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THE LOGIC OF EITHER-OR

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1. The Paradoxes. If there is any reason for calling the logical principles

1 p .˙. if q then p, 2 necessary p .˙. if q then p

paradoxes of the conditional, there is equal reason for calling the logical principles

3 p .˙. either p or q, 4 necessary p .˙. either p or q

paradoxes of the disjunctive.[1] For, just as there are many propositions expressed by sentences of the form 'if p then q' which can be true only if there is an appropriate connection between antecedent and consequent—a connection which is not guaranteed merely by the truth or even the necessary truth of the consequent—, so too there are many propositions expressed by sentences of the form 'either p or q' which can be true only if there is an appropriate connection between the two disjuncts—a connection which is also not guaranteed merely by the truth or even the necessary truth of one of the disjuncts.

In the case of disjunctive propositions the needed connection is of such a nature that it holds only when, in respect to the context in question, the alternatives are exhaustive.[2] In fact, it is typically the case that a sentence of the form 'either p or q' is used to set forth, whether exclusively or non-exclusively, an exhaustive set of alternatives. And such a statement of alternatives does not follow from the fact that one of the alternatives is realized or even that it is necessarily realized.

Suppose that White attempts to list the alternatives exhaustively when he says that Black, who is considered only as a faculty member of a university, is either an assistant or a full professor. Even though Black is, let us say, an assistant professor, the truth of White's statement does not follow. For, since there are further alternatives, his statement is certainly false. Suppose now that one of the alternatives is a necessary proposition. Here, e.g., White attempts to list the alternatives exhaustively by saying of a given integer, say + 2, that it is either positive or negative. Since 0 is an integer and since + 2 is being considered only as a representative of the in-

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tegers, the list of alternatives is not exhaustive and White's statement is false. Consequently his statement does not follow from the necessity of +2's being a positive integer. If it is objected that the necessity of +2's being appositive integer excludes alternatives beyond the two White listed, it can be replied that, by this reasoning, White has not listed alternatives at all, +2's being negative also being excluded by the necessity of its being positive. But since our hypothesis is that White is listing alternatives, it must be admitted that, as alternatives for +2 as a representative of the integers, his list is incomplete and his disjunctive proposition is not true.

The following logical principles, which involve the familiar material conditional connective and the familiar material disjunctive connective, respectively, are unquestionably correct:

5 $p \therefore q \supset p$, 6 $p \therefore p \vee q$.

But it is clear that there are some arguments to which 1 and 3 apply[3] which are invalid. Thus there are some arguments to which 1 and 3 apply which cannot be translated into symbolic arguments to which 5 and 6 apply. Further, on the basis of the following definitions[4] for the strict conditional connective and the strict disjunctive connective respectively:

$p \prec q =_{df} \sim \Diamond (p . \sim q)$, $p \dot{\vee} q =_{df} \sim \Diamond (\sim p . \sim q)$

one can infer the correctness of the following principles:

7 $\sim \Diamond \sim p \therefore q \prec p$, 8 $\sim \Diamond \sim p \therefore p \dot{\vee} q$.

But it is clear that there are some invalid arguments to which 2 and 4 apply. Thus there are some arguments to which 2 and 4 apply which cannot be translated into symbolic arguments to which 7 and 8 apply.

To provide a fuller symbolism we add '$\rightarrow$' to the list of conditional connectives and '$\cup$' to the list of disjunctive connectives. '$\rightarrow$' shall be used for 'if-then' only when 'if-then' is such that no argument made with a sequence of sentences of the form 'p ∴ if q then p' or of the form 'necessary p ∴ if q then p' is valid. To '$\rightarrow$' we apply the admittedly honorific designation "the proper conditional connective." '$\cup$' shall be used for 'either-or' only when 'either-or' is such that no argument made with a sequence of sentences of the form 'p ∴ either p or q' or of the form 'necessary p ∴ either p or q' is valid. This restriction follows from the condition that '$\cup$' be used to symbolize 'either'or' when and only when 'either-or' is used to list alternatives exhaustively, though nonexclusively. To '$\cup$' we apply the designation "the nonexclusive proper disjunctive connective." In sum, not only will each of the following principles be incorrect (i.e. apply to some invalid arguments):

9 $p \therefore q \rightarrow p$, 11 $p \therefore p \cup q$,

10 $\sim \Diamond \sim p \therefore q \rightarrow p$, 12 $\sim \Diamond \sim p \therefore p \cup q$,

but no arguments to which they apply will be valid.

In what follows a characterization of 'either-or' in the exhaustive-alternatives sense is given by means of certain principles formulated with

'$\cup$'. No attempt is made to give a definition of '$\cup$' which would accomplish the same for it as that which the above definition of '$\dot{\vee}$' accomplished for '$\dot{\vee}$'.

2. De Morgan and Tautology. Of the analogues of De Morgan's principles, the following are correct:

13 $p \cup q \therefore \sim(\sim p . \sim q)$, 14 $\sim p . \sim q \therefore \sim(p \cup q)$,

while the following are incorrect:

15 $\sim(\sim p . \sim q) \therefore p \cup q$, 16 $\sim(p \cup q) \therefore \sim p . \sim q$,

since, though it be false that both disjuncts are false, it may yet be false that those disjuncts exhaust the alternatives, and since, though it be false that the two disjuncts exhaust the alternatives, it may yet be false that they are both false.

Further, though

17 $p \cup q \therefore \sim \Diamond(\sim p . \sim q)$

expresses an immediate property of the proper disjunctive, its converse,

18 $\sim \Diamond(\sim p . \sim q) \therefore p \cup q$,

which is related to 15, is incorrect, otherwise 12 would be correct, '$\sim \Diamond \sim p \therefore \sim \Diamond(\sim p . \sim q)$' being correct. And since

19 $\sim(p \cup q) \therefore \Diamond(\sim p . \sim q)$,

which is related to 16, yields 18, it is incorrect.[5]

Of the analogues of principles of tautology,

20 $p \cup p \therefore p$

is correct, while its converse is assuredly incorrect, since, together with 17, the converse of 20 would imply that every proposition entails its own necessity. Yet, though '$p \cup p \therefore \sim \Diamond \sim p$' is correct, its converse '$\sim \Diamond \sim p \therefore p \cup p$' is incorrect, since the latter could be correct only if 18 were correct, there being a proof of 18 based on '$\sim \Diamond \sim p \therefore p \cup p$' and on 36, its converse, and 38, below. Thus the question as to when a '$p \cup p$'-proposition (if it could ever appropriately be expressed) is true remains open.

3. The Exclusive Sense. Since both

21 $\sim p \vee \sim q \therefore \sim(p . q)$

and its converse are correct, exclusive material disjunctive sentences can be symbolized so as to exemplify either the form '$(p \vee q) \cdot (\sim p \vee \sim q)$' or the form '$(p \vee q). \sim(p \cdot q)$', without variation of inferential relations. However, in view of the incorrectness of 15, the forms '$(p \cup q) \cdot (\sim p \cup \sim q)$' and '$(p \cup q). \sim(p \cdot q)$' are nonequivalent, i.e., for our purposes, propositions expressed with sentences exemplifying them are not mutually inferable. Moreover, since 18 is incorrect, it would seem that '$(p \cup q).(\sim p \cup \sim q)$' and '$(p \cup q) . \sim \Diamond(p . q)$' are also nonequivalent. Yet, though

22 $\sim \Diamond(p \cdot q) \therefore \sim p \cup \sim q$

is incorrect,

23 $p \cup q, \sim \Diamond(p \cdot q) \therefore \sim p \cup \sim q$

is correct. For, when the kind of connection needed for the truth of a proposition expressed by a sentence of the form '$p \cup q$' holds, the propositions expressed by the corresponding sentences of the forms '$\sim p$' and '$\sim q$' will not possess the kind of mutual irrelevance sufficient to falsify a proposition expressed by the corresponding sentence of the form '$\sim p \cup \sim q$'. In fact, in such a context, the truth of the last mentioned proposition is guaranteed by the truth of a proposition expressed by the corresponding sentence of the form '$\sim \Diamond (p . q)$'. The equivalence of '$(p \cup q).(\sim p \cup \sim p)$' and '$(p \cup q).\sim \Diamond (p.q)$' is then to be granted, and a disjunctive which is used to express not only a limit on alternatives but also the impossibility of taking them jointly can be symbolized by either of these forms. If, however, a disjunctive sentence is used to express, over and above a limit on the alternatives, only the proposition that they are not jointly realized, then the form '$(p \cup q).(\sim p \vee \sim q)$' or the form '$(p \cup q).\sim (p.q)$', is to be used. The notion of an exclusive proper disjunctive sentence is then ambiguous.

4. *Conditional Contexts*. 'Either -or' shows a certain variability in 'if-then' contexts. The argument

i If Brown either failed or got a D he is ineligible. ∴ If Brown failed he is inelegible, and if he got a D he is ineligible

is valid, while the argument

ii If Member-of-Congress Brown is either a senator or a representative Congress has only two houses. ∴ If Member-of-Congress Brown is a senator Congress has only two houses, and if he is a representative Congress has only two houses

is invalid. In i the disjunctive antecedent of the premiss is not intended as an exhaustive list of alternative grades. The correct principle

24 $(p \vee q) \rightarrow r \therefore (p \rightarrow r).(q \rightarrow r)$

applies to a translation of i.[6] In ii the disjunctive antecedent of the premiss is depended on as an exhaustive list of alternatives. The incorrect principle

25 $(p \cup q) \rightarrow r \therefore (p \rightarrow r).(q \rightarrow r)$

applies to a translation of ii. Clearly, if one thing is implied by the fact that two things exhaust the alternatives, it need not be the case that the former is implied by each of the latter.[7]

The converse of the premiss of i, 'If Brown is ineligible he either failed or got a D', does, however, involve a proper disjunctive consequent, when, as might be expected, it is intended that failing and getting a D are the only paths to ineligibility. It would seem that instances of 'if p then either q or r' with nonproper disjunctive consequents can be found only where the disjunctive is an emphatic disjunctive:

If Brown is ineligible then either his professor had a grudge against the team or my name is not Pat!

5. *Distribution and Association.* Though the distributive principle

26 $p.(q \vee r) \therefore (p.q) \vee (p.r)$

and its converse are both correct, the analogous distributive principle

27 $p.(q \cup r) \therefore (p.q) \cup (p.r)$

is incorrect, while its converse is correct. The conclusion formula of 27 yields '$\sim\Diamond(\sim(p.q).\sim(p.r))$', which yields '$\sim\Diamond\sim p$'. Thus, since an argument to which 27 applies can be valid only if a proposition expressed by the sentence replacing 'p' is necessary, 27 is incorrect. Since '$(p.q) \cup (p.r)$' clearly yields 'p' if it yields '$\sim\Diamond\sim p$', to show that the converse of 27 is correct it remains only to prove that the first yields '$q \cup r$':

α	I	$(p.q) \cup (p.r)$	premiss formula
	II	$\sim(p.q) \rightarrow (p.r)$	from I
	III	$\sim(p.q) \rightarrow r$	from II
	IV	$\sim r \rightarrow (p.q)$	from III
	V	$\sim r \rightarrow q$	from IV
	VI	$\sim q \rightarrow r$	from V
	VII	$q \cup r$	from VI

The legitimacy of the steps to II and to VII will be confirmed in *6*.

Some preliminaries on ternary disjunctive chains are needed before tackling association. Suppose White and Black are drawing cards from a well shuffled deck for the purpose of determining, on the basis of the highest draw, with aces high, who shall deal. As White draws it would certainly not be true to say, in the exhaustive-alternatives sense of 'either-or',

A White will draw either a king or an ace.

Even to say this would be evidence of connivance with stacking the deck. Suppose further that Black has already drawn the queen of spades. It is then true that

B Either White will draw either a king or an ace or else Black will deal.

Now from the assumption that Black will not deal together with B there follows the conclusion that White will draw either a king or an ace, i.e. A. But it is agreed that A is false. So we can know, before White draws and even if Black has not drawn the highest card possible, that Black will deal. This is indeed paradoxical, and in casting about for a solution two possibilities merit consideration.

(1) The first 'or' in B is material. Thus by disjunctive syllogism we do not get the false proper disjunctive A, but its material counterpart, which may be true. The question of Black's dealing would still be open, as it should be if the deck is well-shuffled. (2) The first 'or' in B is proper, but the clause 'or else Black will deal' restricts the contexts in which the binary disjunctive composed of the first 'or' can be true. In a context in which Black's not dealing is assumed, A may be true. Our earlier claim

that A was false was for a context in which this assumption was absent. In the context of inference by disjunctive syllogism, however, this assumption appears as a premiss and, thus, A may be true in such a context.[8] In parallel fashion, 'If oxygen is present then this match will light' may be true when inferred from 'If this match is struck then, if oxygen is present, it will light' and 'This match is struck'. But it is false in a context in which no consideration is given to actions on or with the match.[9]

As to (1), if Black had, instead, drawn the king of spades, it would be true that

C Either White will draw an ace or Black will deal.

Now, using

28 $p \cup r, p \rightarrow q \therefore q \cup r$[10]

together with '$p \rightarrow (p \vee q)$', the conditionalized form[11] of 6, one arrives at

29 $p \cup r \therefore (p \vee q) \cup r.$

Hence, from C one can infer, if (1) is the correct interpretation,

D Either White will draw either an ace or a deuce or else Black will deal.

This clearly misses getting an appropriate alternative to Black's not dealing. Thus, since 29 is correct, we judge that the first 'or' of D is not material. We are left with (2), if the paradox is to be resolved.

Overlooking the possibility of exclusive intent in the use of B, the pattern '$(p \cup q) \cup r$' corresponds to B, while the pattern '$(p \vee q) \cup r$' can safely be viewed as corresponding only to disjunctives with 'not neither-nor' or 'not both not-... and not-...' first parts. The remaining patterns, '$(p \cup q) \vee r$' and '$(p \vee q) \vee r$', correspond, respectively, to the disjunctive chains in

E If either the invasion will or will not take place or it doesn't yet exist to be talked about it is possible to hold to bivalued logic,

F If either Brown failed or got a D or he is on probation he is ineligible.

Thus the question as to whether associativity holds is a manifold one. There seems to be no reason to question either the correctness of

30 $(p \cup q) \cup r \therefore p \cup (q \cup r),$ 31 $(p \vee q) \vee r \therefore p \vee (q \vee r)$

or that of the converses of these principles. Yet for mixed chains associativity fails. The principle

32 $(p \vee q) \cup r \therefore p \vee (q \cup r)$

is incorrect. For 'p', 'q', 'r' put 'White will draw an ace', 'White will draw a deuce', 'Black will deal'. If Black has drawn the king of spades, the premiss, as we noted, is true. Suppose White does not draw an ace. Then, since 'White draws a deuce $\cup$ Black will deal' is manifestly false, even if the 'p'-part of the conclusion is treated as a contextual restriction, the entire conclusion is false. The converse of 32 is also incorrect since the truth of the 'p'-part suffices for the truth of the premiss but not for that of the conclusion of an instancing argument. Likewise,

33 $(p \cup q) \vee r \mathrel{.\because.} p \cup (q \vee r)$

and its converse are incorrect. For, assuming commutativity for both '$\vee$' and '$\cup$', any argument to which 33 or its converse applies transforms into an equivalent argument to which the converse of 32 or 32 itself, respectively, applies.

It is well to consider a further property of ternary chains. Using 28 together with '$(p \cup q) \rightarrow \sim(\sim p. \sim q)$', the conditionalized form of 13, one arrives at

34 $(p \cup q) \cup r \mathrel{.\overset{\circ}{.}.} (p \vee q) \cup r.$

Supplementing this procedure for arriving at 34 by steps involving association for chains with '$\cup$'s and commutation for '$\cup$', one arrives in the end at

35 $(p \cup q) \cup r \mathrel{.\because.} ((p \vee q) \cup r).((q \vee r) \cup p).((p \vee r) \cup q).$

Is the converse of 35 correct?[12] The following argument would decide the matter negatively. Suppose both the 'r'-part and the '$p \cup q$'-part of the conclusion of an argument instancing the converse of 35 are false. The conclusion is then false, but the '$(p \vee q) \cup r$'-part of the premiss may be true, since the converse of 34 is not correct (cf. the discussion of D, above). Now if the following proof is sound:

β	I	$(p \vee q) \cup r$	premiss formula
	II	$r \cup (p \vee q)$	from I
	III	$\sim r \rightarrow (p \vee q)$	from II
	IV	$(\sim q. \sim r) \rightarrow (\sim q.(p \vee q))$	from III
	V	$(\sim q. (p \vee q)) \rightarrow p$	analytic formula
	VI	$(\sim q. \sim r) \rightarrow p$	from IV and V
	VII	$\sim(q \vee r) \rightarrow (\sim q. \sim r)$	analytic formula
	VIII	$\sim(q \vee r) \rightarrow p$	from VI and VII
	IX	$(q \vee r) \cup p$	from VIII

then a similar proof can also be given for '$(p \vee q) \cup r \mathrel{.\because.} (p \vee r) \cup q$' and, hence, the truth of the '$(p \vee q) \cup r$'-part of the premiss is sufficient for the truth of the remaining conjuncts of the premiss. The converse of 35 would then be faced with the possibility of an instancing inference with a true-false combination. But this argument is tentative, since it involves a proof containing transitions between '$\cup$' and '$\rightarrow$' and the analytic formula V, about neither of which has anything been said.

6. *Disjunctives and Contrary-to-fact Conditionals.* There are situations in which 'either-or' and 'if-then' are equally natural. If 'Either you stop or you will be punished' fits a situation, then 'If you don't stop you will be punished' will fit the same situation, and conversely. However, there are situations in which an 'if-then'-sentence is natural but for which there is no corresponding natural 'either-or'-sentence.[13] When I know you did

not go, I choose 'If you had gone you would have won' rather than 'Either you didn't go or you won'. Yet, not knowing whether or not you went, I could choose either the latter or 'If you went you won'. From this it might seem to follow that 'either-or' cannot be used with a true-to-fact force matching the contrary-to-fact force which the conditional can assume.[14] However, an army officer justifying to his superior a movement resulting in heavy casualties might say either 'If I had not attacked they would have taken the pass' or 'Either I had to attack or surrender the pass'. In this context, the fact of the interchangeability of the two forms indicates that the 'either-or'-proposition has a true-to-fact import.

What conclusion in regard to

36 $p \rightarrow q \therefore \sim p \cup q$

and its converse is warranted by these considerations? Though, in the above context, one would say 'If you had gone you would have won' rather than 'If you went you won', the first entails the second.[15] And the second entails 'Either you didn't go or you won'. But in the process the contrary-to-fact force of the conditional does not pass over into a true-to-fact force for the disjunctive. Conversely, 'Either I had to attack or surrender the pass' with true-to-fact import entails 'If I did not attack I surrendered the pass'. Here the true-to-fact does not become the contrary-to-fact. The problem comes in attempting to pass, in the converse direction, from the neutral to the true- or contrary-to-fact. Both of the following arguments:

Either you didn't go or you won. $\therefore$
If you had gone you would have won,

If I did not attack I surrendered the pass. $\therefore$
Either I had to attack or surrender the pass,

with assumed contrary-to-fact and true-to-fact import, respectively, for their conclusions, would be invalid. But can this be taken as a basis for rejecting 36 and its converse?

If it is, why should not the invalidity of

If you went you won. $\therefore$
If you had gone you would have won

be used against '$p \rightarrow q \therefore p \rightarrow q$' or the invalidity of

If Sam is here you will find him.
He was here. $\therefore$ You will find him

be used against *modus ponens*? The problem turns on the use of the letters 'p', 'q', ... A certain variability is to be allowed in replacements for the same letter in the same context. 'Sam is here' can replace one 'p' and 'he is here' can replace another in the same context. But there must be a limit to these variations. In connection with '$p \rightarrow q \therefore p \rightarrow q$', varying 'you went' to 'you had gone' transcends any acceptable limit. Similarly, it must be agreed that, in connection with 36 and its converse, if one 'p' is replaced by a component with contrary-to-fact import the other is to be replaced by a component with true-to-fact import, otherwise both are to be replaced by neutral components. Short of introducing into the symbolism special operators for indicating contrary- and true-to-fact import, a restriction of this kind on replacements for variables is needed to insure the correctness of 36 and its converse and, hence, the soundness of the steps to IX and III in β.

But under this restriction our valid argument

If you had gone you would have won.
$\therefore$ Either you didn't go or you won

does not translate into a symbolic argument to which 36 applies. Its validity goes beyond principles formulated in our symbolism. It is also to be noted that, if to a contrary-to-fact conditional there corresponds no disjunctive which could plausibly have true-to-fact import, then, rather than its being the case that an argument to which 36 applies which embodies such a conditional is invalid, it is the case that there can be no argument to which 36 applies which embodies such a conditional.

7. Contradictory Premisses. The principle

37 $p, \sim p \therefore q$

("From contradictory propositions any proposition follows") would seem to be derivable with the help of disjunctive syllogism for the material disjunctive,

38 $p \vee q, \sim p \therefore q$

(the conditionalized form of a variant of which appears as V in β), in the following simple manner:

γ	I	p	premiss formula
	II	$\sim p$	premiss formula
	III	$p \vee q$	from I
	IV	q	from II and III

Since 37 is incorrect, one might argue that 38 is incorrect and, hence, that V in β is nonanalytic.[16] However, systematic considerations apart, the rejection of 38 and the acceptance of 37 would be on a par as equally repugnant. The falsity of the conclusion of a valid argument can be said to make one of its premisses false. But it cannot be said that the falsity of any proposition whatsoever makes one of any two contradictory propositions false. Thus 37 is clearly incorrect. Yet it is equally clear that if it is false that of two propositions neither is true and if one of them happens to be false, then the remaining one must be true. Thus 38 is correct. The arbitrary step of rejecting 38 in order to reject 37 can be avoided once we have found what can legitimately be taken as a set of premisses.

The very notion of a single line of reasoning precludes the possibility of contradictory premisses. Two propositions which either are of the forms 'p' and '$\sim p$' or lead to corresponding propositions of these forms can be the bases only for separate lines of reasoning. Suppose I first assume A is true and then argue that it is false that neither A nor B is true.

I subsequently argue that from the last and the assumption of the denial of A it follows that B is true. Unless I have simply confused not-A for my original assumption, I can only be regarded as having expressed two arguments, one with A as its sole premiss and the other with not-A as one of its premisses. The two lines of reasoning are not to be joined, since in assuming the denial of A I explicitly turn away from my original assumption of A. It is not that invalidity would result from joining the two lines. Rather, the contradictoriness of the premisses precludes their use as premisses in a single argument. Even when a contradiction is hidden between two or more propositions, a sequence of propositions written after them does not amount to an argument based on them, since they cannot function as premisses for a single argument. In such a case, the original assumption or assumptions are not explicitly turned away from. But, once the contradiction is recognized, it is also recognized that there cannot have been a single line of reasoning.

Now, since proofs of principles, which are composed of formulas, have a status which is derivative of that of arguments, which are composed of propositions, the sequence γ is not a proof of 37. The sequence γ starts with contradictory formulas and, hence, there could be no argument instancing this sequence. This suffices to clear 38 of the objection raised and to vindicate the use of V in β as an analytic formula.

But what of a single premiss which is itself a contradictory proposition? From a single contradictory premiss one can certainly derive two propositions of which one is the negation of the other. Yet, if jointly contradictory premisses cannot be assumed, how can it be legitimate to assume a single contradictory premiss? There is a difference between the two cases. To assume a proposition which contradicts a previous assumption involves giving up the line of reasoning based on the previous assumption and starting a new line of reasoning. But to assume a single contradictory proposition, which one would not wittingly do except for the sake of argument, is not to make two assumptions, the second of which would initiate a second line of reasoning, but only a single assumption, which as single necessarily allows the development of a single line of reasoning. There is, however, need for care in developing the consequences of a single contradictory premiss. After deriving two contradictory propositions from a single contradictory premiss, one cannot proceed to use them together as distinct intermediate premisses in the derivation of some further result.

NOTES

1. The terms 'conditional' and 'disjunctive' are used, instead of the terms 'implication' and 'disjunction', with the understanding that there is an implication only when a conditional proposition is true and that there is a disjunction only when a disjunctive proposition is true.

2. E. E. C. Jones (*Elements of Logic as a Science of Propositions*, Edinburgh, 1890, pp. 117, 121) claims that the alternatives expressed by a

true 'either-or'-proposition must be exhaustive and that the relation between such alternatives is reducible to an inferential relation. J. N. Keynes (*Studies and Exercises in Formal Logic*, 3rd ed, London, 1894, p. 223) offers in objection to her view the "paradoxical" result that 'either p or q' and 'both not-p and not-q' are no longer contradictories. Miss Jones is, it is hoped, vindicated in *2* below.

3. *Logical principle* 1, e.g., is to be understood as an abbreviation for 'Any argument made with a sequence of sentences of the form 'p ∴ if q then p' is valid'. An argument to which 1 *applies* or which *instances* 1 is one made with a sequence of sentences of the form 'p ∴ if q then p'. Finally, a principle is *correct* if all arguments to which it applies are valid.

4. Corresponding to C. I. Lewis' definitions of strict implication and strict logical sum in his *A Survey of Symbolic Logic*, Berkeley, 1918, p. 293. To compare our characterization of what shall be called the proper disjunctive with Lewis' characterization of the strict disjunctive, see theorems 3.31-35, 4.16, 4.21-22 on pp. 302-03, 307 of *A Survey*.

5. Note also that, though '$\sim(\sim p \vee p)$' yields the contradiction-formula '$p . \sim p$', '$\sim(\sim p \cup p)$' does not, since to deny that the alternatives are exhausted by two given propositions is not to imply the denial of each.

6. That 24 applies is further attested to by the fact that from the premiss of i and the proposition 'Brown did fail' there follows the conclusion 'Brown is ineligible'.

7. The results of applying *modus tollens* further illustrate the contrast between the premisses of i and ii. From the premiss of ii and the proposition 'Congress does not have only two houses' it does not follow that Brown is neither a senator nor a representative, where 'neither-nor' has the force of a conjunction of denials, but only that it is not the case that he is either a senator or a representative, where 'not either-or' has the force of a denial of exhaustiveness. (So understood 'neither-nor' and 'not either-or' are nonequivalent, since not both 14 and 16 are correct.) But from the premiss of i and the proposition 'Brown is not ineligible' it does follow that Brown neither failed nor got a D and, hence, by 14, that it is not the case that he either failed or got a D.

8. The above paradox does not arise in connection with 'Either Member-of-Congress Brown is a senator or a representative or else there is a possibility we are overlooking'; for in any context in which 'Either Member-of-Congress Brown is a senator or a representative' is proposed as true there is at least the implicit assumption on the part of the speaker that no possibilities are overlooked. The 'or else'-clause presents an invariant contextual restriction here. The context of disjunctive syllogism cannot in this case differ from others sufficiently to allow for paradox.

9. Cf. W. Sellars, "Counterfactuals, Dispositions, and Causal Modalities," in *Minnesota Studies in the Philosophy of Science*, Vol. II, ed. H. Feigl, M. Scriven, and G. Maxwell, Minneapolis, 1958, pp. 244-45.

10. The correctness of 28 depends on the fact that the exhaustiveness of disjunctive alternatives does not imply that whatever is in fact the case can be expressed only in the manner in which one or the other of these alternatives is expressed. Rather, the exhaustiveness of a set of disjuncts can be destroyed only by replacing one or more of them by components which are not implied by the original ones.

11. The formula $(S_1 . S_2 . \ldots . S_{n-1}) \rightarrow S_n$ is to be called the *conditionalized form* of the principle $S_1, S_2, \ldots, S_{n-1} \therefore S_n$, and the former will be said to be *analytic* if the latter is correct. Analytic formulas can be inserted in proofs without being regarded as additional premiss formulas, except in certain classes of cases, which it is not necessary to list here.

12. Cp. N. D. Belnap, Jr., *A Formal Analysis of Entailment*, Technical Report No. 7, Office of Naval Research, Group Psychology Branch, Contract No. SAR/Nonr-609(16), New Haven, 1960, p. 41.

13. Cf. P. F. Strawson, *Introduction to Logical Theory*, London, 1952, p. 92.

14. We are here excluding emphatic disjunctives ('Either that's blue or it snows in July'), whose true-to-fact force does match the contrary-to-fact force of emphatic conditionals ('If that's not blue it snows in July').

15. Should the objection be accepted that, if one would say A but not B, then A cannot entail B, then it remains unexplained how the claim 'If they had climbed it they would have left their flag' can be met by 'I'm not sure they didn't climb it; but you are right that, if they did, they left their flag'.

16. Cf. A. R. Anderson and N. D. Belnap, Jr., "Tautological Entailments," *Philosophical Studies* (Minnesota), Vol. XIII, 1962, pp. 18-19.

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