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THE POST-LINEAL THEOREMS FOR ARBITRARY RECURSIVELY ENUMERABLE DEGREES OF UNSOLVABILITY*

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Introduction

In 1949 Post and Lineal announced the existence of partial propositional calculi with unsolvable decision problems for theoremhood, completeness and independence of axioms. Unfortunately, they never published the proofs but only indicated in an abstract the idea of how a proof might be given. In 1963, Yntema and Harrop, independently, supplied proofs. As indicated in the abstract and carried out by Yntema, the main idea of the proof is that a semi-Thue system can be represented in a particular partial system in such a way that the word problem for the semi-Thue system is reducible to the decision problem for the partial system. Since the word problem for Thue systems is unsolvable as shown by an earlier theorem of Post (1947), the unsolvability of the corresponding decision problems for the partial systems then follows.

Post's earlier work (1944) on recursively enumerable sets of positive integers discussed recursive sets (those with solvable decision problem) and complete sets (those with decision problem of the highest degree of unsolvability) and tried to find sets whose decision problem was unsolvable but not of the highest degree. Although he did not succeed in this effort, since then and since the Post-Lineal theorems first appeared, Friedberg (1957) and Mučnik (1958) have found such sets. Consequently, old problems which have already been shown to be unsolvable are being reconsidered in the hope of proving that for each recursively enumerable degree of unsolvability there is such a problem.

One such problem is the word problem for Thue systems and Boone has proved that for every recursively enumerable degree of unsolvability there is a Thue system whose word problem is of that degree (1962).[1] The

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1. There are similar results by A. A. Fridman, G. S. Cetin and by C. R. L. Clapham the proofs of which are unknown to us.

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natural question then arises as to whether or not the Post-Lineal theorems can be generalized in a similar way. The main effort of this paper is devoted to showing that they can be. Some of this work duplicates work done by Gladstone (1963) and has subsequently been done by Singletary (1964).

Preliminaries

Definition 1: A *partial propositional calculus* is given by:

Propositional variables: $p, q, r, p_1, q_1, r_1, p_2, \ldots$

Primitive connectives: $\supset, \sim, [,]$

Formation rules:

(i) A propositional variable is a wff.

(ii) If A and B are wffs, $[A \supset B]$ is a wff.

(iii) If A is a wff, $\sim A$ is a wff.

Axioms: a finite number of tautologies

Rules of inference: modus ponens and substitution

Abbreviations:

(i) AB for $\sim [A \supset \sim B]$.

(ii) In order to omit some occurrences of brackets we make the following conventions: (a) the outermost pair of brackets may be omitted—e.g., $X \supset Y$ is an abbreviation of the wff $[X \supset Y]$, and (b) a dot, $\cdot$, immediately after the principal implication-sign may be used so that the pair of brackets around the antecedent and the pair around the consequent may be omitted—e.g., $[X \supset Y \supset \cdot\ U \supset V]$ is an abbreviation for the wff $[[X \supset Y] \supset [U \supset V]]$, and (c) conventions (a) and (b) may be used together so that $X \supset Y \supset \cdot\ U \supset V$ is also an abbreviation for $[[X \supset Y] \supset [U \supset V]]$.

Definition 2: A *semi-Thue system* is given by:

An alphabet: $a_1, a_2, \ldots, a_n$

Operation rules: $G_i \to \bar{G}_i$, $i = 1, 2, \ldots, m$, for G_i, $\bar{G}_i$ fixed words on the alphabet.

We say $C \vdash D$ if there is a finite sequence of statements:

$$C_1 \vdash D_1,\ C_2 \vdash D_2,\ \ldots,\ C_n \vdash D_n$$

where $C = C_1$, $D = D_n$, and each statement holds according to one of the following rules:

(i) $C \vdash C$.

(ii) $G_i \vdash \bar{G}_i$ $i = 1, 2, \ldots,$ or m.

(iii) $CA \vdash DA$ where $C \vdash D$ is a previous statement,

(iv) $AC \vdash AD$ where $C \vdash D$ is a previous statement.

(v) $C \vdash D$ where $C \vdash E$ and $E \vdash D$ are previous statements.

When these rules are used in a proof they will be referred to as rule (i), ..., rule (v).

We will now show that for any semi-Thue system, T, one can effectively construct a partial propositional calculus, P_T, whose theoremhood problem will be of the same recursively enumerable degree of unsolvability as the word problem for T. Thus in P_T there will be certain wffs, W', which are the words of T coded in terms of P_T. Further, the mechanism of proof in P_T will be at least sufficient to encompass that of T. In particular, Theorem 2 below will specify exactly which wffs of P_T may be theorems of P_T. This will be done so as to show (i) that the theoremhood of wffs involving code words and the relationship of those words in T are interdependent and (ii) that any other wffs in P_T are decidable. Thus, not only is the word problem for T reducible (one-one reducible) to the decision problem of P_T, but also the decision problem for P_T is reducible to the word problem for T. (It will be clear after Theorem 2 that the latter reducibility is by unbounded truth-tables.) It will then follow that the two problems are of the same degree of unsolvability.

The Construction

Since a semi-Thue system may have operation rules of the form $A \to 1$ or $1 \to A$, where 1 represents the empty word, and since there is no way to represent the empty word in a partial propositional calculus, we will first construct from any such T a semi-Thue system T^* where both sides of every operation rule are non-empty. The construction will be degree preserving.

Let T be any semi-Thue system. Designate its alphabet by Z and its operation rules by U. Define the semi-Thue system T^* to be the system which has as alphabet both Z and an additional letter q, and operation rules as follows: $qA \to qB$ for A, B words on Z and $A \to B$ in U, and the set $aq \to qa$ and $qa \to aq$ for all a in Z.[2] Define two mappings on words in T^*. First, if W is a word in T^*, then $e(W)$ is the word on Z obtained by erasing all occurrences of q in W. Second, if W is a word in T^*, then $n(W)$ is the non-negative integer which is the number of q's occurring in W.

Lemma 1: (a) $W_1 \vdash_{T^*} W_2$ *implies* $n(W_1) = n(W_2)$.

(b) $W_1 \vdash_{T^*} W_2$ *and* $n(W_1) = n(W_2) = 0$ *implies that* $W_1 = W_2$.

(c) *If* $n(W_1) = n(W_2)$ *and* $e(W_1) = e(W_2)$, *then* $W_1 \vdash_{T^*} W_2$ *and* $W_2 \vdash_{T^*} W_1$.

(d) *If* $W_1 \vdash_{T^*} W_2$, *and* W_1' *and* W_2' *are such that* $e(W_i') = e(W_i)$ *for* $i = 1, 2$, *and* $n(W_1') = n(W_2') \geq n(W_1)$, *then* $W_1' \vdash_{T^*} W_2'$.

Proofs: These are all immediate.

Lemma 2: $W_1 \vdash_{T^*} W_2$ *iff* (*i*) $n(W_1) = n(W_2) = 0$ *and* $W_1 = W_2$, *or* (*ii*) $n(W_1) = n(W_2) > 0$ *and* $e(W_1) \vdash_T e(W_2)$.

Proof: (i) is by Lemma 1(b) and rule (i). If $W_1 \vdash_{T^*} W_2$ and $n(W_1) = n(W_2) > 0$, then by erasing q's in a proof in T^* one obtains a proof in T. For the converse, Lemmas 1(c)(d) mean that it suffices to show that if $W_1 \vdash_T W_2$ then $qW_1 \vdash_{T^*} qW_2$. But this is clear since rules $A_i \to B_i$ in T correspond to

2. This construction is found in "The Word Problem", W. W. Boone, *Annals of Math.*, vol. 70 (1959), p. 250.

rules $qA_i \rightarrow qB_i$ in T^* and rule $aq \rightarrow qa$ means that the q's can always be moved to the left.

Theorem 1: *The word problems of* T *and* T^* *are of the same degree of unsolvability.*

In order to continue the construction of P_T we may now assume that the semi-Thue system, T, has no operation rules which include empty words. Let $a_1, a_2, \ldots, a_n$ be the letters of the alphabet of T. Define a mapping from the words W of T to certain wffs, W', of a partial propositional calculus as follows.

$$\text{Let } a'_1 = \sim\,\sim [\sim p \supset \sim p]$$
$$a'_2 = \sim\,\sim\,\sim\,\sim [\sim p \supset \sim p]$$
$$\vdots$$
$$a'_n = \underbrace{\sim\,\sim \ldots \sim\,\sim}_{2n} [\sim p \supset \sim p]$$
$$(Wa_i)' = \sim [W' \supset \sim a'_i] \text{ for } i = 1, 2, \ldots, n.$$

We note that distinct words in T have distinct images under '.[3]

Thus, for any given semi-Thue system, T, define the corresponding partial propositional calculus, P_T, as follows: P_T has the primitive symbols, formation rules and rules of inference as in definition 1. Let the axioms of P_T be

Identity Axiom: $p \supset p$
Transitivity Axiom: $[\sim p \supset \sim q] \supset . [\sim q \supset \sim r] \supset [\sim p \supset \sim r]$
Right Multiplication Axiom: $[\sim p \supset \sim q] \supset . [\sim p]r \supset [\sim q]\, r$
Left Multiplication Axiom: $[\sim p \supset \sim q] \supset . r[\sim p] \supset r[\sim q]$
Associativity Axioms: $p[qr] \supset [pq]r$
$[pq]r \supset p[qr]$

Semi-Thue Axioms: *If* $G_i \rightarrow \overline{G}_i$, $i = 1, 2, \ldots, m$, *are the operation rules of* T, *then* $(G'_i \supset \overline{G}'_i$, $i = 1, 2, \ldots, m$, *are the semi-Thue axioms of* P_T.

The reader may wonder why we have chosen to include so many not-signs in the axioms. The purpose is to restrict the number of theorems by restricting the possible uses of modus ponens.[4] To indicate that their presence will not interfere with that part of the proof which shows that any semi-Thue system can be represented by this partial propositional calculus, we make two observations. Every a'_i begins with at least two not-signs. Every wff W of the form AB is in reality $\sim [A \supset \sim B]$ and so begins with a not-sign. Hence, any $\sim p$, $\sim q$ or $\sim r$ that occur in an axiom may be replaced by W' for any W in T.

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3. This is an extension of the procedure given by Martin Davis on p. 139 of *Computability and Unsolvability*.

4. I would have preferred the more obvious set of axioms: $[p \supset p], [[p \supset q] \supset . [q \supset r] \supset [p \supset r]], [[p \supset q] \supset [pr \supset qr]], [[p \supset q] \supset [rp \supset rq]], [[pq]r \supset p[qr]], [p[qr] \supset [pq]r]$, and the semi-Thue axioms. However, I have been unable to prove that these axioms without the semi-Thue axioms give a decidable system.

These axioms, in the order given above, are referred to by the abbreviations: id., trans., r.m., l.m., assoc. These abbreviations will also be used to refer to a substitution instance of the same axiom where no confusion can arise. Also, m.p. will refer to modus ponens. Following Church, $\mathbf{S}_B^a W|$ is to mean the result of substituting the wff B for the propositional variable a at every occurrence of a in the wff W.

Definition 3: W is a *minimal conjunct* if W is not of the form $\sim[A \supset \sim B]$. Let $A_1, A_2, \ldots, A_n$ be any non-empty, finite sequence of wffs, possibly with repetitions. We say a wff W is a *conjunction on* $A_1, A_2, \ldots, A_n$ if either (i) $n = 1$ and W is A_1, or (ii) $n = s + t$, $1 \leq s \leq n - 1$, $1 \leq t \leq n - 1$ and W is $\sim [U \supset \sim V]$ where U is a conjunction on $A_1, A_2, \ldots, A_s$ and V is a conjunction on $A_{s+1}, A_{s+2}, \ldots, A_{s+t}$. If W is a conjunction on $A_1, A_2, \ldots, A_n$ and each A_1, $i = 1, 2, \ldots, n$, is a minimal conjunct then $A_1, A_2, \ldots, A_n$ is the *total factorization* of W and W has length n.

We had already noted before definition 3 that we allow $\sim [A \supset \sim B]$ to be abbreviated by AB. However, without further specification the abbreviation of an arbitrary wff is not necessarily unique. For example, $\sim [A \supset \sim [B \supset \sim C]]$ may be abbreviated by (i) $[A[B \supset \sim C]]$ or by (ii) $\sim [A \supset BC]]$. Definition 3 specifies that all abbreviations are to be like those in (i). In abbreviating a wff according to the definition, we work from the outside to the inside and as soon as a conjunct is minimal we do nothing further to it. Because we have specified a recursive procedure for finding the total factorization of any given wff, W, the total factorization and length of W are unique.

Definition 4: If a wff W has total factorization $A_1, A_2, \ldots, A_n$, then define $\mathcal{R}(W)$ to be $\sim[\sim[\ldots \sim[\sim[A_1 \supset \sim A_2] \supset \sim A_3] \ldots \supset \sim A_{n-1}] \supset \sim A_n]$. Using BC for $\sim[B \supset \sim C]$ we may write $\mathcal{R}(W)$ as $[[[\ldots[[A_1A_2]A_3]\ldots]A_{n-1}]A_n]$. If W_1 and W_2 are two wffs such that $\mathcal{R}(W_1) = \mathcal{R}(W_2)$, then we say that W_1 and W_2 are *associates*.

Lemma 3: *If U, V, and W are wffs where each is of length $n \geq 2$, and if* $\vdash_{P_T} [U \supset V]$, *and* $\vdash_{P_T} [V \supset W]$, *then* $\vdash_{P_T} [U \supset W]$.

Proof: U, V, W each of length $n \geq 2$ means that U is of form $\sim[U_1 \supset \sim U_2]$, V is of form $\sim [V_1 \supset \sim V_2]$ and W is of form $\sim [W_1 \supset \sim W_2]$. Since U, V, and W each begin with a not-sign, they can be used for substitution into the transitivity axiom.

Lemma 4:[5] (a) $\vdash_{P_T} W \supset \mathcal{R}(W)$ and (b) $\vdash_{P_T} \mathcal{R}(W) \supset W$.

The proof of (a) is by induction on n, the length of W. Let W have total factorization $A_1, A_2, \ldots, A_n$. If $n = 1$ or 2, the $\mathcal{R}(W) = W$ and so the lemma holds by simply substituting A_1 or A_1A_2 for p in the identity axiom.

5. See Jacobson, *Lectures in Abstract Algebra*, vol. I, p. 20.

If $n > 2$ then let W be $\sim [U \supset\sim V]$, i.e. UV. Suppose U has total factorization $A_1, A_2, \ldots, A_i$ of length i, $i < n$, and V has total factorization $A_{i+1}, \ldots, A_n$ of length $n - i$. Then by the induction hypothesis

(1) $$\vdash_{P_T} V \supset [[\ldots [A_{i+1}A_{i+2}]\ldots]A_n];$$

by substituting U for r, V for $\sim p$, and $[[\ldots [A_{i+1}A_{i+2}]\ldots]A_n]$ for $\sim q$ into the left multiplication axiom and using modus ponens

(2) $$\vdash_{P_T} UV \supset U[[\ldots [A_{i+1}A_{i+2}]\ldots]A_n];$$

by substituting U for p, $[\ldots [A_{i+1}A_{i+2}]\ldots A_{n-1}]$ for q, and A_n for r into the axiom $p[qr] \supset [pq]r$

(3) $$\vdash_{P_T} [U[[\ldots[A_{i+1}A_{i+2}]\ldots A_{n-1}]A_n]] \supset [[U[[\ldots[A_{i+1}A_{i+2}]\ldots]A_{n-1}]]A_n];$$

by the induction hypothesis

(4) $$\vdash_{P_T} [U[[\ldots[A_{i+1}A_{i+2}]\ldots]A_{n-1}]] \supset [\ldots[[A_1A_2]A_3\ldots]A_{n-1}];$$

by substituting $U[[\ldots [A_{i+1}A_{i+2}]\ldots]A_{n-1}]$ for $\sim p$ and $[[[\ldots [A_1A_2]A_3\ldots] A_{n-1}]$ for $\sim q$ and A_n for r into the right multiplication axiom and using modus ponens

(5) $$\vdash_{P_T} [[U[[\ldots[A_{i+1}A_{i+2}]\ldots]A_{n-1}]]A_n] \supset [[[\ldots[A_1A_2]A_3\ldots]A_{n-1}]A_n];$$

and finally, by several uses of Lemma 3

(6) $$\vdash_{P_T} UV \supset [[[\ldots[A_1A_2]A_3\ldots]A_{n-1}]A_n].$$

The proof of part (b) is similar.

Corollary 1: *If W_1 and W_2 are associates, then* $\vdash_{P_T} [W_1 \supset W_2]$ *and* $\vdash_{P_T} [W_2 \supset W_1]$.

Corollary 2: *If U and $\hat{U}$ are associates of length $n \geq 2$, if V and $\hat{V}$ are associates of length n, and if* $\vdash_{P_T} [U \supset V]$, *then* $\vdash_{P_T} [\hat{U} \supset \hat{V}]$.

Proof: Corollary 1 and Lemma 3.

The idea of total factorization gives a unique decomposition of a conjunction into its conjuncts. Lemma 4 and its corollaries mean that intuitively we may disregard the way in which a conjunction is associated.

Definition 5:[6] A wff, W, of P_T is a *regular wff* if (i) W is $a'_1, a'_2, \ldots$ or is a'_n, or (ii) W is of the form $\sim [W_1 \supset\sim W_2]$, where W_1 and W_2 are regular wffs. Let $R, R_1, R_2, \ldots$ be variables whose range is the regular wffs. We write R^A for $\mathbf{S}^p_A R|$, where A is any wff. If W is of the form R^A we say W is a *subregular wff*, and when it is necessary to refer to the specific A involved, we say W is an *A-subregular wff*.

6. These definitions are modifications of those given by M. K. Yntema in "A Detailed Argument for the Post-Lineal Theorems," *Notre Dame Journal of Formal Logic*, vol. 5, no. 1(1964), pp. 37-50.

Definition 6: Since the mapping ' is one-to-one, we may define $<R>$ to be the word in T obtained from the regular wff R in P'_T by taking the pre-image with respect to ' of $\mathcal{R}(R)$.

Although the word $W = a_{i_1}a_{i_2}\ldots a_{i_n}$ in T translates to $W' = [\ldots[[a_{i_1}a_{i_2}a_{i_3}]\ldots a_{i_n}]$ in P_T, Corollary 2 means that, informally, W' may be identified with any of its associates.

Lemma 5: *For any two regular wffs,* R_1 *and* R_2, $<R_1> \vdash_T <R_2>$ *implies* $\vdash_{P_T} [R_1 \supset R_2]$.

Proof: Rules (i) through (v) of definition 2 describe, inductively, the structure of a proof in any semi-Thue system. Thus, for the proof this lemma notice that if $<R_1> \vdash_T <R_2>$, then one of the following is true.

(i) $<R_1>$ is $<R_2>$. Then $\vdash_{P_T} [R_1 \supset R_2]$ by Corollary 2.
(ii) $<R_1>$ is G_i, $<R_2>$ is $\bar{G}_i$ for some operation rule, $G_i \to \bar{G}_i$ in T.

Then $\vdash_{P_T} [R_1 \supset R_2]$ by the corresponding semi-Thue axiom of P'_T and Corollary 2.

(iii) $<R_1>$ is $<R_aR>$ and $<R_2>$ is $<R_bR>$ where $<R_1> \vdash_T <R_2>$ is inferred from a previous statement $<R_a> \vdash_T <R_b>$.

By the induction hypothesis, $\vdash_{P_T} [R_a \supset R_b]$, so by r.m. and m.p. $\vdash_{P_T} [R_aR \supset R_bR]$. Then Corollary 2 gives $\vdash_{P_T} [R_1 \supset R_2]$.

(iv) Where $<R_1>$ is $<RR_a>$ and $<R_2>$ is $<RR_b>$,

the proof is similar to (iii).

(v) $<R_1>$ is $<R_a>$, $<R_2>$ is $<R_b>$ where $<R_1> \vdash_T <R_2>$ is inferred from the previous statements $<R_a> \vdash_T <R_c>$ and $<R_c> \vdash_T <R_b>$.

By the induction hypothesis $\vdash_{P_T} [R_a \supset R_c]$ and $\vdash_{P_T} [R_c \supset R_b]$, so by Lemma 3 and Corollary 2 $\vdash_{P_T} [R_a \supset R_b]$ which is the same as $\vdash_{P_T} [R_1 \supset R_2]$.

Earlier we remarked that the many not-signs in the axioms of P'_T restrict the number of uses of modus ponens. The next two lemmas will show that this is indeed true. This will be done using the following forms:

$$\text{form i:}\quad [\sim X_1 \supset \sim X_2] \supset \cdot [\sim X_3 \supset \sim X_4] \supset [\sim X_5 \supset \sim X_6]$$

$$\text{form ii:}\quad [\sim X_1 \supset \sim X_2] \supset [\sim X_3 \supset \sim X_4]$$

$$\text{form iiia:}\quad [\sim X_1 \supset \sim X_2]$$

Lemma 6: *The following table shows all possible uses of modus ponens among wffs which are of form i, ii, or iiia:*

major premiss	possible minor premiss	result of modus ponens
i	iiia	ii
ii	iiia	iiia
iiia	none possible	---

Proof: That is, if form i is to be the major premiss, then the minor premiss must be $[\sim X_1 \supset \sim X_2]$, but this is, in fact, form iiia. Further, the result, $[[\sim X_3 \supset \sim X_4] \supset [\sim X_5 \supset \sim X_6]]$ is of form ii. If form ii is to be the major premiss, then the minor premiss must be $[\sim X_1 \supset \sim X_2]$ which is of form iiia. The result then is $[\sim X_3 \supset \sim X_4]$, also of form iiia. If the major premiss is to be form iiia, then the minor premiss must be $[\sim X_1]$ which is not form i, ii or iiia.

Lemma 7: $\vdash_{P_T} W$ *implies that* W *is of form i, form ii, form iiia above or of form iiib*: $[X_1 \supset X_1]$.

The proof is by strong induction on the proof of W. $n = 1$: Then W is an axiom. The identity, associativity and semi-Thue axioms are of form iii. The right and left multiplication axioms are of form ii and the transitivity axiom of form i.

$n > 1$: If the nth step is obtained by substitution into form i, ii or iii, then we have these again. Suppose that the nth step is obtained by modus ponens. Without loss of generality we can assume that form iiib is never used as the major premiss. If U is of form iiib and is the minor premiss to be used in modus ponens with a wff of form i or ii as major premiss, then U must itself be of form iiia. Hence the lemma follows from Lemma 6.

Lemma 8: $\mathbf{S}^{a}_{U} R^{A}| = R^{\mathbf{S}^{a}_{U} A|}$, *for any propositional variable a, and any wff U.*

Definition 7: Let V have total factorization $V_1, \ldots, V_n$. Suppose a conjunction, U, on $V_k, \ldots, V_{k+j}$ is an A-subregular wff while V_{k-1} and V_{k+j+1} are not, then U is a *maximal subregular conjunct* of V. If V has total factorization $V^*_1, \ldots, V^*_n$, we define the *Thue factorization*, $V_1, \ldots, V_s$, of V by recursion on n as follows. If $n = 1$, then the Thue factorization, V_1, is V^*_1. For $n = k+1$, suppose V has total factorization $V^*_1, \ldots, V^*_k, V^*_{k+1}$ where, inductively, $\hat{V}$ with total factorization $V^*_1, \ldots, V^*_k$ has Thue factorization $V_1, \ldots, V_t$. Then the Thue factorization of V is either $V_1, \ldots, V_t V^*_{k+1}$ and $s = t$ or $V_1, \ldots, V_t, V^*_{k+1}$ and $s = t + 1$ according as $V_t V^*_{k+1}$ is an A-subregular wff for some A, or not. If V has Thue factorization $V_1, \ldots, V_n$, the V_i are the *Thue factors*. We remark informally that by this procedure every wff has a unique Thue factorization which can be obtained by looking at its total factorization and regrouping these factors into maximal subregular conjuncts and minimal conjuncts.

We are now ready to prove Theorem 2 which shows that the construction of P_T has preserved the degree of unsolvability of T. This is done by showing that only certain, recognizable wffs of P_T can possibly be theorems. Among these, some wffs, W, will contain, in a specified way, a finite number of pairs of subregular wffs, $R^{A_i}_{i,a}$ and $R^{A_i}_{i,b}$. Then, whether or not W is a theorem of P_T depends on whether or not $< R_{i,a} > \vdash_T < R_{i,b} >$ for each pair. For the other wffs, one can decide directly whether or not each is a theorem of P_T.

Theorem 2: $\vdash_{P_T} W$ *iff* W *is one of the following:*

Case 1: *W is the transitivity axiom or a substitution instance thereof, or*

Case 2a: *W is the right or left multiplication axiom or a substitution instance thereof, or*

Case 2b: *W is* $[[\sim X_2 \supset \sim X_3] \supset [\sim X_1 \supset \sim X_3]]$ *where* $[\sim X_1 \supset \sim X_2]$ *falls in case* 3 *below, or*

Case 3: *W is* $[U \supset V]$ *where, for some positive integer n, U has Thue factorization* $U_1, \ldots, U_n$ *and V has Thue factorization* $V_1, \ldots, V_n$ *and either* $U_i = V_i$ *is a minimal conjunct, or there are* A_i, $R_{i,a}$ *and* $R_{i,b}$ *such that* $U_i = R_{i,a}^{A_i}$ *and* $V_i = R_{i,b}^{A_i}$ *and* $\langle R_{i,a} \rangle \vdash_T \langle R_{i,b} \rangle$.

Proof: Before beginning the actual proof of Theorem 2, we prove the following lemma.

Lemma 9: *If* $[U \supset V]$ *falls in case 3, then so does* $\mathbf{S}_B^a[U \supset V]$ *where a is any propositional variable and B is any wff.*

The proof is by induction on, n, the number of Thue factors in U which is the same as the number in V.

Let $n = 1$: Then U is U_1 and V is V_1. We distinguish two cases. Case A: Suppose $U = V$ is a minimal conjunct, then $\mathbf{S}_B^a U| = \mathbf{S}_B^a V|$ has Thue factorization $U_1 = V_1, \ldots, U_m = V_m$ where each $U_i = V_i$ is either a minimal conjunct or the subregular conjunct $R_{i,a}^{A_i} = R_{i,b}^{A_i}$. Then, since the latter alternative implies $\langle R_{i,a} \rangle \vdash_T \langle R_{i,b} \rangle$ by rule (i) of definition 2, the lemma holds. Case B: Suppose U is R_a^A and V is R_b^A where $\langle R_a \rangle \vdash_T \langle R_b \rangle$. Then $\mathbf{S}_B^a U| = R_a^{\mathbf{S}_B^a A|}$ and $\mathbf{S}_B^a V| = R_b^{\mathbf{S}_B^a A|}$ by Lemma 8. Thus, since $\langle R_a \rangle \vdash_T \langle R_b \rangle$, the lemma holds in this case.

Assume that the lemma holds for $n = k - 1$ and show that it holds for $n = k$. Now, U has Thue factorization $U_1, \ldots, U_k$ and V has Thue factorization $V_1, \ldots, V_k$ and $[U \supset V]$ falls in case 3. But then, clearly, so does $[\dot{U} \supset \dot{V}]$ where $\dot{U}$ is any wff with Thue factorization $U_1, \ldots, U_{k-1}$ and $\dot{V}$ is any wff with Thue factorization $V_1, \ldots, V_{k-1}$. This is true since case 3 is specified in terms of corresponding pairs, U_i, V_i, of Thue factors. So by the induction hypothesis of this lemma, $\mathbf{S}_B^a[\dot{U} \supset \dot{V}]|$ falls in case 3. Since $[U \supset V]$ falls in case 3 it is also true that either $U_k = V_k$ is a minimal conjunct or U_k is $R_{a,k}^{A_k}$ and V_k is $R_{b,k}^{A_k}$ where $\langle R_{k,a} \rangle \vdash_T \langle R_{k,b} \rangle$. Let $\mathbf{S}_B^a \dot{U}|$ have Thue factorization $\hat{\hat{U}}_1, \ldots, \hat{\hat{U}}_m$, and $\mathbf{S}_B^a U_k|$ have Thue factorization $\hat{U}_{k,1}, \ldots, \hat{U}_{k,t}$, and $\mathbf{S}_B^a \dot{V}|$ have Thue factorization $\hat{\hat{V}}_1, \ldots, \hat{\hat{V}}_m$, and $\mathbf{S}_B^a V_k|$ have Thue factorization $\hat{V}_{k,1}, \ldots, \hat{V}_{k,t}$. By the induction hypothesis, for each pair $\hat{\hat{U}}_i$, $\hat{\hat{V}}_i$, $i = 1, \ldots, m$, either $\hat{\hat{U}}_i = \hat{\hat{V}}_i$ is a minimal conjunct or $\hat{\hat{U}}_i$ is $R_{i,a}^{A_i}$ and $\hat{\hat{V}}_i$ is $R_{i,b}^{A_i}$ where $\langle R_{i,a} \rangle \vdash_T \langle R_{i,b} \rangle$. Similarly, by case $n = 1$, for each pair $\hat{U}_{k,j}$, $\hat{V}_{k,j}$, $j = 1, \ldots, t$, either $\hat{U}_{k,j} = \hat{V}_{k,j}$ is a minimal conjunct or $\hat{U}_{k,j}$ is $R_{k,j,a}^{A_{k,j}}$ and $\hat{V}_{k,j}$ is $R_{k,j,b}^{A_{k,j}}$ where $\langle R_{k,j,a} \rangle \vdash_T \langle R_{k,j,b} \rangle$. Then either (1) the $\hat{\hat{U}}_i$, $\hat{U}_{k,j}$ and the $\hat{\hat{V}}_i$, $\hat{V}_{k,j}$ are exactly the Thue factors of $\mathbf{S}_B^a U|$ and $\mathbf{S}_B^a V|$, respectively, and so $\mathbf{S}_B^a[U \supset V]|$ falls in case 3; or (2) $\hat{\hat{U}}_m$ is $R_{m,a}^A$, $\hat{\hat{V}}_m$ is $R_{m,b}^A$, $\hat{U}_{k,1}$ is $R_{k,1,a}^A$ and $\hat{V}_{k,1}$ is $R_{k,1,b}^A$ (all for the same A). In this case, there are $m + t - 1$ Thue factors since both $\hat{\hat{U}}_m \hat{U}_{k,1}$ and $\hat{\hat{V}}_m \hat{V}_{k,1}$ collapse to form single maximal subregular conjuncts. That

$\mathsf{S}^a_B\,[U \supset V]|$ falls in Case 3 now follows from the induction hypothesis, case $n = 1$, Corollary 2 and

$$<R_{m,a}> \vdash_T <R_{m,b}> \qquad \text{(induc. hyp.)}$$

$$<R_{k,1,a}> \vdash_T <R_{k,1,b}> \qquad \text{(case } n = 1)$$

$$<R_{m,a}\, R_{k,1,a}> \vdash_T <R_{m,b}\, R_{k,1,a}> \qquad \text{(rule (iii))}$$

$$<R_{m,b}\, R_{k,1,a}> \vdash_T <R_{m,b}\, R_{k,1,b}> \qquad \text{(rule (iv))}$$

$$<R_{m,a}\, R_{k,1,a}> \vdash_T <R_{m,b}\, R_{k,1,b}> \qquad \text{(rule (v))}$$

Proof of Theorem: $\Longrightarrow$(Induction on the number of steps, n, of the proof of W.)

$n = 1$: Then W is an axiom. Case 1 is the transitivity axiom, case 2 includes the right and left multiplication axioms, and case 3 includes the identity, associativity and semi-Thue axioms.

$n > 1$: I. Suppose that the nth step is obtained from an earlier one by substitution. That substitution into each case results again in that case is clear from their forms and from Lemma 9.

II. Suppose that the nth step is obtained from two previous steps by use of modus ponens. By the induction hypothesis both the major and minor premisses fall into case 1, 2, or 3. Since case 3 covers all theorems of form iii of Lemma 7, by Lemma 6 and the induction hypothesis, the minor premiss must fall in case 3. That is, the minor premiss is $[U \supset V]$ where U and V have n Thue factors as described by the conditions of case 3. Again by Lemma 6 we may assume that the major premiss falls in case 1 or case 2.

1. Let the major premiss be a substitution instance of the transitivity axiom, i.e., case 1. The result of modus ponens clearly falls in case 2b, since as we have just remarked, the minor premiss, here $[\sim X_1 \supset \sim X_2]$, falls in case 3.

2a. Let the major premiss be a substitution instance of the right multiplication axiom, say $[U \supset V \supset \cdot\ UX \supset VX]$. Let X have Thue factorization $U_{n+1}, \ldots, U_m$. Then the result of modus ponens is $\vdash_{P_T} [UX \supset VX]$. If U_n is $R^A_{n,a}$ and V_n is $R^A_{n,b}$ and U_{n+1} is R^A, then each pair $U_n U_{n+1}$ and $V_n U_{n+1}$ collapses to a single maximal subregular conjunct as described in the proof of Lemma 9. If there is no collapse, $\vdash_{P_T} [UX \supset VX]$ falls in case 3 because $[U \supset V]$ does. And if there is a collapse, $[U \supset V]$ in case 3 implies that $<R_{n,a}> \vdash_T <R_{n,b}>$ and so $<R_{n,a}R> \vdash_T <R_{n,b}R>$ by rule (iii). Then again $\vdash_{P_T} [UX \supset VX]$ falls in case 3. The argument follows in a similar fashion for the major premiss a substitution instance of the left multiplication axiom.

2b. If the major premiss is 2b, then the minor premiss, $[U \supset V]$, is $[\sim X_2 \supset \sim X_3]$. Case 2b further requires that $[\sim X_1 \supset \sim X_2]$ be in case 3. Consequently, since $\sim X_2$ is the conjunction U, with Thue factorization $U_1, \ldots, U_n$, $\sim X_1$ must be a conjunction Y with Thue factorization $Y_1, \ldots, Y_n$ such that $[Y \supset U]$ falls in case 3. The result of modus ponens is then $\vdash_{\overline{P_T}} [Y \supset V]$. To see that this result falls in case 3, note that $U_i = V_i$ is a minimal conjunct iff $Y_i = U_i$ is, and U_i, V_i are maximal subregular conjuncts, $R^A_{i,a}$ and $R^A_{i,b}$, iff Y_i and U_i are also maximal subregular conjuncts $R^A_{i,c}$ and $R^A_{i,a}$ (all for the same A). Since $[U \supset V]$ and $[Y \supset U]$ fall in case 3, $<R_{i,a}> \vdash_{\overline{\mathbf{T}}} <R_{i,b}>$ and $<R_{i,c}> \vdash_{\overline{T}} <R_{i,a}>$. Hence, by rule (v), $<R_{i,c}> \vdash_{\overline{T}} <R_{i,b}>$, and so $\vdash_{\overline{P_T}} [Y \supset V]$ also falls in case 3.

$\Leftarrow$: Cases 1 and 2a are obvious since they require W to be an axiom or a substitution instance thereof. For W as specified in case 3, the proof is by induction on, n, the number of Thue factors in U equal to the number in V. $n = 1$: Case A: Suppose $<R_{1,a}> \vdash_{\overline{T}} <R_{1,b}>$, then by Lemma 5 and substitution $\vdash_{\overline{P_T}} [R^A_{1,a} \supset R^A_{1,b}]$. Case B: Suppose $U = V$ is a minimal conjunct. Then $\vdash_{\overline{P_T}} [U \supset V]$ by substitution into the identity axiom. Show that this part of the theorem holds for $n = k$ on the assumption that it holds for $n = k - 1$: Let U have Thue factorization $U_1, \ldots, U_k$ and V have Thue factorization $V_1, \ldots, V_k$. Let $\dot{U}$ be any wff with Thue factorization $U_1, \ldots, U_{k-1}$ and $\dot{V}$ be any wff with Thue factorization $V_1, \ldots, V_{k-1}$. Since $[U \supset V]$ falls in case 3, then so does $[\dot{U} \supset \dot{V}]$. But then, by the induction hypothesis, $\vdash_{\overline{P_T}} [\dot{U} \supset \dot{V}]$. Further, since U_n and V_n are Thue factors in U and V where $[U \supset V]$ falls in case 3, either $U_n = V_n$ is a minimal conjunct, or U_n is $R^{A_n}_{n,a}$, and V_n is $R^{A_n}_{n,b}$ and $<R_{n,a}> \vdash_{\overline{T}} <R_{n,b}>$. So.

(1)	$\vdash_{\overline{P_T}} [\dot{U} \supset \dot{V}]$	(induc. hyp.)
(2)	$\vdash_{\overline{P_T}} [U_n \supset V_n]$	(id. or Lemma 5 and substitution)
(3)	$\vdash_{\overline{P_T}} [\dot{U}U_n \supset \dot{V}U_n]$	(r.m. and m.p.)
(4)	$\vdash_{\overline{P_T}} [\dot{V}U_n \supset \dot{V}V_n]$	(1.m. and m.p.)
(5)	$\vdash_{\overline{P_T}} [\dot{U}U_n \supset \dot{V}V_n]$	(Lemma 3)
(6)	$\vdash_{\overline{P_T}} [U \supset V]$	(Cor. 2)

We finish the proof of the theorem by showing that if W is as specified in case 2b, then $\vdash_{\overline{P_T}} W$. In this case we have that $[\sim X_1 \supset \sim X_2]$ falls in case 3, so by the proof just above we have $\vdash_{\overline{P_T}} [\sim X_1 \supset \sim X_2]$. Further, by substituting into the transitivity axiom we have $\vdash_{\overline{P_T}} [\sim X_1 \supset \sim X_2] \supset . [\sim X_2 \supset \sim X_3] \supset [\sim X_1 \supset \sim X_3]$. So by modus ponens, $\vdash_{\overline{P_T}} [\sim X_2 \supset \sim X_3] \supset [\sim X_1 \supset \sim X_3]$.

The Post-Lineal Theorems for arbitrary recursively enumerable degrees of unsolvability

In this section, the Post-Lineal theorems for recursively enumerable degrees of unsolvability are stated and proved. For the first of these we need Boone's theorem and the fact that P_T, as constructed, preserves the degree of $\mathbf{T}$. However, it is of interest to note that neither the second nor the third of our analogues of the Post-Lineal theorems uses the fact that P_T has the same degree of unsolvability as $\mathbf{T}$. And, indeed, Boone's theorem is not actually required although we use it for the sake of convenience.[7] However, we do use the fact that the structure of the theorems of P_{T} is known from Theorem 2; in particular, for U and V in T, $U \vdash_{\mathsf{T}} V$ iff $\vdash_{P_T} [U' \supset V']$.

Boone's Theorem[8]*: For any recursively enumerable degree of unsolvability, D, there exists a Thue system, $\mathsf{T}(D)$, such that the word problem for $\mathsf{T}(D)$ is of degree D.*

Theorem 3:[9] (Analogue of the first Post-Lineal Theorem) *For any recursively enumerable degree of unsolvability, D, there exists a partial propositional calculus, $P_{\mathsf{T}}(D)$, whose decision problem is of degree D.*

Proof: By Boone's theorem, for any recursively enumerable degree of unsolvability, D, we have a Thue system $\mathsf{T}(D)$. (This may equally well be considered as a semi-Thue system.) Using it to start with, we construct $P_{\mathsf{T}(D)}$ in the manner we have described in the previous section. First notice that by case 3 of Theorem 2, $U \vdash_{\mathsf{T}(D)} V$ iff $\vdash_{P_{\mathsf{T}(D)}} [U' \supset V']$ and so the word problem for $\mathsf{T}(D)$ is one-one reducible, as defined by Post,[10] to the decision problem for $P_{\mathsf{T}(D)}$. To see that the decision problem for $P_{\mathsf{T}(D)}$ reduces to the word problem for $\mathsf{T}(D)$, we consider the form of theorems as described in Theorem 2. Let W be a wff of $P_{\mathsf{T}(D)}$. Then, to decide whether or not it is a theorem one checks first to see if it is described by case 1. If so, then $\vdash_{P_{\mathsf{T}(D)}} W$; if not, one tries 2b. Either (a) W is of the form $[[\sim X_2 \supset \sim X_3] \supset [\sim X_1 \supset \sim X_3]]$ and so one must check to see if $[\sim X_1 \supset \sim X_2]$ is in case 3, or (b) W is not of form $[[\sim X_2 \supset \sim X_3] \supset [\sim X_1 \supset \sim X_3]]$ and so W cannot be a theorem according to 2b and one must check to see if it is in case 3. If (a) holds, let Y be $[\sim X_1 \supset \sim X_2]$, and if (b) holds, let Y be W. For Y to be in case 3, it must be of the form $[\sim Y_1 \supset \sim Y_2]$ where $\sim Y_1$ and and $\sim Y_2$ each have the same number of matching Thue factors as described in case 3. If these conditions are not met, then Y is not a theorem. If they are met, then let the m pairs of Thue factors which are maximal subregular conjuncts be called $R^{A_i}_{i,a}$ and $R^{A_i}_{i,b}$, $i = 1, 2, \ldots, m$. Then $\vdash_{P_{\mathsf{T}(D)}} Y$ iff $\langle R_{i,a} \rangle \vdash_{\mathsf{T}(D)} \langle R_{i,b} \rangle$. And so, the decision problem for $P_{\mathsf{T}(D)}$ reduces to the word problem for $\mathsf{T}(D)$. Further, since we have no bound on the number of pairs of maximal subregular conjuncts in any given wff in case 3, the reduction is by unbounded truth tables as defined by Post.[10]

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7. We will indicate the simplification when we prove those theorems. See footnote (12).

8. W. W. Boone, "Partial Results regarding word problems and recursively enumerable degrees of unsolvability", BAMS, vol. 68 (1962), pp. 616-623.

9. A stronger version of this theorem, $\supset$ the unique connective, was proved in 1963 by M. D. Gladstone of the University of Bristol, England. His paper will appear in the *Transactions of the American Mathemathical Society*.

10. E. L. Post, "Recursively enumerable sets of positive integers and their decision problems", *BAMS*, vol. 50 (1944) pp. 296-7 and pp. 299-301.

Definition 8: Let P_1 and P_2 be partial propositional calculi. We say that P_1 *coincides with* P_2 if they have exactly the same theorems.

Definition 9: Let C be a class of partial propositional calculi and F a single, specified partial propositional calculus. Then, the problem of determining of an arbitrary member of C whether or not it coincides with F is called the *class coincidence problem for C relative to F.*

Definition 10: Let C be a class of partial propositional calculi, then the problem of determining of an arbitrary member of C whether or not it is complete is called the *class completeness problem for C.*

Definition 11: Let $\mathbf{T}$ be any semi-Thue system. Let $F_{\mathbf{T}}$ be a partial propositional calculus which includes among its theorems the axioms of $P_{\mathbf{T}}$ (as described in the last section) and the wff $[p \supset [q \supset p]]$. We call such an $F_{\mathbf{T}}$, a $P_{\mathbf{T}}$-*fragment*,

Theorem 4: *For any recursively enumerable degree, D, there is a semi-Thue system,* $\mathbf{T}$, *such that for each* $P_{\mathbf{T}}$*-fragment,* $F_{\mathbf{T}}$, *there is a class* $C_{F_{\mathbf{T}}}$ *such that the class coincidence problem for* $C_{F_{\mathbf{T}}}$, *relative to* $F_{\mathbf{T}}$ *is of degree D.*

Proof: Let $\mathbf{T}$ be a semi-Thue system of degree D and construct $P_{\mathbf{T}}$ as described in the last section. Let $A_1, A_2, \ldots, A_n$ be the axioms of $F_{\mathbf{T}}$.[11] Suppose U and V are a pair of words of $\mathbf{T}$. Let $P(U,V)$ be the partial propositional calculus whose axioms are those of $P_{\mathbf{T}}$ plus the "coincidence" axioms $[[U' \supset V'] \supset A_i]$, $i = 1, \ldots, n$. We will see from the following lemma that the theorem is true for the class $C_{F_{\mathbf{T}}}$ whose members are all such $P(U,V)$ where U and V vary over all pairs of words of $\mathbf{T}$.[12]

Lemma 10: *$P(U,V)$ coincides with $F_{\mathbf{T}}$ iff $U \vdash_{\mathbf{T}} V$.*

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11. This aspect of the proof is not necessarily constructive as we may not be given the axioms of $F_{\mathbf{T}}$. However, in the three corollaries which follow, the axioms can be specifically given.

12. We do not, in fact, need all pairs of words of T. Boone proves his theorem by starting from the following result which is easily inferred from Post ("Recursive Unsolvability of a Problem of Thue," *JSL*, vol. 12(1947) pp. 1-11) and Kleene (chapter 13 of *Introduction to Metamathematics*): *For any recursively enumerable set, S, (and therefore, recursively enumerable degree, D), there is a semi-Thue system,* $\mathbf{T}_1$, *such that* $n \in S$ *iff* $hs_1^{n+1}q_1h \vdash_{\mathbf{T}_1} hqh$, *where* h, s_1, q_1, *and* q *are among the specified letters of the alphabet of* $\mathbf{T}_1$'. (Lemma 1, see (8)). Call such word pairs, $hs_1^{n+1}q_1h, hqh$, the "unsolvable pairs." He then constructs from $\mathbf{T}_1$ a Thue system $\mathbf{T}_4$ and shows that its word problem reduces to the word problem for the "unsolvable pairs." In the analogue of the first Post-Lineal theorem it was necessary to consider the Thue system as a whole and so Boone's theorem was required. In this and Theorem 5 we need only some infinite set of word pairs in a Thue system such that the problem of determining of an arbitrary pair whether or not they are equivalent is of degree D. For such purposes we could take the Thue system, $\mathbf{T}(D)$, corresponding to each D, and take as the required infinite set of word pairs in $\mathbf{T}(D)$ the "unsolvable pairs."

Proof: I. Suppose $U \vdash_{\mathsf{T}} V$. We first show that $\vdash_{\overline{P(U,V)}} W$ implies $\vdash_{\overline{F_{\mathsf{T}}}} W$. The axioms of P_{T} are theorems of F_{T} by definition and the coincidence axioms are theorems of F_{T} because

$\vdash_{\overline{F_{\mathsf{T}}}} [p \supset [q \supset p]]$	(def. of F_{T})
$\vdash_{\overline{F_{\mathsf{T}}}} [A_i \supset [[U' \supset V'] \supset A_i]$	(subst.)
$\vdash_{\overline{F_{\mathsf{T}}}} A_i$	(axiom of F_{T})
$\vdash_{\overline{F_{\mathsf{T}}}} [[U' \supset V'] \supset A_i]$	(m.p.)

Conversely, we see that $\vdash_{\overline{F_{\mathsf{T}}}} W$ implies $\vdash_{\overline{P(U,V)}} W$. Since $U \vdash_{\mathsf{T}} V$, by Theorem 2, $\vdash_{\overline{P_{\mathsf{T}}}} [U' \supset V']$ so $\vdash_{\overline{P(U,V)}} [U' \supset V']$. Since $\vdash_{\overline{P(U,V)}} [[U' \supset V'] \supset A_i]$ is an axiom, $\vdash_{\overline{P(U,V)}} A_i$ by modus ponens. II. Supposing that not $U \vdash_{\mathsf{T}} V$, we now show that the theorems of $P(U,V)$ are exactly the theorems of P_{T}, the coincidence axioms and substitution instances thereof. Recall that the coincidence axioms are $[[U' \supset V'] \supset A_i]$ and that U' and V' each begin with a not-sign. Refer now to the forms of theorems in P_{T} as given in Lemma 7. Let U' be $\sim Y_1$ and V' be $\sim Y_2$. Then consider the following four forms:

form i: $[\sim X_1 \supset \sim X_2] \supset . [\sim X_3 \supset \sim X_4] \supset [\sim X_5 \supset \sim X_6]$
form ii: $[\sim X_1 \supset \sim X_2] \supset [\sim X_3 \supset \sim X_4]$
form iiia: $[\sim X_1 \supset \sim X_2]$ form iiib: $[X_1 \supset X_1]$
coincidence axioms: $[[\sim Y_1 \supset \sim Y_2] \supset A_i]$

It is clear by their form that the only possible uses of modus ponens among these, outside of P_{T}, depend on there being a first use of modus ponens with the minor premiss in form iiia and the major premiss a substitution instance of a coincidence axiom. For there to be such a first use of modus ponens, the theorem of P_{T} used as minor premiss would have to be $[U'^A \supset V'^A]$. But, by Theorem 2, not $\vdash_{\overline{P_{\mathsf{T}}}} [U'^A \supset V'^A]$ since by hypothesis not $U \vdash_{\mathsf{T}} V$. Hence, F_{T} and $P(U,V)$ do not coincide since, in particular, not $\vdash_{\overline{P(U,V)}} [p \supset [q \supset p]]$ although $\vdash_{\overline{F_{\mathsf{T}}}} [p \supset [q \supset p]]$. This completes the proof of Lemma 10 and hence of Theorem 4.

Corollary 1 *to Theorem* 4: (Analogue of the second Post-Lineal Theorem) *For any recursively enumerable degree of unsolvability, D, there exists a class of partial propositional calculi, C, such that the class completeness problem for C is of degree D.*

Proof: The complete propositional calculus includes all tautologies and is thus a P_{T}-fragment for any T.

One can easily show that for any T, the axioms of P_{T} are theorems in the minimal propositional calculus, M, and in the intuitionistic propositional calculus, I. Since $[p \supset [q \supset p]]$ is an axiom in each of these, it follows that each is a P_{T}-fragment for any T. So we have two further corollaries.

Corollary 2 *to Theorem* 4: *For any recursively enumerable degree of unsolvability, D, there exists a class of partial propositional calculi, C, such that the class coincidence problem for C relative to M is of degree D.*

Corollary 3 *to Theorem* 4: *For any recursively enumerable degree of unsolvability, D, there exists a class of partial propositional calculi, C, such that the class coincidence problem for C relative to I is of degree D.*

Definition 12: Let C be a class of partial propositional calculi, then the problem of determining of an arbitrary member whether or not the set of axioms by which it is given is an independent set is called the *class independence problem for C*.

Definition 13: We say that an occurrence of an implies-sign is *superior* if it does not occur in the scope of a not-sign.

Lemma 11: *If T is a semi-Thue system whose operation rules are independent and contain no empty words, then the axioms of* P_T *are independent.*

Proof: To prove this lemma we will consider each subsystem obtained from P_T by removing one axiom. For each such subsystem we will look at the forms of wffs (schemata) which might be theorems in the subsystem so as to ascertain whether or not the deleted axiom can be derived as a theorem from the others. This will be done with the aid of the accompanying chart.

	1	2	3	4
Trans.	$[\sim X_1 \supset \sim X_2] \supset . [\sim X_2 \supset \sim X_3]\ \ [\sim X_1 \supset \sim X_3]$	$[\sim X_2 \supset \sim X_3] \supset [\sim X_1 \supset \sim X_3]$	$\sim X_1 \supset \sim X_3$	$\sim \cancel{X_3}$
R.M.	$\sim X_1 \supset \sim X_2 \supset . [\sim X_1] X_3 \supset [\sim X_2] X_3$	$[\sim X_1] X_3 \supset [\sim X_2] X_3$	$\cancel{[\sim X_2] X_3}$	
L.M.	$\sim X_1 \supset \sim X_2 \supset . X_3 [\sim X_1] \supset X_3 [\sim X_2]$	$X_3 [\sim X_1] \supset X_3 [\sim X_2]$	$\cancel{X_3 [\sim X_2]}$	
1^{st} Assoc.	$X_1 [X_2 X_3] \supset [X_1 X_2] X_3$	$\cancel{[X_1 X_2] X_3}$	–	
2^{nd} Assoc.	$[X_1 X_2] X_3 \supset X_1 [X_2 X_3]$	$\cancel{X_1 [X_2 X_3]}$	–	
semi-Thue	$\mathbf{S}_{A}^{P}\, G_i' \supset \bar{G}_i' \vert$	$\cancel{\mathbf{S}_{A}^{P}\, \bar{G}_i' \vert}$	–	

In the chart the axioms as schemata (except identity) appear in the first column. In each row of the second column is the consequence of a use of modus ponens if the first column entry of that row is taken as major premiss. The third column is the consequence of the second and the fourth of the third in the same way. Thus these entries include all schemata which might have a theorem of P_T as an instance. By Theorem 2 we know immediately that certain of these cannot yield theorems in P_T and so these have been crossed out. Recall also that every schema of the form $[UV]$ is actually of the form $\sim[U \supset \sim V]$ and that, by definition, both G_i' and $\bar{G}_i'$ are associated to the left.

The schema for the identity axiom has not been included in the chart since no other axiom is an instance of it, and, further, the result of modus ponens with an instance of it as major premiss produces nothing new. Since it is also true that the identity axiom, $[p \supset p]$, is neither a substitution instance of any other axiom nor a consequence thereof, it is independent.

To prove each of the other axioms independent we, in effect, cross out the row of the chart in which the axiom in question appears and then look at the remaining entries to see if that axiom could have been derived as a theorem from the others.

The transitivity axiom,

$$[\sim p \supset \sim q] \supset . [\sim q \supset \sim r] \supset [\sim p \supset \sim r],$$

is independent since no entry in the table has five superior implies-signs.

The right multiplication axiom,

$$\sim p \supset \sim q \supset . [\sim p]r \supset [\sim q]r,$$

is independent: In the chart we notice that the only other entries with exactly three superior implies-signs are the left multiplication axiom schema and the second column result from the transitivity axiom schema. Consider the unabbreviated forms of the consequences of both the left multiplication axiom schema and the right multiplication axiom:

$$\sim [X_3 \supset \sim\sim X_1] \supset \sim [X_3 \supset \sim\sim X_2]$$

$$\sim [\sim p \supset \sim r] \supset \sim [\sim q \supset \sim r].$$

They cannot be the same. Nor can the right multiplication axiom be of the form $[[\sim X_2 \supset \sim X_3] \supset [\sim X_1 \supset \sim X_3]]$ since, in particular, here $\sim X_3$ cannot be both $\sim q$ and $[\sim q]r$. That the left multiplication axiom is also independent follows from similar arguments.

The first associativity axiom, $p[qr] \supset [pq]r$, is independent: We consider all entries in the chart which have exactly one superior implies-sign. The first associativity axiom certainly is not an instance of the second associativity axiom schema, nor can it be a semi-Thue axiom schema since in these each $G_i^!$ and $\overline{G}_i^!$ is left associated. In the second column right multiplication result the right conjunct on each side, X_3, is the same, and so the first associativity axiom cannot fit there. Similarly, it cannot fit into the second column result from the left multiplication axiom schema since this has the same left conjunct, X_3, on both sides. To show that the first associativity axiom could not have been derived from the transitivity axiom schema by two uses of modus ponens is somewhat more complicated. Suppose, on the contrary, that the third column transitivity result $[\sim X_1 \supset \sim X_3]$ is $[p[qr] \supset [pq]r]$, and further that we are examining the shortest proof that this is a theorem. For this to be a theorem in the system P_{T} without the first associativity axiom, both (a) $[\sim X_1 \supset \sim X_2]$ and (b) $[\sim X_2 \supset \sim X_3]$ must also be theorems. That is, (a) is the theorem used as minor premiss to derive the second column result from the first, and (b) is the theorem used as minor premiss to derive the third column result from the second. Since $[\sim X_1]$ is $p[qr]$ and $[\sim X_3]$ is $[pq]r$, we must find a $[\sim X_2]$ so that both (a) and (b) are theorems. To see how (a) might be a theorem consider the identity axiom schema and all the entries in the chart with exactly one superior implies-sign. A brief examination of these indicates that there are three ways in which (a) might be a theorem: (1) it is an instance of the identity schema, (2) it is a second column left multiplication result, or (3) it is a third column transitivity result. We consider each in turn. (1). If (a) is an instance of the identity schema, then $[\sim X_2]$ is also $p[qr]$ and so (b) is $[p[qr] \supset [pq]r]$. But this (b) cannot already be a theorem since we are considering the shortest possible proof. (2). If (a) is a theorem as a result of modus ponens with the left multiplication axiom schema as major premiss, then (a) is $[p[qr] \supset p[\sim Y]]$, for some wff Y. But for this modus ponens to occur, it is also necessary that $[qr \supset \sim Y]$ be the theorem acting as minor premiss. However, if $[qr \supset \sim Y]$ is a theorem, Theorem 2 implies that $[\sim Y]$ is also $[qr]$, and so again it follows that (b) is $[p[qr] \supset [pq]r]$. (3).. Let (a) and (b) be redesignated as $[p[qr] \supset \sim V_1]$ and $[\sim V_1 \supset [pq]r]$ respectively. If (a) is itself a third column transitivity result, then in order for it to be a theorem there must be an (a'), $[p[qr] \supset \sim V_2]$, and (b'), $[\sim V_2 \supset \sim V_1]$, which are the theorems used as minor premisses in obtaining (a) from the transitivity schema. Again, (a') can then be a theorem only according to one of the three possibilities given above. Eventually, for some n, (a$^{(n)}$) must be a theorem by (1) or (2) rather than (3). If (a$^{(n)}$) is $[p[qr] \supset \sim V_{n+1}]$ and, as in (1), is an instance of the identity schema, then $[\sim V_{n+1}]$ is $p[qr]$. But then (b$^{(n)}$) is $[p[qr] \supset \sim V_n]$ which is the same as (a$^{(n-1)}$), etc. Suppose, as in (2), $[p[qr] \supset \sim V_{n+1}]$ is a left multiplication result. Then by the same arguments as those given for (2) above, this immediately reduces to the case for ($a^{(n)}$) an instance of the identity schema. The arguments showing the independence of the second associativity axiom follow in a similar way.

Consider the semi-Thue system, T_j, obtained from T by deleting the jth operation rule, $G_j \rightarrow \overline{G}_j$. Theorem 2 applies to any arbitrary semi-Thue system, so that $U \vdash_{\mathsf{T}_j} V$ iff $\vdash_{P_{\mathsf{T}_j}} [U' \supset V']$. Since not $G_j \vdash_{\mathsf{T}_j} \overline{G}_j$, it follows that not $\vdash_{P_{\mathsf{T}_j}} G_j' \supset \overline{G}_j'$. Hence the semi-Thue axioms are also independent.

Modus ponens is independent since there are theorems of P_{T} which are not substitution instances of axioms, for example, case 2b of Theorem 2. And, of course, substitution is independent since without it no theorem could be longer than the longest axiom.

Theorem 5: (Analogue of the third Post-Lineal Theorem) *For any recursively enumerable degree of unsolvability, D, there exists a class of partial propositional calculi, C, whose class independence problem is of degree D.*

Proof: For each D choose some semi-Thue system T_D whose word problem is of degree D. Reduce the set of operation rules of the semi-Thue system to an independent set. That is, take the first operation rule and if it is not independent, discard it, and so on with the second, third, etc.[13] Then for U and V words of T_D, define $P(U, V)$ to be the partial propositional calculus whose axioms are the axioms of P_{T} plus the "independence" axioms $\sim\sim[p \supset p]$ and $[[U' \supset V'] \supset \sim\sim[p \supset p]]$. Notice that not $\vdash_{P_{\mathsf{T}}} \sim\sim[p \supset p]$ and not $\vdash_{P_{\mathsf{T}}} [U' \supset V' \supset . \sim\sim[p \supset p]]$ by Theorem 2. Further, the axioms of P_{T} are independent in P_{T} by construction and Lemma 11, and the independence axioms are independent of each other. The independence axioms cannot be used in modus ponens with each other. The only theorem of P_{T} that could be used in modus ponens with a substitution instance of an

13. This procedure is, of course, highly non-effective However, Boone conjectures that the set of axioms he has given in his theorem—see footnote (8)—could be shown to be independent.

independence axiom is $[U'^A \supset V'^A]$. But by Theorem 2, $\vdash_{P_{\mathsf{T}}} [U'^A \supset V'^A]$ iff $U \vdash_{\mathsf{T}} V$. Hence, if $U \vdash_{\mathsf{T}} V$, $P(U,V)$ can be axiomatized by the axioms of P_{T} and $[U' \supset V' \supset . \sim\sim[p \supset p]]$, and to take $\sim\sim[p \supset p]$ as an axiom would be redundant. If not $U \vdash_{\mathsf{T}} V$, then the theorems of $P(U,V)$ are exactly those of P_{T} plus the independence axioms and substitution instances thereof. (See part II of the proof of Lemma 10.) In this case, then, $\sim\sim[p \supset p]$ is also required as an axiom in order to axiomatize $P(U,V)$. Thus, since for any recursively enumerable degree of unsolvability, D, there is a Thue system, T_D, whose word problem is of degree D, the theorem holds for the class C whose members are $P(U,V)$ where U and V vary over all pairs[12] of words in T_D.

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