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AN AUGMENTED MODAL LOGIC

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The purpose of this paper is to examine the extent to which an augmented modal logic may be used as the formalized meta-theory $\mathfrak{M}$ of a formal system $\mathcal{S}$. The conditions which $\mathcal{S}$ must satisfy are indicated by Roman numerals. Assumptions and theorems of $\mathfrak{M}$ are indicated by Arabic numerals. Comments and examples are enclosed in square brackets.

I Let $\mathcal{S}$ consist of the following: (i) a set **V** of symbols, $v^1, v^2, \ldots v^j, \ldots$, (the vocabulary of $\mathcal{S}$); (ii) formation rules; (iii) axioms; (iv) transformation rules.

II Let **F** be the set of formulae $f^1, f^2, \ldots f^j, \ldots$, of $\mathcal{S}$, where each f^i consists of a finite string of symbols v^i.

III Let the following sub-sets of **F** be selected as follows:

(i) A set **W** of wff selected recursively from **F** by the formation rules.
(ii) A set **A** (the axioms of $\mathcal{S}$) selected recursively from **W** (commonly by giving a finite list).
(iii) A set **P** of provable formulae; a set **D** of disprovable formulae, a set **I** of irresoluble formulae; selected (but not necessarily recursively) from **W** by the transformation rules. Let **P** contain **A**.

IV Let the operations required to select the various sub-sets of **F** be entirely formal. [By this we mean that they depend only on the physical characteristics--shape, position, etc.,--of the v^i].

V Let the sets **P**, **D** and **I** satisfy the following conditions:

(i) If, for a given j, f^j occurs in **P**, then for some definite symbol, v^k say, $v^k f^j$ occurs in **D**; and conversely, if f^j occurs in **D** then $v^k f^j$ occurs in **P**. [Commonly v^k will be the symbol '$\sim$' (interpretable as 'not') and we say that f^j and $v^k f^j$ is each the negation of the other. We leave it open at this stage whether **P** (or **D**) can contain both f^j and $v^k f^j$, for a given j].
(ii) If neither f^j nor $v^k f^j$ occurs in **P** (in which case, by (i) above, neither occurs in **D**) then both occurs in **I**.

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VI Let there be n different interpretations of $\mathcal{S}$, where an interpretation is defined by a set of rules (semantic rules) which specify (i) the ranges of such of the v^i as are variables of $\mathcal{S}$; (ii) the connective words and phrases for such of the v^i as are logical constants of $\mathcal{S}$; (iii) designating words and phrases for such of the v^i as are constants of $\mathcal{S}$. [It is understood that the v^i are classified as variables, logical constants and constants, by the formation rules. It is further understood that two interpretations are different if they differ in any particular at any part of the specification (i)-(iii)].

We say that the formal *system* $\mathcal{S}$, together with a set of rules of interpretation i, is a formal *language* $\mathcal{S}_i$ and is the *logic of* the *model* m_i, which model consists of the "objects" designated by the constants of $\mathcal{S}_i$ and the relations between these "objects". [The "objects" may be abstract: e.g. numbers]. We say further that the n sets of rules of interpretation define *a system of formal languages*, $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$, with a common syntax $\mathcal{S}$.

If we wish to consider the formulae $f^1, f^2, \ldots f^j, \ldots$, of $\mathcal{S}$ under interpretation 1, we represent them as, $f_1^1, f_1^2, \ldots, f_1^j, \ldots$; under interpretation 2, as $f_2^1, f_2^2, \ldots, f_2^j, \ldots$; and so on. If we simply wish to consider them *as* interpreted, and regard the choice of interpretation as arbitrary, we represent them with a letter-subscript, n,m or i. Thus $f_n^1, f_n^2, \ldots, f_n^j, \ldots$, represent formulae of $\mathcal{S}$ interpreted over some arbitrarily chosen model (i.e. with respect to some arbitrarily chosen formal language). [There may be no difference in *appearance* between a given formula f^j considered as an uninterpreted formula of $\mathcal{S}$ and considered as an interpreted formula f_n^j of $\mathcal{S}_n$. The logical difference is that it is significant to say of f_n^j that it is true, false or neither true nor false but not significant to say this of f^j. See VII and comments following].

VII Let the formulae of each formal language $\mathcal{S}_i$ be such that:

(i) It is significant to say of any arbitrarily chosen formula f_i^j that it is true, false or neither true nor false.

[In permitting the significance of the description 'neither true nor false' we have in mind the need to classify certain open interpreted formulae. Thus, suppose that $\mathcal{S}$ is formalized arithmetic, then the formula '$x > 10$' when interpreted over the standard numerical model might be described as neither true nor false. However, we do not wish to restrict the use of this description to open formulae (it may sometimes be useful to describe certain closed formulae of a given $\mathcal{S}_i$ as neither true nor false) nor do we wish to claim in general that all open formulae satisfy the description (it may sometimes be appropriate to describe open interpreted formulae such as '$2x + 2x = 4x$' as true). We therefore make the weakest possible assumption which we wish to hold for all $\mathcal{S}_i$ and allow that more restricted conditions may prevail in a particular $\mathcal{S}_i$; that is, for a given i, $\mathcal{S}_i$ may contain a law of excluded middle which applies to all closed formulae while all its open formulae are classified as neither true nor false.

The use of the description 'neither true nor false' is however limited in its application in that it is only significant of *interpreted* formulae. This is taken to be a consequence of the conditions I-IV. For since the formulae f^j

of $\mathcal{S}$ are strings of uninterpreted symbols, it will not be significant to describe them as true or as false. This being so, it will not be significant to describe them as neither true nor false. If 'not true and not false' were a significant description, so too would each of the descriptions 'not true' and 'not false' be significant and so, therefore, would 'true' and 'false' each be significant].

(ii) The formulae of P go over into true statements and the formulae of D into false statements.

[This is the assumption that $\mathcal{S}$ is plausible. The phrases 'true statement' are here used in a wide sense and in some cases would be more naturally expressed as 'valid formula' and 'invalid formula'. For example, if $\mathcal{S}$ is the sentential calculus, then that interpretation of it under which every theorem designates the universal class is not an interpretation which yields statements in the ordinary sense. We could in this instance turn each theorem into a statement by writing it as '$X = I$' where 'X' is an abbreviation of the original theorem and 'I' designates the universal class, but such manipulation may not always be possible. In view of IV the membership of P, D and I is not affected by specifying an interpretation. Hence, though it is significant to say of a given interpreted formula f^j_n that it is provable (since the classification effected by the transformation rules remains under the interpretation) the information content of 'f^j_n is provable' does not differ from that of 'f^j is provable'. Similarly with the descriptions 'disprovable' and 'irresoluble'].

We propose now to develop $\mathfrak{M}$ as a formal *language*. That is, $\mathfrak{M}$ is itself an interpretation of a formal system and is the logic of the modal defined in I-VII. Hence, some of the following conditions are both formal and interpretive. [E.g. (1) specifies part of the vocabulary of a formal system but also, since it specifies values for certain variables, it is a rule of interpretation]. We leave it open whether there are alternative interpretations of of the formal system of which $\mathfrak{M}$ is an interpretation.

(1) Let $a,b,c,\ldots$ be formulae-variables whose values are the f^j of $\mathcal{S}$. [All letters of the alphabet may be used excluding m to z. If more are required primes may be added].

(2) Let $\begin{Bmatrix} a_1, b_1, c_1, \ldots \\ a_2, b_2, c_2, \ldots \\ \vdots \end{Bmatrix}$ be formulae-variables with constant subscripts whose values are the $\begin{Bmatrix} f^j_1 \\ f^j_2 \\ \vdots \end{Bmatrix}$ of $\begin{Bmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \\ \vdots \end{Bmatrix}$

(3) Let $\begin{Bmatrix} a_m, b_m, c_m, \ldots \\ a_n, b_n, c_n, \ldots \\ a_\nu, b_\nu, c_\nu, \ldots \end{Bmatrix}$ be formulae-variables with variable subscripts whose values are the interpreted $f^1, f^2, \ldots f^j, \ldots$, of an arbitrarily chosen formal language. The values of the subscript variables are the natural numerals.

[If more subscript variables are required they may be chosen from such of the letters m to z which are not specified for other uses in the conditions which follow]

(4) Let P and T be undefined constant operators interpreted respectively as 'it is provable that' and 'it is true that'.

(5) Let the following be wff:

[Though a full vocabulary has not been specified it is taken for granted that all symbols which are either used or mentioned in the following rules are symbols of the vocabulary]

5.1 (a) Formulae-variables, (b) formulae-variables with constant subscripts, (c) formulae-variables with variable subscripts.

.2 Such formulae as can be formed from the formulae-variables, together with the logical constants of the sentential calculus, by applying the formation rules of the sentential calculus as if the formulae-variables were sentential variables. (e.g. $a \& b$, $a \rightarrow \sim b$).

.3 Formulae formed as in 5.2 by using formulae-variables with (constant subscripts, provided that any formula so formed bears the same subscript throughout. (e.g. $a_1 \& b_1$, but *not*, $a_1 \& b_2$)

.4 Formulae formed as in 5.3 by using formulae-variables with variable subscripts. (i.g. $a_n \& b_n$ but *not*, $a_n \& b_m$)

[If a formula is formed as in 5.3 or 5.4, then enclosed in brackets and the subscript deleted at each of its occurrences and written after the closing bracket, we shall regard the result as an alternative way of writing the original formula. (e.g. $a_n \& b_n$ is not different from $(a \& b)_n$]

.5 Formulae which consist of P followed by a wff defined in 5.1 to 5.4. (e.g. Pa, $P(a \& b)$, Pa_n)

[A wff which consists of P followed by a wff of 5.3 or 5.4 does not differ in meaning from that wff which is identical in form except that it contains no subscripts. (e.g. Pa_n does not differ from Pa). See comments following VII(i).]

.6 Formulae which consist of T followed by a wff defined in 5.1(b), 5.1(c), 5.3 and 5.4. (e.g. Ta_1, $T(a \& b)_n$, but *not* $T(a \& b)$).

[See VII(i) and comments following]. We shall call formulae formed as in 5.5 and 5.6, *qualified formulae*; those defined in 5.1 to 5.4, unqualified. With regard to the interpretation of formulae formed as in 5.6, we shall sometimes read '$T\alpha_\nu$', where α is an appropriate wff and ν a subscript, as 'it is true that α with respect to (or in) a formal language $\mathcal{S}_\nu$' and sometimes as 'it is true that α over the modal ν'. We regard these two interpretations as differing only stylistically.

.7 Such formulae as can be formed by treating both qualified formulae and unqualified formulae as if they were wff of the sentential calculus and applying the formation rules of that calculus, provided that the resultant formula contains no part which is excluded by the provisos of 5.3 and 5.4.

[Thus: Ta_n and b_n are wff. Hence, by 5.7, so is $Ta_n \& b_n$. But if we now form a sentential combination of this with the wff c_m say $Ta_n \& b_n \& c_m$, we obtain an expression which is not a wff since the part $b_n \& c_m$ is excluded by the proviso on 5.4. We are nevertheless not precluded by 5.7 from forming formulae which contain more than one subscript. Thus, e.g. $Ta_n \rightarrow a_m$ is a wff formed from Ta_n and a_m, and the excluding condition on 5.7 does not apply to this since it refers only to combinations of unqualified formulae which contain more than one subscript. In general, the formulae excluded by 5.7 are such as are formed from several wff but which, when formed, may be written in the form: qualified wff + unqualified wff ('+' indicates some logical constant), where the unqualified part contravenes one of the provisos in 5.3 and 5.4]

.8 Formulae which are formed from qualified formulae by quantifying some or all of the formulae-variables. (e.g. $(a)Pa$, $(Ea)Ta_n$)

.9 Formulae which are formed from the formulae of 5.7 by quantifying some or all of the formulae-variables in the qualified parts of such formulae. (e.g. $(a)Pa \rightarrow b$)

.10 Formulae which are formed by quantifying any wff containing subscript variables with respect to some or all of those variables. (e.g. $(n)Ta_n$, $(n)(a)Ta_n$, $(Em)(n)(Ta_n \rightarrow a_m)$)

[It follows from 5.5 to 5.10 that quantifiers never occur in a wff on the right-hand side of an operator, and that unqualified formulae, or the unqualified parts of formulae, are not quantified with respect to the formulae-variables in them. In general, formulae such as, $a_1 \& a_2$, $a_n \& b_m$, $P(a)a$ and $(a)(b)(a \& b)$, are excluded from the set of wff because they present difficulties in interpretation. Thus, if we substitute values for the variable in $a_1 \& a_2$ we obtain, say, $f_1^j \& f_2^j$ and thereby express the fact that if f^j is interpreted in two different ways and the two statements so obtained are then conjoined, the result is a significant statement. Since, however, it is not a statement either of $\mathcal{S}_1$ or $\mathcal{S}_2$, or indeed of any other formal language in the system $\mathcal{S}_1, \mathcal{S}_2, \ldots \mathcal{S}_j$, it is difficult to know what kind of statement it is]

(6) Let the following interpretations be put on the following formulae (α stands for any appropriate wff, ν for any subscript):

$\sim P\alpha$;	it is not provable that α.
$P\sim\alpha$;	it is disprovable that α. [Hence, if α takes f^j as a value, $\sim\alpha$ takes $v^k f^j$ as a value]
$\sim P\sim\alpha$;	it is not disprovable that α.
$\sim T\alpha_\nu$;	it is not true that α wrt $\mathcal{S}_\nu$.
$T\sim\alpha_\nu$;	it is false that α wrt $\mathcal{S}_\nu$.
$\sim T\sim\alpha_\nu$;	it is not false that α wrt $\mathcal{S}_\nu$.

(7) Let the following be defined operators:

We say that a formula is,

7.1 *false wrt* $\mathcal{S}_\nu$ if it satisfies $F\alpha_\nu$, defined by $T\sim\alpha_\nu$;

.2 *trivial wrt* $\mathcal{S}_\nu$ if it satisfies $Triv\ \alpha_\nu$, defined by $\sim T\alpha_\nu \& \sim F\alpha_\nu$;

.3 *trivial* if it satisfies $Triv\ \alpha$, defined by $(\nu)Triv\ \alpha_\nu$;

.4 *satisfiable* if it satisfies $S\alpha$, defined by $(E_\nu)T\alpha_\nu$;

.5 *necessary* if it satisfies $N\alpha$, defined by $(\nu)T\alpha_\nu$;

.6 *impossible* if it satisfies $N{\sim}\alpha$, defined by $(\nu)F\alpha_\nu$;
.7 *possible* if it satisfies $M\alpha$, defined by ${\sim}N{\sim}\alpha$;
.8 *contingent* if it satisfies $C\alpha$, defined by ${\sim}N\alpha \,\&\, {\sim}N{\sim}\alpha$;

[A contingent formula is such that its truth-value status is dependent on (contingent on) the choice of a model]

.9 *synthetic wrt* $\mathcal{S}_\nu$ if it satisfies $Syn\alpha_\nu$, defined by $C\alpha \,\&\, (T\alpha_\nu \vee F\alpha_\nu)$;
.10 *synthetic* if it satisfies $Syn\alpha$, defined by $(\nu)Syn\alpha_\nu$;

[A synthetic formula is such that its truth-value status is contingent on the choice of a model but is truth-valued (non-trivial) over every model]

.11 *irresoluble* if it satisfies $I\alpha$; defined by ${\sim}P\alpha \,\&\, {\sim}P{\sim}\alpha$;
.12 *weak irresoluble* if it satisfies $Weak\ I\alpha$, defined by $I\alpha \,\&\, (S\alpha \vee S{\sim}\alpha)$;

[Weak irresoluble formulae are irresoluble, and true over some model or false over some model]
.13 *strong irresoluble* if it satisfies $Strong\ I\alpha$, defined by $I\alpha \,\&\, (N\alpha \vee N{\sim}\alpha)$;

[Strong irresoluble formulae are irresoluble, and true over every model or false over every model]

We say that the formal system $\mathcal{S}$ is,

.14 *incomplete* if $(E\alpha)I\alpha$;
.15 *deductively incomplete in the weak sense* if $(E\alpha)Weak I\alpha$;
.16 *deductively incomplete in the strong sense* if $(E\alpha)Strong I\alpha$;

[A further kind of incompleteness will be defined later. This is closely connected with the existence of irresoluble formulae which are also trivial.]

(8) Let the following be axioms of $\mathfrak{M}$:
8.1 $Pa \rightarrow Ta_n$ Provable formulae are true over an arbitrarily chosen model. See VII(ii).
.2 $Ta_n \rightarrow a_n$ True formulae of an arbitrarily chosen formal language may be asserted with respect to that language .
.3 $P(a \,\&\, b) \leftrightarrow Pa \,\&\, Pb$
.4 $T(a \,\&\, b)_n \leftrightarrow Ta_n \,\&\, Tb_n$

(9) Let $\mathfrak{M}$ contain a set of axiom schemes and rules for the sentential calculus and the predicate calculus of first order.

[The purpose of this assumption is to provide for the manipulation of the formulae of $\mathfrak{M}$ in accordance with the sentential and predicate laws and rules. Thus, we may wish to pass from 8.1 and 8.2 to $Pa \rightarrow a_n$, which is a move made in accordance with a rule derived from the sentential theorem-scheme $(A \rightarrow B \,\&\, B \rightarrow C) \rightarrow (A \rightarrow C)$. Occasionally, however, we may wish to assert as a theorem of $\mathfrak{M}$ a special case of a sentential or predicate theorem scheme. Thus, we may wish to assert $Pa \,\&\, Pb \rightarrow Pa$ (a special case of $A \& B \rightarrow A$) or $(a)Pa \rightarrow Pb$ (a special case of $(\eta)\Phi\eta \rightarrow \Phi\omega$) or $(n)Ta_n \rightarrow Ta_m$ (again a special case of $(\eta)\Phi\eta \rightarrow \Phi\omega$). When one or more steps in a proof depend upon a rule of the sentential or predicate calculus, the details will not be given but the move will be indicated by the phrase 'by SC' (by the sent. calc.) or by 'by PC' (by the pred. calc.). When a special case of an SC or PC theorem is required it will be indicated by 'a case of SC' or 'a case of PC'.

It should be noted that in allowing sentential combinations of unqualified formulae as wff (e.g. $a \& b$) and in allowing that certain of these are theorems of $\mathfrak{M}$ (e.g. $a \& b \rightarrow a$), we are thereby presupposing that $\mathcal{S}$ contains symbols which can be interpreted as the logical constants of the sentential calculus, and hence that $\mathcal{S}$ contains the sentential calculus. For since the formulae-variables of $\mathfrak{M}$ take the f^j of $\mathcal{S}$ as values, the theorem, $a \& b \rightarrow a$, say, of $\mathfrak{M}$ expresses the fact that a similar sentential relation holds between the formulae of $\mathcal{S}$. That is, formulae of the form $f^i v^p f^j v^t f^i$, where v^p and v^t are definite symbols of $\mathcal{S}$, are acceptable formulae of $\mathcal{S}$. If the formulae of $\mathfrak{M}$ were restricted to qualified formulae and sentential combinations of these, we should not be committed to this presupposition since the sentential connectives would then only be used in $\mathfrak{M}$ to combine statements *about* the formulae of $\mathcal{S}$, rather than the formulae of $\mathcal{S}$ themselves.

In allowing that certain of the theorems of $\mathfrak{M}$ are special cases of the theorems of the predicate calculus we are *not* thereby presupposing that $\mathcal{S}$ contains the predicate calculus since the quantifiers are only used within $\mathfrak{M}$ to make statements *about* the formulae or models of $\mathcal{S}$. On the other hand, there is nothing in the conditions laid down for $\mathcal{S}$ which precludes its containing the predicate calculus. That is, the formulae-variables of $\mathfrak{M}$ may take quantified formulae of $\mathcal{S}$ as values. However, since we do not permit the explicit use of quantifiers on the RHS of an operator, no cognisance is taken of the special characteristics of such of the f^j as are quantified. The theorems of $\mathfrak{M}$, that is, though they remain valid over quantified f^j, are valid for reasons which are independent of the quantification. (We cannot, for example, express the fact that $\mathcal{S}$ contains a rule of inference which permits the universal quantification of theorems which contain free variables: i.e. $Pa(x) \rightarrow P(x)a(x)$). If we wish to specify that $\mathcal{S}$ contains the predicate calculus, $\mathfrak{M}$ may be extended to express this. Such extension is indicated later.]

(10) Let the following be transformation rules of $\mathfrak{M}$ (in addition to those of (9)).

10.1 If a wff not containing subscripts is substituted for a formula-variable throughout a theorem, including those occurrences of formula-variables of the same shape but which bear subscripts, the result is a theorem provided it is a wff.

["Higher-order" modal expressions such as *NNa*, *PSa*, are not wff. Hence, the result of substituting say *Na* for *a* throughout 8.1, say, would not be a theorem since it would not be a wff]

.2 If a subscript variable is replaced by another, or by an arbitrarily chosen numeral, throughout a theorem, the result is a theorem.

.3 If α is a theorem containing no operators, no quantifiers and no subscripts, then $P\alpha$ and $N\alpha$ are likewise theorems. If α_ν is a theorem containing no operators and no quantifiers then $T\alpha_\nu$ is a theorem.

.4 If $\alpha \leftrightarrow \beta$ is a theorem which contains no operators, no quantifiers and no subscript variables, then $P\alpha \leftrightarrow P\beta$ and $N\alpha \leftrightarrow N\beta$ are likewise theorems. If $\alpha_\nu \leftrightarrow \beta_\nu$ is a theorem which contains no operators and no quantifiers then $T\alpha_\nu \leftrightarrow T\beta_\nu$ is a theorem.

[It should be noted that though the rules of (9) are rules of $\mathfrak{M}$, their use is sometimes restricted by the special characteristics of $\mathfrak{M}$. Thus, a theorem of $\mathfrak{M}$ which contains free formulae-variables may be universally quantified wrt those variables to yield a further theorem provided that the result is a wff; that is, provided that the original theorem was a quantified formula (5.8)]

(11) Following are some theorems of $\mathfrak{M}$.

11.1 $Pa \rightarrow Na$ (By PC universally quantify consequent of 8.1 wrt n. Result by 7.5)

.2 $Na \rightarrow Ta_n$ (A case of PC is $(m)Ta_m \rightarrow Ta_n$ (cf. comments following (9)). Result by 7.5)

.3 $Ta_n \rightarrow Sa$ (A case of PC is $Ta_n \rightarrow (Em)Ta_m$. Result by 7.4)

.4 $Na \rightarrow Sa$ (By SC from 11.2 and 11.4)

.5 $Na \rightarrow a_n$ (By SC from 11.2 and 8.2)

.6 $N(a\&b) \leftrightarrow Na\&Nb$ (By PC universally quantify 8.4 wrt n, distribute quantifier through '$\leftrightarrow$' and then '&'. Result by 7.5)

[In view of 11.5, 11.6, 10.3 and 10.4 (insofar as these refer to N) and the provision for SC in (9), it follows that von Wright's system M is a sub-logic of $\mathfrak{M}$ (in the sense that the axioms and rules of M are among the theorems and rules of $\mathfrak{M}$) with the variation that all unqualified formulae bear a subscript. See pp 84-85 of [1]. We shall call this sub-logic the system M_n. Theorems of $\mathfrak{M}$ which are theorems of M_n will be taken for granted without proof

.7 $Pa \rightarrow a_n$ (By SC from 8.1 and 8.2)

[In view of 11.7, 8.3, 10.3 and 10.4 (insofar as these refer to P) and the provision for SC in (9), it follows that a system structurally isomorphic to M_n is a sub-logic of $\mathfrak{M}$. This system will be called P_n (it is in fact an alternative interpretation of the formal system of which M_n is an interpretation). Theorems of P_n may be obtained by writing P for N throughout a M_n theorem. Proofs of such theorems will not be given. In view of 8.2, 8.4, 10.3 and 10.4 (insofar as these refer to T) and the provision for SC in (9), it follows that a system structurally isomorphic to M_n, with the variation that *all* formulae-variables bear subscripts, is a sub-logic of $\mathfrak{M}$. This system will be called T_n. Theorems of T_n may be obtained by writing T for N throughout a M_n theorem and by adding the same subscript to each formula-variable in the theorem.]

.8(1) $Fa_n \rightarrow {\sim} Ta_n$
(2) $P{\sim}a \rightarrow {\sim}Pa$ $\Big\}$ (Each obtained from the M_n theorem $N{\sim}a \rightarrow {\sim}Na$. Result of (1) by 7.1)

.9 ${\sim}(En)(Ea)(Fa_n \& Ta_n)$ (By SC, ${\sim}(Fa_n \& Ta_n)$ from 11.8(1). By PC universally quantify wrt a and n. Result by PC)

[This is the law of non-contradiction for each formal language $\mathcal{S}_n$]

.10 $\sim(Ea)(P{\sim}a \,\&\, Pa)$ (Proof similar to 11.9)

[This expresses the fact that $\mathcal{S}$ is simply consistent. It can be proved directly from the axiom 8.1 and the theorem 11.8(1), thus indicating that the simple consistency follows from the assumption of plausibility. Alternatively, $\mathcal{S}$ is proved to be simply consistent because we presuppose in 8.1 and 11.3 (which together by SC give $Pa \rightarrow (En)Ta_n$) the existence of at least one model of $\mathcal{S}$.]

.11 $P(a \rightarrow b) \,\&\, Pa \rightarrow Pb$ (A P_n theorem)

[This expresses the fact that detachment is a rule of inference of $\mathcal{S}$, and is a further indication that the sentential calculus is contained in $\mathcal{S}$. (See comments following (9))]

.12 $P(a \rightarrow b) \rightarrow (Pa \rightarrow Pb)$ (By SC from 11.11)

[The converse fails to hold. Thus, let $\mathcal{S}$ be the sentential calculus and let f^1 be p and f^2 be $p \,\&{\sim}p$. Then $Pf^1 \rightarrow Pf^2$ holds (since $p \,\&{\sim}p$ could be obtained from p by substitution if p were a theorem) but $(f^1 \rightarrow f^2)$ fails to hold (since $p \rightarrow p \,\&\, {\sim}p$ is not a theorem).

.13 $Pa \,\&\, (Pa \rightarrow Pb) \rightarrow P(a \rightarrow b)$ (A P_n theorem)

[Thus the converse of 11.12 holds under the condition that Pa. Loosely read, 11.13 is: if f^j is provable, and a proof of f^j implies a proof of f^k, then $f^j \rightarrow f^k$ is provable. Hence we can always establish in $\mathcal{S}$ a theorem of the form $f^j \rightarrow f^k$ whenever f^k can be obtained from f^j by some rule (e.g., substitution) provided f^j is a theorem. In view of this we may write $Pa \,\&\, (Pa \rightarrow Pb)$ as $a \vdash b$, and $P(a \rightarrow b)$ as $\vdash (a \rightarrow b)$. 11.13 is then: if $a \vdash b$ then $\vdash (a \rightarrow b)$, a deduction theorem for $\mathcal{S}$. In view of 11.12, $Pa \rightarrow Pb$ by itself cannot be written as $a \vdash b$; instead, it gives rise to counterfactuals of the sort: if f^j were provable, f^k would be. There is a similar difference between $P(a \leftrightarrow b)$ and $Pa \leftrightarrow Pb$. If the first holds of two formulae of $\mathcal{S}$, they are deductively equivalent but not necessarily equivalent.]

.14 $(Pa \rightarrow Pb) \,\&P{\sim}\, b \rightarrow {\sim}\, Pa$ (A P_n theorem)

[If f^j were provable then f^k would be, but f^k is disprovable, hence f^j is *unprovable*. We do not by such an argument establish that f^j is disprovable. For that we require the hypothesis $P(a \rightarrow b)$ in place of $Pa \rightarrow Pb$ (see 11.15)]

.15 $P(a \rightarrow b) \,\&\, P{\sim}b \rightarrow P \sim a$ (A P_n theorem)

[The *modus tollens* type of argument. For a discussion of the differences between 11.14 and 11.15 when applied to the predicate calculus see [2]]

.16 $Ta_n \vee Fa_n \vee Triva_n$ (A T_n theorem is $Ta_n \vee T{\sim}a_n \vee ({\sim}Ta_n \,\&\, \sim T{\sim}a_n)$. Result by 7.1 and 7.2)

[The law of excluded fourth (see VII(i)). We avoid being committed to the law of excluded middle by not accepting as an axiom the converse of 8.2]

The theorems 11.5 to 11.16 each contain only one of the operators P, N and T, and are essentially alternative interpretations of the von Wright system M. Certain others are of interest. In particular, T_n theorems such as $Ta_n \& Tb_n \rightarrow T(avb)_n$ can be interpreted as rules for the construction of truth-tables for the sentential formulae of $\mathcal{S}$. Hence the techniques of the decision procedure for such formulae are expressible within $\mathfrak{M}$.

We now consider theorems which relate two or more of the operators P, N and T. These are not alternative versions of the system M theorems.

.17 $Ca \rightarrow Ia$ (By SC from 11.1 we have $\sim Na \rightarrow \sim Pa$; also, by substituting $\sim a$ for a in this we have $\sim N \sim a \rightarrow \sim P \sim a$. Result by SC, 7.8 and 7.11)

[Formulae of $\mathcal{S}$ which are such that, when interpreted, their truth-value status is contingent on the choice of a model, are irresoluble]

.18 $Triva_n \rightarrow Ca$ (Proof similar to above using 11.2 in place of 11.1)

.19 $Triva_n \rightarrow Ia$ (By SC from 11.18 and 11.17)

[An interpreted formula of $\mathcal{S}$ which is trivial over an arbitrarily chosen model is irresoluble]

.20 $(En) Triva_n \rightarrow Ia$ (By PC from 11.19)

[A particularization of 11.19. A formula which is trivial over a specific model (rather than an arbitrarily chosen one) is irresoluble]

.21 $Sa \rightarrow Ma$ (By PC universally quantify 11.8(1) wrt n and use PC to obtain $(En)Fa_n \rightarrow (En) \sim Ta_n$; i.e., by PC and 7.1, $(En) T\sim a_n \rightarrow \sim (n) Ta_n$. By 7.4 and 7.5 this is $S \sim a \rightarrow \sim Na$. Result by subst. $\sim a$ for a in this and using 7.7)

.22 $Sa \& S \sim a \rightarrow Ca$ (In the proof of 11.21 we have, $S \sim a \rightarrow \sim Na$. Subst. $\sim a$ for a in this to obtain $Sa \rightarrow \sim N \sim a$. Result by SC and 7.8)

.23 $Sa \& S \sim a \rightarrow Ia$ (By SC from 11.17 and 11.22)

[This is, using 7.4, $(En) Ta_n \& (En) Fa_n \rightarrow Ia$, which expresses the fact that if a formula f^j of $\mathcal{S}$ is interpreted in two different ways (over two models) and is true under one interpretation, false under the other (in view of 11.9 the models must be different) then we may conclude that it is irresoluble. In general, the theorems 11.17, 11.19, 11.20 and 11.22 validate special cases of the general technique of demonstrating irresolubility by exhibiting a model or models of $\mathcal{S}$. (See e.g. [3])]

We now introduce an extra condition on $\mathcal{S}$ in order to consider a different technique for demonstrating irresolubility:

VIII Let $\mathcal{S}$ contain such formal apparatus as is required for constructing formulae which, under a given interpretation of $\mathcal{S}$, may be considered as referring to some syntactical properties of themselves when uninterpreted.

(12) Let α be any wff of $\mathfrak{M}$ which does not contain operators, quantifiers or subscripts. Let Φ be any operator and ν any subscript. Let 'A' be a constant relation-term to be interpreted as 'asserts that'. Let '$-$' be a constant such that if α takes f^j as a value then α takes the name of f^j as a value. [We assume that the formation rules are extended to permit the construction of such formulae as are required in this section and which involve the new constants]

12.1 Let the following be an axiom scheme of $\mathfrak{M}$:

$\bar{\alpha}_\nu A \Phi \alpha \rightarrow (\alpha_\nu \leftrightarrow \Phi\alpha)$

[This expresses the fact that an interpreted formula which makes some statement about its own formal (syntactical) properties is truly-functionally equivalent to the statement it makes. It is an extension of the fact that any statement is truth-functionally equivalent to itself]

Following are some theorems which depend on 12.1

.2 $(\bar{a}_n A \sim Pa) \rightarrow Ia$ (A P_n theorem is, $(a_n \leftrightarrow \sim Pa) \rightarrow \sim Pa \& \sim P \sim a$. A special case of 12.1 is, $(\bar{a}_n A \sim Pa) \rightarrow (a_n \leftrightarrow \sim Pa)$. Result by SC and 7.11)

[Thus if an interpreted formula f^j_n of $\mathcal{S}_n$ asserts that f^j is unprovable in $\mathcal{S}$, then f^j is irresoluble wrt $\mathcal{S}$. The Gödel condition, that this theorem only holds on the hypothesis that $\mathcal{S}$ is consistent, is embraced by the fact that according to 11.10 $\mathcal{S}$ is consistent]

.3 $(\bar{a}_n AP \sim a) \rightarrow Ia$ (A P_n theorem is $(a_n \leftrightarrow P \sim a) \rightarrow \sim Pa \& \sim P \sim a$. Remainder of proof similar to 12.2)

[If an interpreted f^j_n of $\mathcal{S}_n$ asserts that f^j is disprovable in $\mathcal{S}$, then f^j is irresoluble in $\mathcal{S}$. The irresolubility of other variants of the Gödel formula can be demonstrated in a similar way]

We now consider the truth-value status of formulae which satisfy 12.2 and 12.3.

.4 $(\bar{a}_n A \sim Pa) \rightarrow \sim Fa_n$ (A case of SC is, $((Pa \rightarrow a_n) \& (a_n \rightarrow \sim Fa_n)) \rightarrow ((a_n \leftrightarrow \sim Pa) \rightarrow \sim Fa_n)$. By SC we have $a_n \rightarrow \sim Fa_n$ from 8.2 using 7.1. Now detach antecedent using this and 11.7. Result by SC on consequent and a special case of 12.1)

[Hence, a formula f^j_n which asserts the unprovability of f^j is not false wrt $\mathcal{S}_n$. It may therefore be either true or trivial. To establish that it is not trivial, some supplementary argument is required (e.g. that f^j_n is closed and that $\mathcal{S}_n$ contains a restricted law of excluded middle which applies to all closed formulae (cf. comments following VII(i)). This is made explicit in 12.5]

.5 $(\bar{a}_n A \sim Pa) \& (Ta_n \vee Fa_n)) \rightarrow Ta_n$ (A special case of an SC theorem is, 12.4 $\rightarrow$ 12.5. Result by detaching 12.4)

[If f^j_n asserts the unprovability of f^j and is truth-valued wrt $\mathcal{S}_n$, then it is true wrt $\mathcal{S}_n$.]

.6 $(\bar{a}_n AP{\sim}a) \rightarrow {\sim}Ta_n$ (Proof similar to 12.4)

[If f_n^j asserts the disprovability of f^j then it is not true wrt $\mathcal{S}_n$. As above, that f_n^j is false rather than trivial depends on a supplementary argument.]

We now modify 12.1 slightly and write the same subscript on each occurrence of α. This enables us to consider the liar sentence.

.7 $(\bar{a}_n A{\sim}Ta_n) \rightarrow Triva_n$ (Proof similar to 12.2 using the corresponding T_n theorem)

[If f_n^j asserts that f_n^j is not true wrt $\mathcal{S}_n$, then it is neither true nor false wrt $\mathcal{S}_n$. Hence, if it can be shown that f_n^j in fact satisfies such conditions as are imposed on a restricted law of excluded middle (e.g. that it is closed) then we conclude that $\mathcal{S}_n$ is inconsistent; or, equivalently, that since $\mathcal{S}$ is consistent (11.10), it has no predicate which can be consistently interpreted as 'true'.]

We now consider the various kinds of incompleteness that $\mathcal{S}$ may have.

Suppose that f^j is an unprovable formula of $\mathcal{S}$ which is necessary then, since it has the modal status of a theorem, it could be taken as an additional axiom without affecting the number of models of $\mathcal{S}$. That it is not provable, therefore, is a defect of the system and for this reason we say that $\mathcal{S}$ is deductively incomplete in the strong sense (7.16). Suppose, now, that f^j is true over some models but not all (satisfiable but not necessary). In this case f^j could be taken as an additional axiom but the new system $\mathcal{S}'$ so formed would only have some models in common with those of $\mathcal{S}$, viz those over which f^j is true. Such an extension of $\mathcal{S}$ would be undesirable if the preferred or standard model were excluded (assuming that there is one) since f^j, not being true over this model, is not wanted as a theorem. Hence, if $\mathcal{S}$ is deductively incomplete in the weak sense (7.15) it may nevertheless be desirably incomplete and f^j desirably unprovable. In this case we say that f^j is *refutable wrt the preferred model*, and we represent this concept within $\mathfrak{M}$ by using the operator R_p. [Since refutability *wrt* the preferred model is closely connected with the purposes for which the system $\mathcal{S}$ is wanted, R_p may be interpreted as: it is pragmatically refutable that]. In general we may say that a formal system $\mathcal{S}'$, which is formed from $\mathcal{S}$ by the addition to f^j to the axiom set, interlocks with $\mathcal{S}$ if it has one or more models in common with $\mathcal{S}$; and f^j is undesirably unprovable wrt $\mathcal{S}$ if the two systems interlock over all the models of $\mathcal{S}$ or over the standard model, otherwise f^j is desirably unprovable. The question now arises whether any formal techniques within $\mathcal{S}$ are relevant to the determination of refutable formulae. First, suppose that $\mathcal{S}$ becomes inconsistent when f^j is taken as an additional axiom. Since each of the original theorems of $\mathcal{S}$ is true over every model, and since all interpreted forms of the new system $\mathcal{S}'$ formed by the addition of f^j are inconsistent (i.e. $\mathcal{S}'$ has no models), f^j must be incompatible with the criteria of truth for each $\mathcal{S}_n$. That is, f^j, when interpreted, is not true over any model of $\mathcal{S}$. Hence it is not true over the preferred model. Such a formula we say is *formally refutable*. We represent this concept within $\mathfrak{M}$ by using the operator R_f and we affirm $R_f a \rightarrow {\sim}Ta_n$.

This enables us to conclude $R_f a \rightarrow R_p a$. [For example, let $\mathcal{S}$ be the sentential calculus and let f^j be p. Then f^j is formally refutable, hence pragmatically refutable]. Formal refutability is not of course an exhaustive analysis of pragmatic refutability since an unprovable formula may not be wanted as a theorem even though it can be consistently taken as an additional axiom. This will be so if it is not true over the preferred model but is true over some other model. [For example, let $\mathcal{S}$ be the predicate calculus of first order and let f^j be $(Ex)fx \rightarrow (x)fx$. Then f^j is pragmatically refutable but not formally refutable.]

Let it now be supposed, however, that at least one formula f^j of $\mathcal{S}$ has been classified as refutable (either by demonstrating that it is formally refutable or by showing that it fails to satisfy the criterion of truth for formulae interpreted over the preferred model) and let it further be supposed that if f^k were a theorem, f^j would be; that is, f^j follows by rule from f^k. Then f^k is refutable. For if it were a theorem, or if it were taken as an additional axiom, f^j would be demonstrable and $\mathcal{S}$ would therefore no longer be interpretable over the preferred model. Hence we may affirm that $\mathcal{S}$ contains the *rule of refutation* which is expressed in $\mathfrak{M}$ by $(Pa \rightarrow Pb)$ & $R_p b \rightarrow R_p a$. By applying this rule to the known refutable formulae of $\mathcal{S}$, the refutability of other formulae is thus demonstrable. [For example, let $\mathcal{S}$ be the predicate calculus of first order, let f^j be $(Ex)fx \rightarrow (x)fx$ and let f^k be $(x)fx$. Then since we know that, for $\mathcal{S}$, $P(\bar{f}^k \rightarrow \bar{f}^j)$, we have by 11.12, $Pf^k \rightarrow Pf^j$; and since we also have $R_p f^j$, we conclude $R_p f^k$.]

We shall say that $\mathcal{S}$ is a *fully-exploited* formal system if it has an axiomatic theory of refutation as well as an axiomatic theory of affirmation. By this we mean that certain formulae of $\mathcal{S}$ are assumed to be refutable (or are shown to be refutable either by showing that they are formally refutable or that they are non-true over the preferred model) and the refutability of other formulae is then demonstrated by means of rules of refutation.

We now develop these points formally within $\mathfrak{M}$.

IX Let $\mathcal{S}$ be a fully-exploited formal system and let the logic of the preferred model be given by $\mathcal{S}_1$

(13) Let the following be further definitions of $\mathfrak{M}$.

[It is assumed in what follows that appropriate extensions have been made to the formation rules of $\mathfrak{M}$]

We say that a formula is,

13.1 *pragmatically refutable wrt* $\mathcal{S}$ if it satisfies $R_p\alpha$, defined by $\sim T\alpha_1$;

.2 *independent of the axioms of* $\mathcal{S}$ if it satisfies $Ind\,\alpha$, defined by $\sim R_f\,\alpha \,\&\, \sim R_f \sim \alpha$;

[An independent formula is such that either it or its negation may be taken as an additional axiom without inconsistency]

.3 *strongly-complete wrt* $\mathcal{S}$ if it satisfies $StrongComp\,\alpha$, defined by $R_f\,\alpha \,\&\, R_f \sim \alpha$;

[A strongly-complete formula is such that both it and its negation lead to inconsistency when taken (separately) as additional axioms.]

We say that the formal system $\mathcal{S}$ is,

.4 *strongly-complete* if all its irresoluble formulae are strongly-complete; i.e. if it satisfies, $(\alpha)(I\alpha \rightarrow R_f\,\alpha \,\&\, R_f \sim \alpha)$. [*cf* pp 42-43 of [4]]

.5 *deductively incomplete wrt the preferred model* if it contains an irresoluble formula which is truth-valued over the preferred model; i.e. if it satisfies, $(E\alpha)(I\alpha \,\&\, (T\alpha_1 \vee F\alpha_1))$.

[*Cf.* the definition of deductive incompleteness in the weak sense (7.16) which can be written, after defined operators are eliminated, as $(E\alpha)(I\alpha \& (E\nu)(T\alpha_\nu \vee F\alpha_\nu))$. Hence, $\mathcal{S}$ is deductively incomplete in the weak sense if one of its irresoluble formulae is truth-valued over some model; deductively incomplete wrt the standard model if, specifically, one of its irresoluble formulae is truth-valued over that model. If it is deductively incomplete wrt the standard model, it is deductively incomplete in the weak sense. The converse does not hold in general].

(14) Let the following be additional axioms of $\mathfrak{M}$.

14.1 $R_f\,a \rightarrow \sim Ta_n$

.2 $(Pa \rightarrow Pb) \,\&\, R_p b \rightarrow R_p a$

(15) Following are further theorems of $\mathfrak{M}$.

[It is assumed that the rules UI (universal instantiation) and EG (existential generalization) are rules of $\mathfrak{M}$ if these are not already included in (9)]

15.1 $R_f a \rightarrow \sim Na$ (By SC we have $\sim Ta_n \rightarrow \sim Na$ from 11.2. Result by SC from this and 14.1).

.2 $R_f a \rightarrow \sim Pa$ (By SC we have $\sim Na \rightarrow \sim Pa$ from 11.1. Result by SC from this and 15.1)

.3 $StrongCompa \rightarrow Ia$ (Subst. $\sim a$ for a in 15.2. Use this and 15.2 to obtain $R_f\,a \& R_f \sim a \rightarrow \sim Pa \,\&\, \sim P{\sim}a$ by SC. Result by 13.3 and 7.11)

[This indicates that a formula which is such that neither it nor its negation can be taken as an additional axiom without inconsistency, is irresoluble. This is a further technique for demonstrating irresolubility. According to 15.4 such formulae are trivial]

.4 $StrongCompa \rightarrow Triva_n$ (Subst. $\sim a$ for a in 14.1. Use this with 14.1 to get $R_f\,a \,\&\, R_f \sim a \rightarrow \sim Ta_n \,\&\, \sim T{\sim}a_n$ by SC. Result by 7.1, 7.2 and 13.3)

.5 $R_f\,a \rightarrow R_p a$ (By PC universally quantify consequent of 14.1 wrt n. From this we have by UI $R_f a \rightarrow \sim Ta_1$. Result by 13.1)

.6 $(Pa \rightarrow Pb) \& R_f\,b \rightarrow R_p a$ (By SC we have $R_p b \rightarrow ((Pa \rightarrow Pb) \rightarrow R_p a$ from 14.2. Subst b for a in 15.5. Result by SC)

.7 $R_p a \rightarrow \sim Ta_1$ (A case of SC is $R_p a \rightarrow R_p a$. Replace consequent by defined equivalent using 13.1)

.8 $R_p a \& R_p \sim a \rightarrow \sim Ta_1 \,\&\, \sim Fa_1$ (Subst $\sim a$ for a in 15.7. Result by SC from this and 15.7 using 7.1)

.9 $(Ia \rightarrow R_p a \,\&\, R_p \sim a) \rightarrow (Ia \rightarrow \sim Ta_1 \& \sim Fa_1)$ (By SC from 15.8)

.10 $(a)(Ia \rightarrow R_p a \,\&R_p \sim a) \rightarrow \sim(Ea)(Ia \,\&\, (Ta_1 \vee Fa_1))$ (Universally quantify 15.9 wrt a. Result by PC and SC)

[$\mathcal{S}$ is deductively complete wrt the preferred model if all its irresoluble formulae are such that both they and their negations are pragmatically refutable]

.11 $R_p a \rightarrow \sim Na$ (By EG we have $R_p a \rightarrow (En) \sim Ta_n$ from 15.7. Result by PC and 7.5)

.12 $R_p a \& R_p \sim a \rightarrow Ca$ (Subst. $\sim a$ for a in 15.11. Result by SC from this and 15.11 using 7.8)

.13 $(Ia \rightarrow R_p a \,\&R_p \sim a) \rightarrow (Ia \rightarrow Ca)$ (By SC from 15.12)

.14 $(a)(Ia \rightarrow R_p a \,\&R_p \sim a) \rightarrow \sim (Ea) Strong Ia$ (Universally quantify 15.13 wrt a. Result by PC and SC using 7.8 and 7.13)

[$\mathcal{S}$ is deductively complete in the strong sense if all its irresoluble formulae are such that both they and their negations are pragmatically refutable. Hence if $\mathcal{S}$ satisfies $(a)(Ia \rightarrow R_p a \,\&R_p \sim a)$ it is not incomplete in any significant sense in view of 15.10 and 15.14. It may however contain an irresoluble formula which is true over some model other than the standard model. In view of 15.17 this will not be so if $\mathcal{S}$ is strongly-complete.]

.15 $R_f a \& R_f \sim a \rightarrow \sim (En)(Ta_n \vee Fa_n)$ (Universally quantify consequent of 15.4 wrt n. Result by 13.3, 7.2 and PC)

.16. $(Ia \rightarrow R_f a \,\&R_f \sim a) \rightarrow (Ia \rightarrow \sim (En)(Ta_n \vee Fa_n))$ (By SC from 15.15)

.17 $(a)(Ia \rightarrow R_f a \,\&R_f \sim a) \rightarrow \sim (Ea) Weak Ia$ (Universally quantify 15.16 wrt a. Result by PC using 7.4 and 7.12)

[If $\mathcal{S}$ is strongly-complete it is not deductively incomplete in the weak sense]

We consider finally how $\mathfrak{M}$ may be extended if it is specified that $\mathcal{S}$ contains the predicate calculus of first order. It was noted in (9) that none of the conditions so far imposed on $\mathcal{S}$ exclude its containing the PC; that is, all the existing theorems of $\mathfrak{M}$ hold of such an $\mathcal{S}$. On the other hand, if it is known that $\mathcal{S}$ does in fact contain the PC, then certain special theorems hold in $\mathfrak{M}$ which would not hold of all other $\mathcal{S}$ which satisfy the previous conditions.

X Let $\mathcal{S}$ contain the predicate calculus of first order.

(16) Let the following be additional formation rules of $\mathfrak{M}$:

16.1 A formula-variable followed by one or more of the letters 'x', 'y', 'z' in parentheses is a wff (e.g. $a(x)$, $b(x,y)$). [If more letters are required, primes may be added]. The values of these expressions are formulae of $\mathcal{S}$ which contain free individual variables, the number of different individual variables in the f^i is indicated by the number of different letters in the parentheses adjoined to the formulae-variables.

.2 Let such formulae as are defined in 16.1 be wff if the formulae-variables bear subscripts (e.g. $a_1(x)$, $b_n(x,y)$).

.3 If α is a wff defined in 16.1 then $(\;)\alpha$ and $(E\;)\alpha$ are wff when the blanks are filled by one or more of the letters which occur in the parentheses of α (e.g. $(x)a(x)$, $(Ey)b(x,y)$). The values of such wff are respectively universally quantified and existentially quantified formulae of $\mathcal{S}$. The values are closed formulae wrt the individual variables in them if all letters which occur in the parentheses of α occur in $(\;)$ or $(E\;)$; otherwise open wrt as many individual variables as there are letters in the parentheses of α which do not occur in $(\;)$ or $(E\;)$.

.4 Let such formulae as are defined in 16.3 be wff if α bears a subscript (i.e. apply a rule similar to 16.3 to the formulae of 16.2)

[The rules 16.3 and 16.4 may be extended to admit $(\;)(E\;)\alpha$, $(E\;)(\;)\alpha$, etc., which take as values formulae of $\mathcal{S}$ with mixed quantifiers,]

.5 Let the formation rules of (5)(which apply to formulae-variables apply also to the formulae of 16.1 and 16.3 (e.g. $a(x)\,\&\,b(y)$, $Pa(x)$, $P(Ex)a(x)$).

.6 Let the formation rules of (5) which apply to formulae-variables with subscripts apply also to the formulae of 16.2 and 16.4 (e.g. $a_1(x)\,\&\,b_1(y)$, $Ta_n(x)$, $T(x)a_n(x)$, but not $a_1(x)\,\&\,b_2(y)$ etc)

[Formulae such as $(Ex)Pa(x)$ which have quantifiers on the LHS of an operator have not been defined as wff. Such formulae may be admitted but have to be interpreted with care]

(17) The following conditions govern the use of the formulae of (16):

17.1 Let the definitions of (7) apply to the formulae of (16).

.2 Let the interpretations put on the various operators of $\mathfrak{M}$ remain the same when these occur in conjunction with formulae of (16).

.3 Let $(\;)\alpha$ and $(E\;)\alpha$ be interpreted respectively as, for all ... α—and, there is a ... such that α ---, where the dots are replaced by the letters in $(\;)$ of $(E\;)$ and the dashes by the letters in the parentheses of α.

.4 Let the transformation rules of (10) apply to the formulae of (16), where it is understood that when wff containing quantifiers are excluded from the operation (e.g. (10.3), the reference is specifically to subscript quantifiers and quantifiers over formulae-variables, not to the quantifiers introduced in (16).

.5 Let the transformation rules of PC apply to the formulae of (16). In particular, let the letters 'x', 'y' and 'z' be treated for the purposes of substitution etc as the individual variables of the PC are treated. [In this way we mirror within $\mathfrak{M}$ the operations which can be carried out on such of the f^j of $\mathcal{S}$ as are PC formulae]

(18) Let the following be additional axioms of $\mathfrak{M}$.

18.1 $Pa(x) \rightarrow P(x)a(x)$

.2 $Ta_n(x) \rightarrow T(x)a_n(x)$

[18.1 expresses the fact that a provable open formula of $\mathcal{S}$ may be universally quantified wrt to its free individual variables to yield a further theorem. 18.2 expresses the essential meaning of 'true open interpreted formula'. In view of 16.4, formulae such as $Ta_1(x)$ are admitted; and in view of 17.2, the operator T takes its previous interpretation. Hence we are presupposing that it is *significant* to say of an open interpreted formula of $\mathcal{S}$ that it is true (or false). This is consistent with VII(i) and is inevitable if we wish to say that $\sim Ta_1(x) \& \sim Fa_1(x)$, e.g., is significant; i.e. if we wish to say of an open interpreted formula of $\mathcal{S}$ that it is neither true nor false. Using 18.2 we are now able to explicate the meaning of 'true(false) open interpreted formula'. We show that such a formula is true if, and only if, the closed formula which is obtained from it by universally quantifying all the free individual variables is true; false, if, and only if, the formula which is obtained by existentially quantifying the free individual variables is false.]

(19) Following are further theorems of $\mathfrak{M}$.

19.1 $Ta_n(x) \leftrightarrow T(x)a_n(x)$ (A case of PC is $(x)a_n(x) \rightarrow a_n(y)$. By 10.3 we have $T((x)a_n(x) \rightarrow a_n(y))$. A T_n theorem is $T(a_n \rightarrow b_n) \rightarrow (Ta_n \rightarrow Tb_n)$ (*cf* 11.12). Now derive $T(x)a_n(x) \rightarrow Ta_n(y)$. Subst. x for the free y (if necessary re-writing the bound variable twice) to obtain $T(x)a_n(x) \rightarrow Ta_n(x)$. Result by SC on this and 18.2)

.2 $Fa_n(x) \leftrightarrow F(Ex)a_n(x)$ (Subst. $\sim a(x)$ for $a(x)$ in 19.1. Result by PC and 7.1)

.3 $\sim Ta_n(x) \& \sim Fa_n(x) \leftrightarrow \sim T(x)a_n(x) \& \sim F(Ex)a_n(x)$ (By SC from 19.1 and 19.2)

.4 $T(Ex)\sim a_n(x) \& T(Ex)a_n(x) \rightarrow \sim Ta_n(x) \& \sim Fa_n(x)$ (Subst. $(x)a(x)$ for a in 11.8(1) and use 7.1 and PC to obtain $T(Ex)\sim a_n(x) \rightarrow \sim T(x)a_n(x)$. Now subst. $(Ex)a(x)$ for a in 11.8(1) and transpose to obtain $T(Ex)a_n(x) \rightarrow \sim F(Ex)a_n(x)$. Result by SC and 19.3)

[Let $\mathcal{S}$ be formalized arithmetic and $\mathcal{S}_1$ the (standard) numerical interpretation. Let f_1^1 be $2x+2x=4x$; let f_1^2 be $2x+2x=5x$; let f_1^3 be $x > 10$. Then by 19.1, to say that f_1^1 is true is just to say that $(x)(2x+2x=4x)$ is true; by 19.1, to say that f_1^2 is false is just to say that $(Ex)(2x+2x=5x)$ is false; by 19.4, f_1^3 is neither true nor false, since it is true that $\sim(5 > 10)$ and $(11 > 10)$; from which, by EG, we conclude that $(Ex)\sim(x > 10)$ and $(Ex)(x > 10)$ are both true. In general, an open interpreted formula is true if it holds for all values of the variables, false if it fails for all values, and neither true nor false if there is a value for which it holds and a value for which it fails.]

Theorems similar to 19.1 to 19.4 with P in place of T and $P\sim$ in place of F may be obtained if the above proofs are repeated using formulae-variables without subscripts and 11.8(2) in place of 11.8(1), 11.12 in place of the T_n theorem in 19.1, and 18.1 in place of 18.2. These theorems may be numbered 19.5 to 19.8. 19.7 and 19.8 indicate criteria for demonstrating the irresolubility of certain open formulae of $\mathcal{S}$.

No further theorems of $\mathfrak{M}$ will be given here. Further extensions suggest themselves. In particular, let the free individual variables of $\mathcal{S}$ range over a denumerable set of constants, $v^1, v^2, v^3, \ldots$, which are represented in $\mathfrak{M}$ by the numerals, 1,2,3,..., and let $a(1)$, $a(2)$, $b(1)$, $a(1,2)$, $b(x,1)$, etc., take as values the f^j of $\mathcal{S}$ which result from open formulae when some or all of the individual variables are replaced throughout by constants selected from the $v^1, v^2, \ldots,$ (the numeral(s) in the parenthesis indicating which constants(s) is chosen). Then we may within $\mathfrak{M}$ investigate the concepts of ω-completeness and ω-consistency. For $\mathcal{S}$ is ω-complete if it satisfies $Pa(1) \,\&\, Pa(2) \,\&\, \ldots \rightarrow P(x)a(x)$ for all a; ω-inconsistent if it satisfies $P(Ex)a(x) \,\&\, P \sim a(1) \,\&\, P \sim a(2) \,\&\, \ldots$ for some a. [Such extension requires the admission of formulae of infinite length within $\mathfrak{M}$].

Again, if we permit individual-variable quantifiers on the LHS of an operator, then we may express the fact that, e.g., there is a value for which an open f^j is provable, by the notation $(Ex)Pa(x)$. Such a formula of $\mathfrak{M}$ will be a consequence of each of $Pa(1)$, $Pa(2)$, etc. Similarly, we can express the principle UG (Universal generalization) by $(y)Ta_n(y) \rightarrow T(x)a_n(x)$, thus indicating the difference between it and the non-law $a_n(y) \rightarrow (x)a_n(x)$. The principle EI (Existential instantiation) is expressed by $T(Ex)a_n(x) \rightarrow (Ey)Ta_n(y)$. [$P$-versions of these, if taken as theorems, indicate that UG and EI are rules of $\mathcal{S}$].

It is not claimed that the theorems of $\mathfrak{M}$ which have been given establish new results in methodology. The purpose of this paper as stated at the beginning has been to axiomatize $\mathfrak{M}$ rather than to develop the meta-theory of a given $\mathcal{S}$.

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