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A SET OF AXIOMS FOR THE PROPOSITIONAL CALCULUS WITH IMPLICATION AND CONVERSE NON-IMPLICATION

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It is well-known that implication and converse non-implication constitute a complete system of independent primitive connectives for the propositional calculus. In this article it is the author's intention to give a set of independent axioms for the propositional calculus by means of the two connectives mentioned above, the rules of inference being substitution and *modus ponens*[1]. In setting up the axioms the purpose of the author has been to achieve simplicity of individual axioms while preserving their independence. In §1 we give the set of axioms and prove some preliminary theorems. In §2 we solve the decision problem. Finally, in §3, we establish the independence of the axioms and rules. In the matter of notation and style of presenting proofs of theorems we shall follow Church.

§1. AXIOMS AND PRELIMINARY THEOREMS. The axioms of our logistic system, say **P**, are the six following

Axiom 1. $p \supset . q \supset p$
Axiom 2. $s \supset [p \supset q] \supset . s \supset p \supset . s \supset q$
Axiom 3. $p \supset q \supset p \supset p$
Axiom 4. $p \supset [p \not\subset q] \supset . q \supset . p \not\subset q$
Axiom 5. $p \not\subset q \supset q$
Axiom 6. $p \not\subset q \supset . p \supset s$

In fact, as is evident from the above set, any formulation of the implicational propositional calculus and Axioms 4-6 will suffice. We note that from the present formulation the deduction theorem—to be henceforth referred to as **D.T.**—follows immediately. We now go on to prove some theorems.

Theorem 1. $p \not\subset p \supset s$

1. This is suggested as an open problem in Church's *Introduction to Mathematical Logic*, I. Princeton, N. J., 1956. p. 139.

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Proof. By Axiom 5, $p \not\subset p \vdash p$. By Axiom 6, $p \not\subset p \vdash p \supset s$. Hence $p \not\subset p \vdash s$ Hence by **D.T.**, $\vdash p \not\subset p \supset s$.

Theorem 2. $p \supset [r \not\subset r] \supset . \; q \supset . \; p \not\subset q$

Proof. We have $p \supset [r \not\subset r], p \vdash r \not\subset r$. Hence by Theorem 1, $p \supset [r \not\subset r]$, $p \vdash p \not\subset q$. Hence by **D.T.**, $p \supset [r \not\subset r] \vdash p \supset . \; p \not\subset q$. Hence by Axiom 4, $p \supset [r \not\subset r] \vdash q \supset . \; p \not\subset q$. Hence by **D.T.**, $\vdash p \supset [r \not\subset r] \supset . \; q \supset . \; p \not\subset q$.

Theorem 3. $p \supset . \; q \not\subset r \supset . \; p \supset q \not\subset r$

Proof. By Axiom 6, $q \not\subset r \vdash q \supset . \; s \not\subset s$. Hence $p, q \not\subset r, p \supset q \vdash s \not\subset s$. Hence by **D.T.**, $p, q \not\subset r \vdash p \supset q \supset . \; s \not\subset s$. Hence by Theorem 2, $p, q \not\subset r \vdash r \supset . \; p \supset q \not\subset r$. Again by Axiom 5, $p, q \not\subset r \vdash r$. Hence $p, q \not\subset r$ $\vdash p \supset q \not\subset r$. Hence by **D.T.**, $\vdash p \supset . \; q \not\subset r \supset . \; p \supset q \not\subset r$.

Theorem 4. $p \supset . \; r \supset . \; p \not\subset q \not\subset r$

Proof. By Axiom 6, $p \not\subset q \vdash p \supset . \; s \not\subset s$. Hence $p, p \not\subset q \vdash s \not\subset s$. Hence by **D.T.**, $p \vdash p \not\subset q \supset . \; s \not\subset s$. Hence by Theorem 2, $p \vdash r \supset . \; p \not\subset q \not\subset r$. Hence by **D.T.**, $\vdash p \supset . \; r \supset . \; p \not\subset q \not\subset r$.

Theorem 5. $q \supset [s \not\subset s] \supset . \; p \not\subset q \supset . \; s \not\subset s$

Proof. By Axiom 5, $p \not\subset q \vdash q$. Hence $q \supset [s \not\subset s], p \not\subset q \vdash s \not\subset s$. Hence by **D.T.**, $\vdash q \supset [s \not\subset s] \supset . \; p \not\subset q \supset . \; s \not\subset s$.

Theorem 6. $q \not\subset r \supset . \; p \not\subset q \not\subset r$

Proof. By Axiom 6, $q \not\subset r \vdash q \supset . \; s \not\subset s$. Hence by Theorem 5, $q \not\subset r \vdash$ $p \not\subset q \supset . \; s \not\subset s$. Hence by Theorem 2, $q \not\subset r \vdash r \supset . \; p \not\subset q \not\subset r$. Again by Axiom 5, $q \not\subset r \vdash r$. Hence $q \not\subset r \vdash p \not\subset q \not\subset r$. Hence by **D.T.**, $\vdash q \not\subset r \supset .$ $p \not\subset q \not\subset r$.

Theorem 7. $p \not\subset r \supset . \; q \supset . \; p \not\subset q$

Proof. By Axiom 6, $p \not\subset r \vdash p \supset . \; r \not\subset r$. Hence by Theorem 2, $p \not\subset r \vdash q \supset .$ $p \not\subset q$. Hence by **D.T.**, $\vdash p \not\subset r \supset . \; q \supset . \; p \not\subset q$.

Theorem 8. $p \supset q \supset . \; q \supset r \supset . \; p \supset r$

Proof. By Axiom 1, $q \supset r \vdash p \supset . \; q \supset r$. Hence by Axiom 2, $q \supset r \vdash p \supset q$ $\supset . \; p \supset r$. Hence $p \supset q, q \supset r \vdash p \supset r$. Hence by **D.T.**, $\vdash p \supset q \supset . \; q \supset r \supset .$ $p \supset r$.

Theorem 9. $p \not\subset q \supset s \supset . \; p \supset s \supset . \; q \supset s$

Proof. By Theorem 8, $p \supset s, s \supset [r \not\subset r] \vdash p \supset [r \not\subset r]$. Hence by Theorem 2, $p \supset s, s \supset [r \not\subset r] \vdash q \supset . \; p \not\subset q$. Hence $p \supset s, q, s \supset [r \not\subset r] \vdash p \not\subset q$. Hence $p \not\subset q \supset s, p \supset s, q, s \supset [r \not\subset r] \vdash s$. Hence by **D.T.**, $p \not\subset q \supset s, p \supset s$, $q \vdash s \supset [r \not\subset r] \supset s$. Hence by Axiom 3, $p \not\subset q \supset s, p \supset s, q \vdash s$. Hence by **D.T.**, $\vdash p \not\subset q \supset s \supset . \; p \supset s \supset . \; q \supset s$.

§2. THE DECISION PROBLEM

METATHEOREM 1. *Every theorem of* **P** *is a tautology.*

Proof. This Metatheorem can be established easily. We omit the proof.

METATHEOREM 2. *Let* B *be a wff of* P, *let* $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ *be distinct variables among which are all the variables occurring in* B, *and let* $a_1, a_2, \ldots, a_n$ *be truth-values. Let* C *be any theorem of* P, *i.e.,* $\vdash \mathsf{C}$. *Further, let* A_i *be* $\mathfrak{a}_i$ *or* $\mathfrak{a}_i \not\subset \mathsf{C}$ *according as* a_i *is* T *or* F; *and let* B' *be* B *or* $\mathsf{B} \not\subset \mathsf{C}$ *according as the value of* B *for the values* $a_1, a_2, \ldots, a_n$ *of* $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ *is* T *or* F. *Then* $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vdash \mathsf{B}'$.

Proof. In order to prove that

(1) $$\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vdash \mathsf{B}'$$

we proceed by mathematical induction with respect to the number of occurrences of $\supset$ and $\not\subset$ in B.

If there are no occurrences of $\supset$ and $\not\subset$ in B, then B is one of the variables $\mathfrak{a}_i$. Hence B' is the same wff as A_i, and (1) follows trivially.

Suppose that there are occurrences of $\supset$ or $\not\subset$ or both in B. Then B is either $\mathsf{B}_1 \supset \mathsf{B}_2$ or $\mathsf{B}_1 \not\subset \mathsf{B}_2$. By the hypothesis of induction,

(2) $$\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vdash \mathsf{B}_1'$$

(3) $$\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vdash \mathsf{B}_2'$$

where B_1' is B_1 or $\mathsf{B}_1 \not\subset \mathsf{C}$ according as the value of B_1 for the values $a_1, a_2, \ldots, a_n$ of $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ is T or F, and B_2' is B_2 or $\mathsf{B}_2 \not\subset \mathsf{C}$ according as the value of B_2 for the values $a_1, a_2, \ldots, a_n$ of $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ is T or F.

CASE I. B is of the form $\mathsf{B}_1 \supset \mathsf{B}_2$.

In case B_2' is B_2, we have that B' is $\mathsf{B}_1 \supset \mathsf{B}_2$, and (1) follows from (3) by Axiom 1. In case B_1' is $\mathsf{B}_1 \not\subset \mathsf{C}$, we have again that B' is $\mathsf{B}_1 \supset \mathsf{B}_2$ and (1) follows from (2) by Axiom 6. There remains only the case that B_1' is B_1 and B_2' is $\mathsf{B}_2 \not\subset \mathsf{C}$ and in this case B' is $\mathsf{B}_1 \supset \mathsf{B}_2 \not\subset \mathsf{C}$, and (1) follows from (2) and (3) by Theorem 3.

CASE II. B is of the form $\mathsf{B}_1 \not\subset \mathsf{B}_2$.

In case B_1' is B_1, we have that B' is $\mathsf{B}_1 \not\subset \mathsf{B}_2 \not\subset \mathsf{C}$ and (1) follows from (2) by Theorem 4. (It is to be noted here that $\vdash \mathsf{C}$). In case B_2' is $\mathsf{B}_2 \not\subset \mathsf{C}$, we have again that B' is $\mathsf{B}_1 \not\subset \mathsf{B}_2 \not\subset \mathsf{C}$, and (1) follows from (3) by Theorem 6. There remains only the case that B_1' is $\mathsf{B}_1 \not\subset \mathsf{C}$ and B_2' is B_2, and in this case B' is $\mathsf{B}_1 \not\subset \mathsf{B}_2$ and (1) follows from (2) and (3) by Theorem 7.

Therefore Metatheorem 2 is proved by mathematical induction.

METATHEOREM 3. *If* B *is a tautology,* $\vdash \mathsf{B}$.

Proof. Let $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ be the variables of B, and for any system of values $a_1, a_2, \ldots, a_n$ of $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ let $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ be as in Metatheorem 2. The B' of Metatheorem 2 is B, because B is a tautology. Therefore by Metatheorem 2,

$$\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vdash \mathsf{B}$$

This holds for either choice of a_n, i.e., whether a_n is F or T, and so we have both

$$\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_{n-1}, \mathfrak{a}_n \not\subset \mathsf{C} \vdash \mathsf{B}$$

and

$$A_1, A_2, \ldots, A_{n-1}, a_n \vdash B$$

By the deduction theorem,

$$A_1, A_2, \ldots, A_{n-1} \vdash a_n \not\subset C \supset B$$

$$A_1, A_2, \ldots, A_{n-1} \vdash a_n \supset B$$

Hence, by Theorem 9,

$$A_1, A_2, \ldots A_{n-1} \vdash C \supset B$$

Hence, since $\vdash C$,

$$A_1, A_2, \ldots, A_{n-1} \vdash B.$$

This shows the elimination of the hypothesis A_n. The same process may be repeated to eliminate the hypothesis A_{n-1}, and so on, until all the hypotheses are eliminated. Finally we obtain $\vdash B$.

In Metatheorem 1 and Metatheorem 3, together with the algorithm for determining whether a wff is a tautology, we have a solution of the decision problem of P. The consistency and completeness of P, now follow as corollaries of this solution.

§3. INDEPENDENCE. The independence of the axioms and rules of P, with the exception of the rule of substitution, is established by the standard device of generalised systems of truth-values (see tables below).

The independence of the rule of substitution can be established by a well-known argument. For the proof of independence of *modus ponens*, it is necessary to supply also an example of a theorem of P which is not a tautology according to the truth-table (Table No. 1) used. One such example is $p \supset p$. Lastly, since the calculations are extremely long to prove the independence of Axiom 2, the author wishes to point out for the convenience of the reader that when s, p, q take the values *4, 5, 3* respectively the axiom yields a non-designated value according to the truth-table (Table No. 3) used.

TABLE NO. 1. (MODUS PONENS)

$\supset$	0	1	2
*0	0	0	0
1	0	2	0
2	0	0	0

$\not\subset$	0	1	2
*0	2	2	2
1	2	2	2
2	2	2	2

TABLE NO. 2. (AXIOM 1)

$\supset$	0	1	2	3	4
*0	0	1	2	3	4
*1	0	1	3	3	4
*2	0	1	0	3	4
3	0	1	0	0	1
4	0	1	0	0	1

$\not\subset$	0	1	2	3	4
*0	4	4	4	4	4
*1	4	4	4	4	4
*2	4	4	4	4	4
3	0	0	2	4	4
4	0	0	2	4	4

TABLE NO. 3. (AXIOM 2)

$\supset$	0	1	2	3	4	5
*0	0	1	2	3	5	5
*1	0	1	2	3	5	5
*2	2	1	0	3	5	5
3	0	1	0	2	4	4
4	0	0	0	3	0	0
5	1	1	1	1	1	1

$\not\subset$	0	1	2	3	4	5
*0	5	5	5	5	5	5
*1	5	5	5	5	5	5
*2	5	5	5	5	5	5
3	5	5	5	5	5	5
4	3	3	3	3	5	5
5	0	0	0	3	5	5

TABLE NO. 4. (AXIOM 3)

$\supset$	0	1	2
*0	0	1	2
1	0	0	2
2	0	0	0

$\not\subset$	0	1	2
*0	2	2	2
1	2	2	2
2	0	1	2

TABLE NO. 5. (AXIOM 4)

$\supset$	0	1
*0	0	1
1	0	0

$\not\subset$	0	1
*0	1	1
1	1	1

TABLE NO. 6. (AXIOM 5)

$\supset$	0	1
$*0$	0	1
1	0	0

$\not\subset$	0	1
$*0$	1	1
1	0	0

TABLE NO. 7 (AXIOM 6)

$\supset$	0	1
$*0$	0	1
1	0	0

$\not\subset$	0	1
$*0$	0	1
1	0	1

Remark. Axiom 1, Axiom 2, Axiom 5, Axiom 6 and Theorem 9 also constitute a complete set. For, (1) Axiom 4 follows immediately from Theorem 9 by substitution and *modus ponens*, and (2) in order to prove the completeness of P, we need Axiom 3 only in one place: to prove Theorem 9.

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