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TRUTH VALUE ASSIGNMENT IN PREDICATE CALCULUS OF FIRST ORDER

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In this paper we shall try to present a theory of testing of validity and consistency in predicate calculus of first order.

This method of testing used here is advantageous for practical and instructive purposes, because of its very simplicity and very easiness.

First it is illustrated by the following examples.

Example 1. For testing the validity of

$$\left[(x)(Ax \supset Bx) \cdot (\exists x)(Cx \cdot Ax)\right] \supset (\exists x)(Cx \cdot Bx)$$

let us assume that this formula is false under some interpretation of predicate 'A', 'B', 'C'. Then '$(x)(Ax \supset Bx)$' is true, '$(\exists x)(Cx \cdot Ax)$' is true, and '$(\exists x)(Cx \cdot Bx)$' is false. Therefore '$Ax \supset Bx$' is true, '$Ca \cdot Aa$' is true, and '$Cx \cdot Bx$' is false. Here 'a' is an individual constant about which we do not know anything except that '$Ca \cdot Aa$' is true. Then we are led to: 'Ca' is true, 'Aa' is true and '$Aa \supset Ba$' is true. Therefore 'Ba' is true. On the other hand, since '$Ca \cdot Ba$' is false, 'Ca' is false. Then 'Ca' is true and false at the same time. This self-contradiction proves that the formula tested is valid, because no such an interpretation exists that the formula tested is false. This can be shown by the following schema.

$$\left[(x)(Ax \supset Bx) \cdot (\exists x)(Cx \cdot Ax)\right] \underset{\mathsf{F}}{\supset} (\exists x)(Cx \cdot Bx)$$

$$\begin{array}{ccc}
(x)(Ax \supset Bx) \underset{\mathsf{T}}{\cdot} (\exists x)(Cx \cdot Ax) & & (\exists x)\underset{\mathsf{F}}{(Cx \cdot Bx)} \\
(x)\underset{\mathsf{T}}{(Ax \supset Bx)} & (\exists x)\underset{\mathsf{T}}{(Cx \cdot Ax)} & Cx \underset{\mathsf{F}}{\cdot} Bx \\
Ax \underset{\mathsf{T}}{\supset} Bx & Ca \underset{\mathsf{T}}{\cdot} Aa & Ca \underset{\mathsf{F}}{\cdot} Ba \\
Aa \underset{\mathsf{T}}{\supset} Ba & \underset{\mathsf{T}}{Ca} \quad \underset{\mathsf{T}}{Aa} & \underset{\mathsf{F}}{Ca} \quad \underset{\mathsf{T}}{Ba} \\
\underset{\mathsf{T}}{Aa} \quad \underset{\mathsf{T}}{Ba} & \text{self-contradiction} &
\end{array}$$

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Example 2. Test of validity

$$\underset{\mathbf{F}}{\left[(\exists x)(Ax \cdot Bx) \cdot (\exists x)(Ax \cdot Cx)\right] \supset (\exists x)(Bx \cdot Cx)}$$

$$\begin{array}{ccc}
\multicolumn{2}{c}{\underset{\mathbf{T}}{(\exists x)(Ax \cdot Bx) \cdot (\exists x)(Ax \cdot Cx)}} & \underset{\mathbf{F}}{(\exists x)(Bx \cdot Cx)} \\[2ex]
\underset{\mathbf{T}}{(\exists x)(Ax \cdot Bx)} & \underset{\mathbf{T}}{(\exists x)(Ax \cdot Cx)} & \underset{\mathbf{F}}{Bx \cdot Cx} \quad \text{substitution} \\[2ex]
\underset{\mathbf{T}}{Aa \cdot Ba} & \underset{\mathbf{T}}{Ab \cdot Cb} & \underset{\mathbf{F}}{Ba \cdot Ca} \\[2ex]
\underset{\mathbf{T}}{Aa} \quad \underset{\mathbf{T}}{Ba} & \underset{\mathbf{T}}{Ab} \quad \underset{\mathbf{T}}{Cb} & \underset{\mathbf{T}}{Ba} \quad \underset{\mathbf{F}}{Ca} \\[2ex]
 & & \underset{\mathbf{F}}{Bb \cdot Cb} \\[2ex]
 & & \underset{\mathbf{F}}{Bb} \quad \underset{\mathbf{T}}{Cb}
\end{array}$$

Here the following interpretation is possible: Aa, Ba, Ab, Cb are true, Bb, Ca are false, and $Bx \cdot Cx$ is false. For instance when x is not a, b truth values of Bx, Cx are respectively the same as truth value of Ba, Ca. Under this interpretation by starting from truth value assignments of atomic formula and by proceeding reversely truth value assignments, we can reach the assignment of the formula tested. Therefore the formula tested is invalid since there exists such an interpretation that the formula tested is false.

Next we shall here give the general formulation for the above method.

Truth value assignments of this method is as follows.

(I) If the formula whose truth value has been assigned has a connective in the outermost side, then for the component formulas which are connected by the connective their truth values are assigned by the following way.

The assignment for the formula must be deduced from assignments for the components. But such assignments for the components are not always unique.

Disjunction of all of such assignments for the components must be deduced from the assignments for the formula. In case where assignments for the components satisfying the above conditions are impossible, we cannot use this way of assignments.

For instance while we can get $\underset{\mathbf{T}}{(x)Ax} \quad \underset{\mathbf{T}}{(\exists y)By}$; $\underset{\mathbf{T}}{(x)Ax} \quad \underset{\mathbf{F}}{(\exists y)By}$; $\underset{\mathbf{F}}{(x)Ax} \quad \underset{\mathbf{T}}{(\exists y)By}$ from $(x)Ax \underset{\mathbf{T}}{\vee} (\exists y)By$ as all possible cases, we cannot, use this way in the case of $Ax \underset{\mathbf{T}}{\vee} Bx$; We shall hereafter call a sequence of truth value assignments which adopt only one from all possible cases 'a succession of the test.'

(II) If a formula whose assigned truth value is 'T' ('F') has a universal (existential) quantifier in the outermost side, for the formula which is got by eliminating the quantifier, 'T' ('F') is assigned. If a formula whose assigned truth value is 'F' ('T') has a universal (existential) quantifier in the outermost side, for the formula which is obtained by eliminating the quantifier and by substituting an individual constant having no prior occurrence in the succession, into the individual variable which has been bounded by the quantifier, 'F' ('T') is assigned, except the cases where the predicate which has the individual variable bounded by the quantifier has other individual free variables. For instance we can not get $\underset{\mathsf{T}}{Aay}$ from $\underset{\mathsf{T}}{(\exists x)Axy}$.

(III) This is used only if the use of (I), (II) is impossible. For a non-elementary formula[1] whose truth value has been assigned, if neither (I) nor (II) can be used, then the formula has at least one individual free variable. For one of those individual variables, the following procedure is done. When a predicate having the individual free variable within the formula, has in the same argument place an individual constant in the very succession, for the formula which is got by substituting the individual constant into the individual free variable in the formula, the same truth value is assigned. If such individual constants have not yet occurred in the argument place in the succession, any individual constant is substituted. If such an occurrence of an individual constant appear later, then similar procedure must be done.

For all formula except elementary formulas we can always apply one of the three. But truth value assignments do not always finish at truth value assignment for finite elementary formulas since occurrences of infinite elementary formulas are possible as examples later-stated. The successions which finish at finite steps are called *'finite succession.'* The successions which continue infinitely are called *'infinite succession.'*

We now state the following theorem about testing of validity and consistency in predicate calculus of first order.

Theorem. The tested formula whose assigned truth value is 'F' ('T') *is valid (inconsistent) if self-contradiction appears in its all successions. The tested formula whose assigned truth value is* 'F' ("T') *is invalid (consistent) if self-contradiction does not appears in truth value assignments for all elementary formulas in its at least one succession.*

Proof. The former part of this theorem is evident, since proceeding to next step by (I), (II), (III) is valid. As the proof of the latter part, we show the following; Under a suitable interpretation of all predicate contained in the formula tested, the formula tested has the truth value assigned, if the above stated condition is satisfied.

Let us interpret all predicates contained in the formula tested according to one succession as follows. Truth value assignments for all elementary formula is adopted as part of our interpretation. Next following interpretation is added. For a predicate 'A' having an argument place '*', when 'x' is not 'i', 'j', ... , '$A \ldots x \ldots$' has the same truth value as

'$A \ldots i \ldots$'. Here 'i', 'j', ... are all individual constants which occur in '*' of 'A' in the succession; 'i' is initial among 'i', 'j', ... in alphabetical order.

Let us assume that for

$$(A \ldots \underset{*}{x} \ldots, \ B \ldots \underset{\mathrm{o}}{x} \ldots, \ \ldots\ldots) \text{————————} x$$

Which contains predicates 'A', 'B' etc., (III) is used for 'x'.[2] Then there are truth value assignments for the following formulas which have the same truth value as x, in later steps.

$$(A \ldots \underset{*}{i} \ldots, \ B \ldots \underset{\mathrm{o}}{i} \ldots, \ \ldots\ldots) \text{————————} i$$

$$(A \ldots \underset{*}{j} \ldots, \ B \ldots \underset{\mathrm{o}}{j} \ldots, \ \ldots\ldots) \text{————————} j$$

$$\vdots$$

Here 'i', 'j', ... are individual constants which occur in any of the following———— '*' of 'A', 'o' of 'B' etc.————in the very succession. It will be understood that all individual constants which occur in '*' of 'A' in the succession are identical with all individual constants which occur in 'o' of 'B' in the succession etc. Then we can show that truth value assignments for x is deduced validily from truth value assignments for i, j, ... Let us assume that 'i' is initial among 'i', 'j', ... in alphabetical order. If 'x' in x is 'i' or 'j' or ....., the truth value of x is the same as that of i, because i, j ..... have the same truth value. If 'x' in x is deferent from any one of 'i', 'j', ..., by the interpretation above-stated the truth value of x is the same as that of i. Thus all substituation instances of x have the same truth value as i. Therefore x has same truth value as i. Thus we have right to proceed reversely truth value assignments in cases where (III) is applied. Since it is evident that we have right to proceed reversely in cases where (I) or (II) are applied, we have reached the above-stated end.

Most formulas which usually appear in text books of logic as theorems and exercises are very simply and very easily tested by our method of testing.

Next peculiar examples are shown.

Example 3. Test of validity

$$\underset{\mathbf{F}}{(x)(\exists y)Axy \supset (\exists x)(y)Axy}$$

$$\underset{\mathbf{T}}{(x)(\exists y)Axy} \qquad\qquad \underset{\mathbf{F}}{(\exists x)(y)Axy}$$

$$\underset{\mathbf{T}}{(\exists y)Axy} \ldots\ldots\ldots(1) \qquad \underset{\mathbf{F}}{(y)Axy} \ldots\ldots\ldots\ldots(2)$$

Since we cannot use (II) for (1), (2), we use (III)

$$(\exists y)Aay \qquad\qquad (y)Aay$$
$$\mathbf{T} \qquad\qquad\qquad \mathbf{F}$$

$$Aab \qquad\qquad Aac$$
$$\mathbf{T} \qquad\qquad \mathbf{F}$$

Therefore the formula tested is invalid.

Example 4. Test of validity

$$(x)(\exists y)Axy \supset (\exists y)(x)Axy$$
$$\mathbf{F}$$

$$(x)(\exists y)Axy \qquad\qquad (\exists y)(x)Axy$$
$$\mathbf{T} \qquad\qquad\qquad \mathbf{F}$$

$$(\exists y)Axy \qquad\qquad (x)Axy$$
$$\mathbf{T} \qquad\qquad \mathbf{F}$$

$$(\exists y)Aay \qquad\qquad (x)Axb$$
$$\mathbf{T} \qquad\qquad \mathbf{F}$$

$$Aac \qquad\qquad Adb$$
$$\mathbf{T} \qquad\qquad \mathbf{F}$$

$$(\exists y)Ady \qquad\qquad (x)Axc$$
$$\mathbf{T} \qquad\qquad \mathbf{F}$$

$$Ade \qquad\qquad Afc$$
$$\mathbf{T} \qquad\qquad \mathbf{F}$$

$$(\exists y)Afy \qquad\qquad (x)Axe$$
$$\mathbf{T} \qquad\qquad \mathbf{F}$$
$$\vdots \qquad\qquad \vdots$$

Where '↓' shows that (III) is used there. Similar processes repeat infinitely. Since elementary formulas occurring are different from each other, self-contradiction does not appear among truth value assignments for them. Therefore the formula tested is invalid.

Example 5. Test of consistency

$$(y)\left[(\exists x)(Ax \cdot By) \vee (\exists x)(Ay \cdot Bx)\right]$$
$$\mathbf{T}$$

$$(\exists x)(Ax \cdot By) \vee (\exists x)(Ay \cdot Bx) \cdots\cdots\cdots\cdots (1)$$
$$\mathbf{T}$$

$$(\exists x)(Ax \cdot Ba) \vee (\exists x)(Aa \cdot Bx)$$
$$\mathbf{T}$$

We adopt the following assignment as one of possible cases.

$$\begin{array}{cc} (\exists x)(Ax \cdot Ba) & (\exists x)(Aa \cdot Bx) \\ \mathbf{T} & \mathbf{T} \\ Ab \cdot Ba & Aa \cdot Bc \\ \mathbf{T} & \mathbf{T} \\ \begin{array}{cc} Ab & Ba \\ \mathbf{T} & \mathbf{T} \end{array} & \begin{array}{cc} Aa & Bc \\ \mathbf{T} & \mathbf{T} \end{array} \end{array}$$

b, c are substituted into (1).

$$\begin{array}{cc} (\exists x)(Ax \cdot Bb) \vee (\exists x)(Ab \cdot Bx) & (\exists x)(Ax \cdot Bc) \vee (\exists x)(Ac \cdot Bx) \\ \mathbf{T} & \mathbf{T} \end{array}$$

We adopt the following assignments as one of possible cases.

$$\begin{array}{cccc} (\exists x)(Ax \cdot Bb) & (\exists x)(Ab \cdot Bx) & (\exists x)(Ax \cdot Bc) & (\exists x)(Ac \cdot Bx) \\ \mathbf{T} & \mathbf{T} & \mathbf{T} & \mathbf{T} \\ Ad \cdot Bd & Ab \cdot Be & Af \cdot Bc & Ac \cdot Bc \\ \mathbf{T} & \mathbf{T} & \mathbf{T} & \mathbf{T} \\ \begin{array}{cc} Ad & Bb \\ \mathbf{T} & \mathbf{T} \end{array} & \begin{array}{cc} Ab & Be \\ \mathbf{T} & \mathbf{T} \end{array} & \begin{array}{cc} Af & Dc \\ \mathbf{T} & \mathbf{T} \end{array} & \begin{array}{cc} Ac & Bg \\ \mathbf{T} & \mathbf{T} \end{array} \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

If we adopt

$$\begin{array}{cc} (\exists x)(Ax \cdot Bi) & (\exists x)(Ai \cdot Bx) \\ \mathbf{T} & \mathbf{T} \end{array}$$

as the assignments of the next step of

$$\begin{array}{c} (\exists x)(Ax \cdot Bi) \vee (\exists x)(Ai \cdot Bx), \\ \mathbf{T} \end{array}$$

then we get an infinite succession. Here 'i' is any individual constant. Among assignments for all elementary formulas occurring in the succession no self-contradiction appears, because truth values assigned are always '**T**'. Therefore the formula tested is consistent.

Example 6. Test of consistency

$$\begin{array}{c} (y)\left[(\exists x)(Ax \cdot -Ay) \vee (\exists x)(Ay \cdot -Ax)\right] \\ \mathbf{T} \\ (\exists x)(Ax \cdot -Ay) \vee (\exists x)(Ay \cdot -Ax) \\ \mathbf{T} \\ (\exists x)(Ax \cdot -Aa) \vee (\exists x)(Aa \cdot -Ax) \\ \mathbf{T} \end{array}$$

We adopt the following assignments as one of possible cases.

$$\begin{array}{ccc}
\multicolumn{3}{c}{(\exists x)(Ax \cdot -Aa) \ldots\ldots\ldots\ldots (1)} \\
\multicolumn{3}{c}{\mathsf{T}} \\
& Ab \cdot -Aa & \\
& \mathsf{T} & \\
Ab & & -Aa \\
\mathsf{T} & & \mathsf{T} \\
& & Aa \\
& & \mathsf{F} \longleftarrow \text{self-contradiction}
\end{array}
\qquad
\begin{array}{ccc}
\multicolumn{3}{c}{(\exists x)(Aa \cdot -Ax) \ldots\ldots\ldots\ldots (2)} \\
\multicolumn{3}{c}{\mathsf{T}} \\
& Aa \cdot -Ac & \\
& \mathsf{T} & \\
\rightarrow Aa & & -Ac \\
\mathsf{T} & & \mathsf{T}
\end{array}$$

If we adopt assignments in which one of (1), (2) is true and the other is false, we are led to self-contradiction. Therefore the formula tested is inconsistent. Note that these successions are infinite.

Lastly we show a method of proving invalidity and consistency of formulas, as a variant of the above method of testing. This is illustrated by a few examples, as this seems easily to be understood without general explanation, though (II), (III) are slightly varied.

Example 7. Proof of consistency of the formula in Example 5.

Let us assume that the individual domain contains only one individual which is designated by 'a'.

$$\begin{array}{c}
(y)\left[(\exists x)(Ax \cdot By) \vee (\exists x)(Ay \cdot Bx)\right] \\
\mathsf{T} \\
(\exists x)(Ax \cdot By) \vee (\exists y)(Ay \cdot Bx) \\
\mathsf{T} \\
(\exists x)(Ax \cdot Ba) \vee (\exists y)(Aa \cdot Bx) \\
\mathsf{T}
\end{array}$$

We adopt the following assignments as one of possible cases.

$$\text{by means of (II)} \qquad
\begin{array}{ccc}
\multicolumn{3}{c}{(\exists x)(Ax \cdot Ba)} \\
\multicolumn{3}{c}{\mathsf{T}} \\
& Aa \cdot Ba & \\
& \mathsf{T} & \\
Aa & & Ba \\
\mathsf{T} & & \mathsf{T}
\end{array}
\qquad
\begin{array}{ccc}
\multicolumn{3}{c}{(\exists y)(Aa \cdot Bx)} \\
\multicolumn{3}{c}{\mathsf{T}} \\
& Aa \cdot Ba & \\
& \mathsf{T} & \\
Aa & & Ba \\
\mathsf{T} & & \mathsf{T}
\end{array}$$

Thus the given formula is true under the above interpretation in the above individual domain. Therefore the formula is consistent.

Example 8. Proof of invalidity of the formula in Example 4.

Let us assume that the individual domain contains only one individual which is designated by 'a'.

$$\begin{array}{c}
(x)(\exists y)Axy \supset (\exists y)(x)Axy \\
\mathsf{F}
\end{array}$$

$$\begin{array}{cc}
(x)(\exists y)Axy & (\exists y)(x)Axy \\
\mathbf{T} & \mathbf{F} \\[1ex]
(\exists y)Axy & (x)Axy \\
\mathbf{T} & \mathbf{F} \\[1ex]
(\exists y)Aay & (x)Axa \\
\mathbf{T} & \mathbf{F}
\end{array}$$

by means of (II)

$$\begin{array}{ccc}
Aaa & & Aaa \\
\mathbf{T} & \text{self-contradiction} & \mathbf{F}
\end{array}$$

Next let us assume that the individual domain contains only two individuals which are respectively designated by 'a', 'b'.

$$\begin{array}{cc}
(\exists y)Axy & (x)Axy \\
\mathbf{T} & \mathbf{F}
\end{array}$$

by means of (III)

$$\begin{array}{cccc}
(\exists y)Aay & (\exists y)Aby & (x)Axa & (x)Axb \\
\mathbf{T} & \mathbf{T} & \mathbf{F} & \mathbf{F}
\end{array}$$

by means of (II)

$$\boxed{\begin{matrix}Aaa\\ \mathbf{T}\end{matrix}} \cdots \begin{matrix}Aab\\ \mathbf{T}\end{matrix} \quad \begin{matrix}Aba\\ \mathbf{T}\end{matrix} \cdots \boxed{\begin{matrix}Abb\\ \mathbf{T}\end{matrix}} \quad \begin{matrix}Aaa\\ \mathbf{F}\end{matrix} \cdots \boxed{\begin{matrix}Aba\\ \mathbf{F}\end{matrix}} \quad \boxed{\begin{matrix}Aab\\ \mathbf{F}\end{matrix}} \cdots \begin{matrix}Abb\\ \mathbf{F}\end{matrix}$$

'. . . .' shows that at least one of two truth value assignments for formulas connected by it holds. Here, for instance, if we adopt truth value assignments enclosed by $\square$ as an interpretation of 'A', then that the formula is false in the domain is shown. Therefore the formula is invalid.

NOTES

1) "An elementary formula" means a formula which contains neither any other formulas nor individual variables as its parts.

2) In the case of more than one argument places of 'A', 'B' etc. which x occurs, we can proceed in similar manner.

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