C) \therefore \supset . A \varepsilon \mathrm{Kl}\,(a)$ [*T21*1*, $\mathfrak{B}$, *T21*4*, *T21*5*]

*T21*7.* (= *AD1*). [*T21*6*, *T21*2*, *T21*3*]

*T21*8.* $[A\ B] : A \varepsilon \mathrm{el}\,(B) . \supset . B \varepsilon B$ [*BD1*]

*T21*9.* $[A\ a\ E] :: E \varepsilon a \therefore [B] : B \varepsilon a . \supset . B \varepsilon \mathrm{el}\,(A) \therefore [B] : B \varepsilon \mathrm{el}\,(A) . \supset . [\exists\ C\ D] . C \varepsilon a . D \varepsilon \mathrm{el}\,(B) . D \varepsilon \mathrm{el}\,(C) \therefore \supset . A \varepsilon \mathrm{Kl}\,(a)$ [*T21*8*, *T21*6*]

*T21*10.* $[A\ B\ a] : A \varepsilon \mathrm{el}\,(B) . B \varepsilon \mathrm{Kl}\,(a) . \supset . A \varepsilon \mathrm{elKl}\,(B\ a)$ [*BD1*, *T21*1*, $\mathfrak{B}$]

*T21*11.* $[A\ B\ a] : A \varepsilon \mathrm{elKl}\,(B\ a) . \supset . A \varepsilon \mathrm{el}\,(B)$ [$\mathfrak{B}$, *T21*1*, *BD1*]

*T21*12.* $[A\ B\ a] : A \varepsilon \mathrm{elKl}\,(B\ a) . \supset . B \varepsilon \mathrm{Kl}\,(a)$ [$\mathfrak{B}$, *T21*1*, *BD2*]

*T21*13.* (= *AD2*). [*T21*10*, *T21*11*, *T21*12*]

*T21*14.* $[A\ B\ a] : A \varepsilon \mathrm{Kl}\,(a) . B \varepsilon \mathrm{Kl}\,(a) . \supset . A \varepsilon B$ [*BD2*, *T21*1*, *BD1*, *T21*3*, $\mathfrak{B}$]

*T21*15.* $[A\ B\ a] : A \varepsilon \mathrm{el}\,(B) . B \varepsilon a . \supset . A \varepsilon \mathrm{el}\,(\mathrm{Kl}\,(a))$ [*BD1*, *T21*1*, $\mathfrak{B}$, *T21*11*, *T21*12*, *T21*14*]

*T21*16.* $[A\ B\ f\ a] \vdots\vdots A \varepsilon \mathrm{el}\,(B) :\cdot: [C] :: C \varepsilon f\,(a) . \equiv \therefore [D] : D \varepsilon a . \supset . D \varepsilon \mathrm{el}\,(C) \therefore [D] : D \varepsilon \mathrm{el}\,(C) . \supset . [\exists\ E\ F] . E \varepsilon a . F \varepsilon \mathrm{el}\,(D) . F \varepsilon \mathrm{el}\,(E) :\cdot: B \varepsilon a :\cdot: \supset . A \varepsilon \mathrm{el}\,(f\,(a))$ [*T21*7*, *T21*9*, *T21*15*]

*T21*17*. $[A\ B] \vdots\because B \varepsilon B \because\vdots [f\ a] \vdots\vdots [C] :: C \varepsilon f(a) . \equiv .\therefore [D] : D \varepsilon a . \supset . D \varepsilon \mathrm{el}(C) .\therefore [D] : D \varepsilon \mathrm{el}(C) . \supset . [\exists E\ F] . E \varepsilon a . F \varepsilon \mathrm{el}(D) . F \varepsilon \mathrm{el}(E) :\cdot: B \varepsilon \mathrm{el}(B) . B \varepsilon a :\cdot: \supset . A \varepsilon \mathrm{el}(f(a)) \vdots\because \supset . A \varepsilon \mathrm{el}(B)$ [*T21*7, T21*9, T21*1, BD1, BD2, T21*14*]

*T21*18*. (= $\mathfrak{A}$). [*T21*8, T21*16, T21*17*]

From *T21*18*, *T21*7*, and *T21*13* we see that the system $\{\mathfrak{B}$, *BD1*, *BD2*$\}$ implies the system $\{\mathfrak{A}$, *AD1*, *AD2*$\}$, and this completes the proof that the two systems are inferentially equivalent.

Early this year I noticed that Sobociński's original problem was not as difficult to solve as it had appeared at first. A slight modification of $\mathfrak{A}$ gets rid of the quantification over functorial variables, and leaves us with an axiom of the following form.

$\mathfrak{C}$ $[A\ B] \vdots\because A \varepsilon \mathrm{el}(B) . \equiv \vdots\because B \varepsilon B \because\vdots [C\ a] \vdots\vdots [D] :: D \varepsilon C . \equiv .\therefore [E] : E \varepsilon a . \supset . E \varepsilon \mathrm{el}(D) .\therefore [E] : E \varepsilon \mathrm{el}(D) . \supset . [\exists F\ G] . F \varepsilon a . G \varepsilon \mathrm{el}(E) . G \varepsilon \mathrm{el}(F) :\cdot: B \varepsilon \mathrm{el}(B) . B \varepsilon a :\cdot: \supset . A \varepsilon \mathrm{el}(C)$

The proof that $\mathfrak{A}$ and $\mathfrak{C}$ are inferentially equivalent is very easy. In order to show that $\mathfrak{A}$ implies $\mathfrak{C}$ we continue our deductions as follows.

T22. $[A\ B\ C\ a] \vdots\vdots A \varepsilon \mathrm{el}(B) :\cdot: [D] :: D \varepsilon C . \equiv .\therefore [E] : E \varepsilon a . \supset . E \varepsilon \mathrm{el}(D) .\therefore [E] : E \varepsilon \mathrm{el}(D) . \supset . [\exists F\ G] . F \varepsilon a . G \varepsilon \mathrm{el}(E) . G \varepsilon \mathrm{el}(F) :\cdot: B \varepsilon a :\cdot: \supset . A \varepsilon \mathrm{el}(C)$ [*T11, T21*15*]

T23. $[A\ B] \vdots\because B \varepsilon B \because\vdots [C\ a] \vdots\vdots [D] :: D \varepsilon C . \equiv .\therefore [E] : E \varepsilon a . \supset . E \varepsilon \mathrm{el}(D) .\therefore [E] : E \varepsilon \mathrm{el}(D) . \supset . [\exists F\ G] . F \varepsilon a . G \varepsilon \mathrm{el}(E) . G \varepsilon \mathrm{el}(F) :\cdot: B \varepsilon \mathrm{el}(B) . B \varepsilon a :\cdot: \supset . A \varepsilon \mathrm{el}(C) \because\vdots \supset . A \varepsilon \mathrm{el}(B)$ [*T11, T2, T4, T21*14*]

T24. (= $\mathfrak{C}$). [*T1, T22, T23*]

Now, in order to show that $\mathfrak{C}$ implies $\mathfrak{A}$ we subjoin *CD1* (= *AD1*) to $\mathfrak{C}$, and proceed as indicated below.

*T24*1*. (= *T1*). [$\mathfrak{C}$]

*T24*2*. (= *T2*). [$\mathfrak{C}$]

*T24*3*. (= *T3*). [*T24*1, T24*2*]

*T24*4*. (= *T4*). [*T24*2, T24*3, CD1*]

*T24*5*. (= *T11*). [*CD1, T24*1*]

*T24*6*. (= *T21*16*). [$\mathfrak{C}$, *T24*5*]

*T24*7*. (= *T21*17*). [*T24*5, T24*2, T24*1, T24*4*]

*T24*8*. (= $\mathfrak{A}$) [*T24*1, T24*6, T24*7*]

Similarly, proposition

$\mathfrak{D}$ $[A\ B] \because\vdots A \varepsilon \mathrm{el}(B) . \equiv \vdots\vdots B \varepsilon B \vdots\vdots [C\ D\ a] :\cdot: [E] .\therefore E \varepsilon D . \equiv : [F] : [\exists G] . G \varepsilon \mathrm{el}(E) . G \varepsilon \mathrm{el}(F) . \equiv . [\exists H\ I] . H \varepsilon a . I \varepsilon \mathrm{el}(F) . I \varepsilon \mathrm{el}(H) :: B \varepsilon \mathrm{el}(B) . B \varepsilon \mathrm{el}(C) . C \varepsilon a :: \supset . A \varepsilon \mathrm{el}(D)$

can be shown to be inferentially equivalent to $\mathfrak{A}$. But I have not yet been able to simplify the other single axioms of mereology in the same way, that is to say, without making them longer.

NOTES

1. For a general and historical introduction to mereology see B. Sobociński, 'Studies in Leśniewski's Mereology', *Polish Society of Arts and Sciences Abroad, Yearbook for 1954-55*, London 1955, pp. 34-43. See also B. Sobociński, 'L'analyse de l'antinomie Russellienne par Leśniewski', *Methodos*, Vol. I (1949) pp. 94-107, 220-228, 308-316, and Vol. II (1950) pp. 237-257. An unpublished result of A. Grzegorczyk helped Sobociński to construct his axiom $\mathfrak{A}$.
2. See C. Lejewski, 'A Contribution to Leśniewski's Mereology', *Polish Society of Arts and Sciences Abroad, Yearbook for 1954-55*, London 1955, pp. 43-50, C. Lejewski, 'A New Axiom of Mereology', *ibid., Yearbook for 1955-56*, London 1956, pp. 65-70, and B. Sobociński, 'On Well Constructed Axiom Systems', *ibid.*, pp. 54-65.
3. By 'ontological unit' one understands an expression of the form '$\alpha \varepsilon \beta$', where 'α' and 'β' stand for any nominal expression.

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