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ON GÖDEL'S PROOF THAT V=L IMPLIES THE GENERALIZED CONTINUUM HYPOTHESIS

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Chapter VIII, p. 53-61 of Gödel's book *"The consistency of the axiom of choice and of the generalized continuum hypothesis with the axioms of set theory,"* Princeton, Third printing, 1953, is devoted to the derivation of the generalized continuum hypothesis from V=L and the axioms of Σ. It is perhaps the most strenuous part of the book. While retaining most of the ideas and steps of Gödel's proof we present here a shorter version of this proof involving some simplifications.

We use the results, notations and numberings of Gödel's book up to 11.7 page 52.

11.8 Dfn $\langle yx\rangle \in \mathrm{As}. \equiv: y \in x \cdot (z)[\mathrm{Od}'z < \mathrm{Od}'y \cdot \supset \cdot \sim z \in x] \cdot \mathfrak{Rel}(\mathrm{As})$.

As'x is what may be called the "designated" element of x.

11.81 Dfn $\quad \mathrm{C}'\alpha = \mathrm{Od}'[\mathrm{As}'(\mathrm{F}'\alpha)] \cdot \mathrm{C}\, \mathfrak{Fn}\, \mathrm{On}.$

11.82 Dfn $\quad \mathrm{C}_1'\alpha = \mathrm{Od}'[\mathrm{As}'(\mathrm{F}'\alpha - \{\mathrm{F}'\mathrm{C}'\alpha\})] \cdot \mathrm{C}_1\, \mathfrak{F}\dot{\mathfrak{n}}\, \mathrm{On}.$

C'α is the order of the designated element of F'α. $\mathrm{C}_1'\alpha$ is the order of the "next designated" element of F'α.

12.1 Dfn *If* $m \subset \mathrm{On}$ *and* m *is closed with respect to* $\mathrm{C}, \mathrm{C}_1, \mathrm{K}_1, \mathrm{K}_2$ *and with respect to* $\mathrm{J}_0, \ldots, \mathrm{J}_8$ *as triadic relations, define recursively a function* H *on* On *as follows*:

$$\eta \in \mathfrak{W}(\mathrm{J}_0) \cdot \supset \cdot \mathrm{H}'\eta = \mathrm{H}''(m\eta)$$

$$\eta = \mathrm{J}_i'\langle\beta\gamma\rangle \cdot \supset \cdot \mathrm{H}'\eta = \mathfrak{F}_i(\mathrm{H}'\beta, \mathrm{H}'\gamma) \qquad \text{for } i = 1, \ldots, 8.$$

12.11. *If* $\eta \in m$, *then every element* x *of* H'η *is of the form* H'α *with* $\alpha \in m\eta$.

Proof. If $\eta \in \mathfrak{W}(\mathrm{J}_0)$ then $x \in \mathrm{H}''(m\eta)$ and the statement is evident. If $\eta = \mathrm{J}_1'\langle\beta\gamma\rangle$ then, by the closure properties of m w.r. to $\mathrm{K}_1, \mathrm{K}_2$ and by 9.25 we have $\beta,\gamma \in m\eta$. Then $x = \mathrm{H}'\beta$ or $\mathrm{H}'\gamma$. If $\eta = \mathrm{J}_i'\langle\beta\gamma\rangle$, $i = 2, \ldots 8$, then again $\beta \in m\eta$. We have $x \in \mathrm{H}'\beta$ and the proof follows by induction.

12.12. *If* m *satisfies the conditions of Dfn. 12.1 then* $\alpha \in m \cdot \supset \cdot \mathrm{Od}'\mathrm{F}'\alpha \in m$.

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For, by the closure property of m w.r. to J_1 we have $\{F'\alpha\} = F'\alpha_1$ with $\alpha_1 = J_1{}'\langle\alpha\alpha\rangle \epsilon\, m$. Put $\alpha' = Od'F'\alpha$. Then $\alpha' = Od'[As'\{F'\alpha\}] = C'\alpha_1 \epsilon\, m$, by the closure property of m w.r. to C.

12.2 1) $F'\alpha \epsilon F'\eta \cdot\equiv\cdot H'\alpha \epsilon H'\eta$ for $\eta \epsilon m, \alpha \epsilon m\eta$.

2) $F'\alpha = F'\eta \cdot\equiv\cdot H'\alpha = H'\eta$ for $\eta \epsilon m, \alpha \epsilon m\eta$.

The proof is by induction on η i.e. we prove 12.2 under the hypotheses

I) $F'\alpha \,\epsilon\, F'\beta \,.\equiv.\, H'\alpha \,\epsilon\, H'\beta$ for $\beta \epsilon m\eta$, $\alpha \epsilon m\beta$.

II) $F'\alpha = F'\beta \cdot\equiv\cdot H'\alpha = H'\beta$ for $\beta \epsilon m\eta$, $\alpha \epsilon m\beta$.

Observe that I) and II) imply

I') $F'\alpha \epsilon F'\beta \cdot\equiv\cdot H'\alpha \epsilon H'\beta$ for $\alpha, \beta \epsilon m\eta$

II') $F'\alpha = F'\beta \cdot\equiv\cdot H'\alpha = H'\beta$ for $\alpha, \beta \epsilon m\eta$.

II') is the same as II) by the symmetry of equality. To prove I') suppose $F'\alpha \epsilon F'\beta$, then writing $\alpha' = Od'F'\alpha$ we have $F'\alpha' = F'\alpha \epsilon F'\beta$. By 12.12 and 9.52, $\alpha' \epsilon m\beta$. Then, by II'), $H'\alpha' = H'\alpha$, and by I), $H'\alpha' \epsilon H'\beta$. Hence $H'\alpha \epsilon H'\beta$. Next suppose $H'\alpha \epsilon H'\beta$, then, by 12.11 there is an $\alpha' \epsilon m\beta$ such that $H'\alpha' = H'\alpha$. We conclude as before $F'\alpha \epsilon F'\beta$.

12.21 *Let* $\eta = J_i{}'\langle\beta\gamma\rangle$, $i = 1, \ldots, 8$, $\eta \epsilon m$. *Under* I), II), *if* $F'\alpha = F'\alpha'$ *with* $\alpha' \epsilon m\beta$ [or $\alpha' \epsilon m\gamma$] *and if* $F'\alpha$ *is an ordered pair or an ordered triple, then, respectively,* $F'\alpha = \langle F'\lambda F'\mu\rangle$ *and* $H'\alpha = \langle H'\lambda H'\mu\rangle$, *or* $F'\alpha = \langle F'\lambda F'\mu F'\nu\rangle$ *and* $H'\alpha = \langle H'\lambda H'\mu H'\nu\rangle$ *with* $\lambda, \mu, \nu \epsilon m\alpha'$.*-- There is a similar statement if in the hypothesis* F *is replaced by* H.

Proof. By II') we have $H'\alpha = H'\alpha'$. Suppose $F'\alpha' = \{\{F'\lambda_1\}\{F'\lambda_1 F'\mu_1\}\}$. Put $\alpha_1 = Od'\{F'\lambda_1\}$, $\alpha_2 = Od'\{F'\lambda_1 F'\mu_1\}$, $\lambda = Od'F'\lambda_1$, $\mu = Od'F'\mu_1$. By the closure properties of m w.r. to C, C_1, we get successively $\alpha_1, \alpha_2 \epsilon m, \lambda, \mu \epsilon m$. By 9.52 we have $\lambda < \alpha_1 < \alpha' < \beta$ [or $<\gamma$]; $\lambda, \mu < \alpha_2 < \alpha' < \beta$ [or $<\gamma$]. But $F'\alpha' = \{F'\alpha_1 F'\alpha_2\} = F'J_1{}'\langle\alpha_1\alpha_2\rangle$. Since $J_1{}'\langle\alpha_1\alpha_2\rangle < J_i{}'\langle\beta\gamma\rangle = \eta$ we have, by II'), $H'\alpha' = H'J_1{}'\langle\alpha_1\alpha_2\rangle = \{H'\alpha_1 H'\alpha_2\}$. Also $F'\alpha_2 = F'J_1{}'\langle\lambda\mu\rangle$. Since $J_1{}'\langle\lambda\mu\rangle < J_i{}'\langle\beta\gamma\rangle = \eta$, we have again, by II') $H'\alpha_2 = H'J_1{}'\langle\lambda\mu\rangle = \{H'\lambda H'\mu\}$ and similarly $H'\alpha_1 = \{H'\lambda\}$. This gives $F'\alpha' = \langle F'\lambda F'\mu\rangle$ and $H'\alpha' = \langle H'\lambda H'\mu\rangle$. Suppose next that $F'\alpha'$ is an ordered triple. Then $F'\mu$ is an ordered pair; since $\mu \epsilon m\beta$, [or $\mu \epsilon m\gamma$], we may write $F'\mu = \langle F'\mu' F'\nu'\rangle$, $H'\mu = \langle H'\mu' H'\nu'\rangle$, where $\mu', \nu' \epsilon m\mu \subset m\alpha'$, and the required forms for $F'\alpha'$, $H'\alpha'$ follow. If the starting hypotheses concern H instead of F the treatment is similar but simpler we use 12.11 instead of Od' and 9.52.

We now prove 12.2 1) under I'), II'). We have different cases.

1.- $\eta \epsilon \mathfrak{W}(J_0)$. We must show that $H'\alpha \epsilon H''(m\eta) \equiv\cdot F'\alpha \epsilon F''\eta$. The equivalence holds for the two terms are true.

2.- $\eta \epsilon \mathfrak{W}(J_1)$. Then $\eta = J_1{}'\langle\beta\gamma\rangle$ where $\beta, \gamma \epsilon m$, by the closure properties of m and $\beta, \gamma < \eta$, by 9.25. Also $F'\eta = \{F'\beta F'\gamma\}$ and $H'\eta = \{H'\beta H'\gamma\}$. Now $H'\alpha \epsilon H'\eta \cdot\equiv\cdot H'\alpha = H'\beta \vee H'\alpha = H'\gamma$. By II') the r.h.s. is equivalent to $F'\alpha = F'\beta \vee F'\alpha = F'\gamma$, and therefore equivalent to $F'\alpha \epsilon F'\eta$.

3.- If $\eta \epsilon \mathfrak{M}(J_2)$ then we have as before $\eta = J_2{}'\langle\beta\gamma\rangle, \beta, \gamma \epsilon m\eta$. By 9.32, $F'\eta = F'\beta \cdot E$, $H'\eta = H'\beta \cdot E$. Now $F'\alpha \epsilon F'\beta \cdot \equiv \cdot H'\alpha \epsilon H'\beta$, by I'). Also, by 12.21, if $F'\alpha$ is an ordered pair then $F'\alpha = \langle F'\lambda\, F'\mu\rangle$, $H'\alpha = \langle H'\lambda\, H'\mu\rangle$, with $\lambda, \mu \epsilon m\alpha$. If $F'\alpha \epsilon E$ then $F'\lambda \epsilon F'\mu$, so that, by I'), $H'\lambda \epsilon H'\mu$, hence $H'\alpha \epsilon E$, and the same if $H'\alpha$ is an ordered pair. Hence $H'\alpha \epsilon H'\eta \cdot \equiv \cdot F'\alpha \epsilon F'\eta$.

4.- If $\eta \epsilon \mathfrak{M}(J_3)$ we get in the same fashion, by 9.33 $F'\eta = F'\beta - F'\gamma$, $H'\eta = H'\beta - H'\gamma$, with $\beta, \gamma \epsilon m\eta$. Assume $F'\alpha \epsilon F'\eta$ and I') applied to $F'\alpha$ with $F'\beta$ and $F'\gamma$ gives $H'\alpha \epsilon H'\eta$. Assume $H'\alpha \epsilon H'\eta$, we get in the same way $F'\alpha \epsilon F'\eta$.

5.- Suppose $\eta \epsilon \mathfrak{M}(J_4)$. As above $\eta = J_4{}'\langle\beta\gamma\rangle$, with $\beta, \gamma \epsilon m\eta$, and $F'\eta = F'\beta \cdot V \times F'\gamma$, $H'\eta = H'\beta \cdot V \times H'\gamma$. Assume first that $F'\alpha \epsilon F'\eta$, that is $F'\alpha \epsilon F'\beta \cdot V \times F'\gamma$. By I') $H'\alpha \epsilon H'\beta$. Now $F'\alpha$ being an ordered pair we have, by 12.21, $F'\alpha = \langle F'\lambda\ F'\mu\rangle$, $H'\alpha = \langle H'\lambda\ H'\mu\rangle$, with $\lambda, \mu \epsilon m\alpha$. Since $F'\mu \epsilon F'\gamma$, we get by I'), $H'\mu \epsilon H'\gamma$. Hence $H'\alpha \epsilon V \times H'\gamma$, and finally $H'\alpha \epsilon H'\eta$. If $H'\alpha \epsilon H'\eta$ we get in the same way $F'\alpha \epsilon F'\eta$.

6.- Suppose $\eta \epsilon \mathfrak{M}(J_5)$. As above $\eta = J_5{}'\langle\beta\gamma\rangle$, with $\beta, \gamma \epsilon m\eta$, and $F'\eta = F'\beta \cdot \mathfrak{D}(F'\gamma)$, $H'\eta = H'\beta \cdot \mathfrak{D}(H'\gamma)$. Assume $F'\alpha \epsilon F'\eta$. Then $F'\alpha \epsilon F'\beta$, so that, by I'), $H'\alpha \epsilon H'\beta$. Put $\alpha_1 = J_1{}'\langle\alpha\,\alpha\rangle$. Then $\alpha_1 \epsilon m$, $F'\alpha_1 = \{F'\alpha\}$, $H'(\alpha_1) = \{H'\alpha\}$. Consider the set $F'\gamma \cdot V \times \{F'\alpha\} = F'\alpha_2$, with $\alpha_2 = J_4{}'\langle\gamma\,\alpha_1\rangle \epsilon m$. $F'\alpha_2$ is not empty, since $F'\alpha \epsilon \mathfrak{D}(F'\gamma)$. Hence, putting $\alpha' = C'\alpha_2$ we have $F'\alpha' \epsilon F'\gamma$ and $\alpha' \epsilon m\gamma$. By 12.21 we may write $F'\alpha' = \langle F'\lambda' F'\mu'\rangle$, $H'\alpha' = \langle H'\lambda' H'\mu'\rangle$, with $\mu' \epsilon m\alpha'$. Also since $F'\alpha' \epsilon F'\alpha_2$ we have $F'\mu' = F'\alpha$, so that, by II') $H'\mu' = H'\alpha$. By I'), $H'\alpha' \epsilon H'\gamma$. Hence $H'\alpha' \epsilon H'\gamma \cdot V \times \{H'\alpha\}$. This means $H'\alpha \epsilon \mathfrak{D}(H'\gamma)$. Finally $H'\alpha \epsilon H'\eta$. If $H'\alpha \epsilon H'\eta$ we get in the same way $F'\alpha \epsilon F'\eta$.

7.- $\eta \epsilon \mathfrak{M}(J_i)$, $i = 6,7,8$. Consider e.g. $i = 7$. As above $\eta = J_7{}'\langle\beta\gamma\rangle$, with $\beta, \gamma \epsilon m\eta$ and $F'\eta = F'\beta \cdot \mathfrak{Cnv}_2(F'\gamma)$, $H'\eta = H'\beta \cdot \mathfrak{Cnv}_2(H'\gamma)$. Assume $F'\alpha \epsilon F'\eta$, that is $F'\alpha \epsilon F'\beta$, $F'\alpha \epsilon \mathfrak{Cnv}_2(F'\gamma)$. By I'): $H'\alpha \epsilon H'\beta$. Now $F'\alpha$ being an ordered triple we have by 12.21: $F'\alpha = \langle F'\lambda\, F'\mu\, F'\nu\rangle$, $H'\alpha = \langle H'\lambda\, H'\mu\, H'\nu\rangle$ with $\lambda, \mu, \nu \epsilon m\alpha$ and $\langle F'\mu\, F'\nu\, F'\lambda\rangle \epsilon F'\gamma$ by 4.41. Put $\alpha' = Od'\langle F'\mu\, F'\nu\, F'\lambda\rangle$. By the closure properties of m and by 12.12 we have $\alpha' \epsilon m$. Also, since $F'\alpha' \epsilon F'\gamma$ we have $\alpha' \epsilon \gamma$. Then, by 12.21 there are ordinals $\lambda', \mu', \nu' \epsilon m\alpha'$ such that $F'\alpha' = \langle F'\lambda' F'\mu' F'\nu'\rangle$, $H'\alpha' = \langle H'\lambda' H'\mu' H'\nu'\rangle$. Since $F'\mu = F'\lambda'$, $F'\nu = F'\mu'$, $F'\lambda = F'\nu'$ we have by II') $H'\mu = H'\lambda'$, $H'\nu = H'\mu'$, $H'\lambda = H'\nu'$, and since $F'\alpha' \epsilon F'\gamma$ we have by I') $H'\alpha' \epsilon H'\gamma$. This is $\langle H'\mu\, H'\nu\, H'\lambda\rangle \epsilon H'\gamma$, i.e. $H'\alpha \epsilon \mathfrak{Cnv}_2(H'\gamma)$. Finally $H'\alpha \epsilon H'\eta$. If we assume $H'\alpha \epsilon H'\eta$ we get in the same way $F'\alpha \epsilon F'\eta$. This completes the proof of 12.2. 1).

To conclude the proof of 12.2 we show that 2) is a consequence of 1) and I'). For suppose that $F'\alpha \neq F'\eta$. Then either $F'\eta - F'\alpha \neq O$, or $F'\alpha - F'\eta \neq O$. If $F'\eta - F'\alpha \neq O$, put $\alpha' = C'J_3{}'\langle\eta\,\alpha\rangle$. Then $\alpha' \epsilon m$ by the closure properties of m w.r. to J_3 and C. Also $F'\alpha' \epsilon F'J_3{}'\langle\eta\,\alpha\rangle = F'\eta - F'\alpha$, i.e. $F'\alpha' \epsilon F'\eta$, $\sim(F'\alpha' \epsilon F'\alpha)$. Hence $\alpha' \epsilon m\eta$ by 9.52. We conclude by 1) $H'\alpha' \epsilon H'\eta$ and by I') $\sim(H'\alpha' \epsilon H'\alpha)$. Hence $H'\eta - H'\alpha \neq O$. If $F'\alpha - F'\eta \neq O$ we prove in the same way $H'\alpha - H'\eta \neq O$. Next suppose $H'\eta \neq H'\alpha$, for example $H'\eta - H'\alpha = H'J_3{}'\langle\eta\,\alpha\rangle \neq O$. Then, by 12.11 there is an $\alpha' \epsilon m\, J_3{}'\langle\eta\,\alpha\rangle$ such that

$H'\alpha' \epsilon H'\eta$, $\sim(H'\alpha' \epsilon H'\alpha)$. By 12.11 we may suppose $\alpha' \epsilon m\eta$ and we conclude as before $F'\eta - F'\alpha \neq O$.

12.3 *If* G *is an isomorphism from* m *to an ordinal* o *with respect to* E, *then* $H'\eta = F'G'\eta$ *for* $\eta \epsilon m$.

Proof. By the definition of an isomorphism w.r. to E we have $\alpha\epsilon\beta \equiv G'\alpha \epsilon G'\beta$, for $\alpha,\beta\epsilon m$. Hence, for $\alpha,\beta,\gamma,\delta\epsilon m$, by definition 7.8, $<\alpha\beta>$Le$<\gamma\delta> \cdot \equiv \cdot <G'\alpha G'\beta>Le<G'\gamma G'\delta>$. Likewise, by definition 7.81 $<\alpha\beta>$ R $<\gamma\delta> \cdot \equiv \cdot <G'\alpha G'\beta>$ R $<G'\gamma G'\delta>$. It follows then by definition 9.2 that $<k\alpha\beta>$S$<i\gamma\delta> \cdot \equiv \cdot <kG'\alpha G'\beta>$ S$<iG'\gamma G'\delta>$, for $i,k = 0,1, \ldots 8$. Hence, for $\alpha,\beta,\gamma,\delta \epsilon m$, $i, k = 0, 1, \ldots, 8$:

(1) $\quad J_k' <\alpha\beta> \epsilon J_i'<\gamma\delta> \cdot \equiv \cdot J_k'<G'\alpha G'\beta> \epsilon J_i'<G'\gamma G'\delta>$,

and by the closure properties of m, (1) holds for $J_k' <\alpha\beta>$, $J_i'<\gamma\delta> \epsilon m$.

We now prove that

(2) $\quad \eta \epsilon m, \eta = J_i'<\gamma\delta> \cdot \supset \cdot G'\eta = J_i'<G'\gamma G'\delta>$,

under the induction hypothesis

(3) $\quad \eta' \epsilon m\eta, \ \eta' = J_k'<\alpha\beta> \cdot \supset \cdot G'\eta' = J_k'<G'\alpha G'\beta>$.

So suppose first that $J_i' <G'\gamma G'\delta> \epsilon G'\eta$. Since G is an isomorphism of m, then $J_i'<G'\gamma G'\delta> = G'\eta'$, with $\eta' \epsilon m\eta$. Let $\eta' = J_k'<\alpha\beta>$. Then by (3) and by (1) $G'\eta' = J_k'<G'\alpha G'\beta> \epsilon J_i'<G'\gamma G'\delta> (= G'\eta')$, against 1.6. Suppose next that $G'\eta \epsilon J_i'<G'\gamma G'\delta>$. Let $G'\eta = J_k'<\alpha'\beta'>$. By 9.25 $\alpha',\beta' \leq G'\eta$; hence $\alpha' = G'\alpha, \beta' = G'\beta$, with $\alpha,\beta\epsilon m$. Then $J_k' <G'\alpha G'\beta> = G'\eta \epsilon J_i' <G'\gamma G'\delta>$. By (1), $J_k'<\alpha\beta> \epsilon J_i'<\gamma\delta> (= \eta)$. Hence $G'J_k'<\alpha\beta> \epsilon G'\eta$ (*), and by (3) $G'J_k'<\alpha\beta> = J_k'<G'\alpha G'\beta> (= G'\eta)$. This contradicts (*). Hence (2) is proved.

We now prove 12.3 by induction on η. If $\eta \epsilon \mathfrak{W}(J_0)$, then $H'\eta = H''(m\eta)$. Also, by (2) $G'\eta \epsilon \mathfrak{W}(J_0)$, so that $F'G'\eta = F''G'\eta = F''G''(m\eta)$. By the induction hypothesis this last is $H''(m\eta)$. If $\eta = J_i'<\beta\gamma>$, $i = 1, \ldots, 8$, then $\beta,\gamma \epsilon m\eta$. By the induction hypothesis and (2) $H'\eta = \mathfrak{F}_i(H'\beta H'\gamma) = \mathfrak{F}_i(F'G'\beta F'G'\gamma) = F'J_i' <G'\beta G'\gamma> = F'G'J_i' <\beta\gamma> = F'G'\eta$.

12.4 $\quad \overline{\overline{F''\omega_\alpha}} = \omega_\alpha \qquad$ (Gödel 12.1)

Proof. $\overline{\overline{F''\omega_\alpha}} \leq \overline{\overline{\omega_\alpha}} = \omega_\alpha$, by 8.31. On the other hand, there exists a subset of ω_α, namely $\omega_\alpha \cdot \mathfrak{W}(J_0)$, such that the values of F over this subset are all different, since if $\gamma \neq \delta$ and $\gamma,\delta\epsilon\,\omega_\alpha \cdot \mathfrak{W}(J_0)$, assume $\gamma < \delta$, then $F'\gamma \epsilon F'\delta$, by 9.3, so that $F'\gamma \neq F'\delta$. But $\overline{\overline{\omega_\alpha \cdot \mathfrak{W}(J_0)}} \geq \omega_\alpha$, because $J_0''(\omega_\alpha{}^2) \subseteq \omega_\alpha \mathfrak{W}(J_0)$, by 9.26 and J_0 is one-to-one. Hence $\overline{\overline{F''\omega_\alpha}} \geq \omega_\alpha$.

By 12.4 the generalized continuum hypothesis follows immediately from the following theorem:

12.5 $\quad \mathfrak{P}(F''\omega_\alpha) \subset F''\omega_{\alpha+1} \qquad$ (Gödel 12.2).

This theorem is proved as follows. Consider $u \epsilon \mathfrak{P}(F''\omega_\alpha)$, that is $u \subset F''\omega_\alpha$. By V = L there is a δ such that $u = F'\delta$; form the closure of the set $\omega_\alpha + \{\delta\}$ w.r. to C, C_1, K_1, K_2 and w.r. to the J_i, $i = 0, 1, \ldots 8$, as triadic relations, according to 8.73 and let the closure be denoted by m. Now by 8.73 m is a set and $\overline{\overline{m}} = \omega_\alpha$. m is a set of ordinals, hence m is well-ordered by E by 7.161 and is isomorphic to some ordinal o by 7.7. Let the isomorphism be denoted by G, so that $G''m = o$. Hence $\overline{\overline{o}} = \overline{\overline{m}} = \omega_\alpha$. By 12.3, since $\delta \epsilon m$, we have $H'\delta = F'G'\delta$, so that $\overline{Od'(H'\delta)} \leq \overline{G'\delta} < \omega_{\alpha+1}$. Now $\omega_\alpha \subseteq m$ and by 12.1, $F'\beta = H'\beta$ for $\beta \epsilon \omega_\alpha$. We may suppose $\delta \geq \omega_{\alpha+1}$, otherwise there is nothing to prove. Then by 12.2, for $\beta \epsilon \omega_\alpha$: $F \beta \epsilon F \delta \cdot \equiv \cdot H'\beta \epsilon H'\delta$. Hence $F'\delta$ and $H'\delta$ have exactly the same elements with $F''\omega$ in common, i.e. $F'\delta \cdot F''\omega_\alpha = H'\delta\, F''\omega_\alpha$; but $u = F'\delta \subseteq F''\omega_\alpha$ by assumption, therefore $F'\delta = H'\delta\, F''\omega_\alpha$. Also $\omega_\alpha \epsilon \mathfrak{M}(J_0)$, by 9.27, therefore, by 9.35 $F''\omega_\alpha = F'\omega_\alpha$; hence $F'\delta = H'\delta\, F'\omega_\alpha$. Therefore, by 9.611, $\overline{Od'u} < \omega_{\alpha+1}$, in other words $u \epsilon F''\omega_{\alpha+1}$, q.e.d.

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